Tutorial on Semantics
Part II
Domain Theory

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Outline

What we need

- We saw that we needed fixed-point theory at *all* types.
- We therefore need to define models of data types that support this.
- We also need functions between data types to be data types.
- Since we are looking at properties of all data types together we need to look at the *category* of data types.

Definition

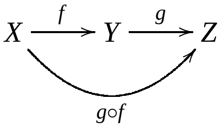
A **category** $\mathcal{C}$ consists of two collections: $\mathcal{C}_0$ **objects** and $\mathcal{C}_1$ **morphisms**.

There are functions $dom, cod : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ and a partial function $\circ : \mathcal{C}_1 \times \mathcal{C}_1 \longrightarrow \mathcal{C}_1$ called **composition**.

The function $g \circ f$ is defined if and only if $cod(f) = dom(g)$ and when it is defined $dom(g \circ f) = dom(f)$, $cod(g \circ f) = cod(g)$.

For every $X \in \mathcal{C}_0$ there is a unique morphism id_X which is an identity for composition.

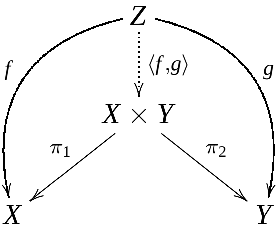
Composition is associative.



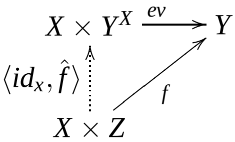
Some categorical concepts

- The collection of objects can be a set: small category.
- The collection of morphisms between two objects can be a set: locally small category. We write $Hom(A,B)$ or $\mathcal{C}(A,B)$: homset.
- A **functor** $\mathcal{F}$ relates two categories: it maps objects to objects and morphisms to morphisms and it preserves identities and composition.
- $$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \overset{\mathcal{F}}{\longmapsto} \quad \mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Z)$$
  $$g\circ f \qquad\qquad \mathcal{F}(g\circ f)=\mathcal{F}(g)\circ\mathcal{F}(f)$$

Products



A category may or may not have products.



This makes the concept of homset (or function space) *internal*; i.e. there are objects that behave like the homsets.

Terminal Objects

Definition

An object in a category is **terminal** if there is a unique morphism to it from every object.

Definition

An object in a category is **initial** if there is a unique morphism from it to every object.

Cartesian closed categories

- A CCC has finite products,
- a terminal object
- and exponentials.
- We want our domains to form a CCC.

Basic properties of domains

- Domains should capture the idea of *partial information*.
- This is expressed *qualitatively* through a domain.
- A domain should be a poset with a least element.
- A directed set $X \subseteq D$ satisfies: $\forall x, y \in X \exists z \in X$ with $x, y \leq z$. It represents a *consistent* collection of data. Every directed set should have a least upper bound (sup, $\bigvee$).
- Such posets are called **dcpo**s for directed-complete posets.
- Henceforth, all domains will be dcpos; more conditions later.
- Functions between domains should be monotone.
- Functions between domains should preserve sups of directed sets: continuity.

Approximation

- We want some concept of "piece of information".
- We say that b is an *essential approximation of* y if whenever there is a directed set X with $y \leq \bigvee X$ then for some $x \in X$ we have $b \leq x$; we write $b \ll y$.
- Any limiting process that passes y must pass b *at some finite stage*.
- Example: consider the domain consisting of subsets of the integers. Then an essential approximation of the set of positive even numbers is $\{2, 6, 8\}$ but the set of positive powers of two is an approximation but not an essential approximation.
- We will write $\Downarrow(x)$ for the set of essential approximations to x.

Bases for domains

- We would like to have a collection of “tractable” elements that allow one to represent everything in the domain.
- A **basis** B for a domain D is a (countable) family of elements such that for every $d \in D$ the set of elemnts $B_d = B \cap \twoheaddownarrow(d)$ is directed and $\bigvee B_d = d$.
- A domain with a (countable) basis is said to be (ω-)**continuous**.
- We say that e is *finite* (compact) if $e \ll e$.
- Sometimes we do not have enough finite elements but we can often find enough essential approximations.
- Example: $[0, 1]$ with the usual order has only one finite element but the rational form a nice countable basis.

Examples of continuous domains

- The set of all subsets of positive integers, ordered by inclusion. Take the *finite* subsