Interest Rates & Credit Risk Midterm Test

Solutions

1. (a) Eq'n 3.29

[2] $$r_t = r_0 e^{-kt} + \theta(1-e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dW_s.$$

(b) $$Y_T = \int_0^T [(r_0-\theta)e^{-kt} + \theta] dt + \sigma \int_0^T \left[\int_0^t e^{-k(t-s)} dW_s\right] dt.$$

[2]

$$\text{Term 1} = \frac{r_0-\theta}{k}\left[1-e^{-kT}\right] + \theta T$$

[1]

$$\text{Term 2} = \sigma \int_0^T \left[\int_s^T e^{-k(t-s)} dt\right] dW_s$$

[1] $$= \frac{\sigma}{k}\int_0^T \left(1-e^{-k(T-s)}\right) dW_s.$$

[1] Thus $\mu_y := E[Y_T] = \left(\frac{r_0-\theta}{k}\right)(1-e^{-kT}) + \theta T$

$$\sigma_y^2 := Var[Y_T] = \frac{\sigma^2}{k^2}\int_0^T \left(1-e^{-k(T-s)}\right)^2 ds \quad (\text{Itô isometry})$$

[1] $$= \frac{\sigma^2}{k^2}\left[T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})\right].$$

$$\mu_r := E[r_T] = (r_0-\theta)e^{-kT} + \theta$$

[1]

$$\sigma_r^2 := Var[r_T] = \sigma^2 \int_0^T e^{-2k(T-s)} ds = \frac{\sigma^2}{2k}(1-e^{-2kT})$$

[1]

$$\rho\sigma_y\sigma_r := E[(Y_T-\mu_y)(r_T-\mu_r)] = \frac{\sigma^2}{k}\int_0^T \left(1-e^{-k(T-s)}\right)e^{-k(T-s)} ds$$

[1]

$$= \frac{\sigma^2}{k}\left[\frac{1}{k}(1-e^{-kT}) - \frac{1}{2k}(1-e^{-2kT})\right]$$

[1] Conclude $(r_T, Y_T) \sim N(\mu, \Sigma)$

$$\mu = \begin{bmatrix}\mu_r \\ \mu_y\end{bmatrix}, \quad \Sigma = \begin{bmatrix}\sigma_y^2 & \rho\sigma_y\sigma_r \\ \rho\sigma_y\sigma_r & \sigma_r^2\end{bmatrix}.$$

1 c) Let $Z_t = E\left[\frac{dQ^T}{dQ}\middle|\mathcal{F}_t\right] := e^{\int_0^t (\phi_s dW_s^Q - \frac{1}{2}\phi_s^2 ds)}$ (2)

Then $-\phi_t$ is drift of W^Q under Q^T.

Then $Z_t = E\left[\frac{e^{-Y_T}}{P_0(T)}\middle|\mathcal{F}_t\right] = e^{-Y_t} \times E\left[\frac{e^{-(Y_T-Y_t)}}{P_0(T)}\middle|\mathcal{F}_t\right]$

Note $Y_T - Y_t = \left(\frac{r_t-\theta}{k}\right)(1-e^{-k(T-t)}) +$ a term indep. of $\mathcal{F}_t$

$\therefore \quad Z_t = e^{-Y_t - \frac{1}{k}(1-e^{-k(T-t)})r_t} \times$ deterministic fn.

Now $dZ_t = Z_t\left[\underbrace{-dY_t}_{\text{pure drift}} - \frac{\sigma}{k}(1-e^{-k(T-t)})dW_t^Q + \text{pure drift}\right]$

$= \phi_t Z_t \, dW_t^Q$

$\therefore \quad \boxed{-\phi_t = +\frac{\sigma}{k}(1-e^{-k(T-t)})}$ = drift of BM W^Q under Q^T.

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Alternative $\quad Z_t = \frac{e^{-Y_t} \cdot e^{A(t,T) + B(t,T)r_t}}{e^{A(0,T) + B(0,T)r_0}}$ (Prop 3.2.2)

$dZ_t = Z_t\left[B(t,T)\,dr_t + \text{drift terms}\right]$.

[3] $\quad \therefore \quad \phi_t = \sigma B(t,T) = \frac{\sigma}{k}\left[e^{-k(T-t)} - 1\right] = -$ drift of BM

2.
$$E_t = BSCall(A_t, K, r, \sigma, T-t) = S_t N. \quad (\star)$$

$$dE_t = \frac{\partial E_t}{\partial A} \cdot dA_t + \frac{\partial E_t}{\partial t} dt + \frac{1}{2} \frac{\partial^2 E_t}{\partial A^2} \cdot d[A,A]_t$$

$$= N(d_{1,t}) \cdot \sigma A_t \, dW_t^{\circ} + \text{drift}$$

$$:= \sigma_t^{(s)} E_t \, dW_t^{\circ} + \text{drift}$$

$$\therefore \quad \sigma_t^{(s)} = \frac{\sigma A_t N(d_1)}{E_t}$$

[3]

where E_t given by $(\star)$

$$d_1(A, K, r, \sigma, T-t) = \frac{\log(A/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

For fixed K, r, σ let $F(A,t) = BSCall(A, K, r, \sigma, t)$

for $t>0$. $F(A,t)$ is monotone increasing in A

$F(-\infty, t) = 0$, $F(+\infty, t) = +\infty$

Let $A = G(x,t)$ be solution of $F(A,t) = x$, $x \in (0,\infty)$

Then
$$f(T-t, S) = \frac{\sigma G(NS, T-t) N(d_1)}{NS}$$

[1]

where $d_1 = d_1(G(NS, T-t), K, r, \sigma, T-t)$.

$$\therefore \quad \frac{dS_t}{S_t} = r\,dt + f(T-t, S_t)\,dW_t^{\circ}$$

Yes S is a Markov process, since it solves

[1]

an Ito SDE of type Eqn A.1

(Note: SDE is not time-homogeneous).

3. (a) $P[\tau > t \mid \mathcal{F}_s] = E[H_t^c \mid \mathcal{F}_s] = H_s^c\, E\left[e^{-\int_s^t \lambda_u du} \mid \mathcal{G}_s\right]$

[2]

[2] $= H_s^c\, f(t-s, r_s, \lambda_s, 0, 1)$

(b) $\bar{P}_t^0(T) = E\left[H_T^c\, e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right] = H^c\, E\left[e^{-\int_t^T (r_s + \lambda_s) ds} \mid \mathcal{G}_t\right]$

[2]

[2] $= H_t^c\, f(T-t, r_t, \lambda_t, 1, 1)$

(c) f, by Feynman-Kac, solves

NOTE: sign change because of T−t

[3] $$\begin{cases} -\partial_t f + \mathcal{L}f - (ur + v\lambda) f = 0, & t > 0 \\ f(0, r, \lambda, u, v) = 1 \end{cases}$$

where $\mathcal{L}f = (a - br)\partial_r f + (a' - b'r - c'\lambda)\partial_\lambda f$

[1] $+ \frac{1}{2}\left[\sigma^2 r\, \partial_r^2 f + \sigma'^2 \lambda\, \partial_\lambda^2 f\right]$

$(-\dot{A} - \dot{B} r - \dot{C}\lambda) + (a - br)B + (a' - b'r - c'\lambda)C$

[1] $+ \frac{1}{2}\left[\sigma^2 r B^2 + \sigma'^2 \lambda C^2\right] - (ur + v\lambda) = 0$

[2] $$\begin{cases} \dot{A} = aB + a'C, & A(0) = 0 \\ \dot{B} = -bB - b'C + \frac{1}{2}\sigma^2 B^2 - u = 0, & B(0) = 0 \\ \dot{C} = -c'C + \frac{1}{2}\sigma'^2 C - v = 0, & C(0) = 0 \end{cases}$$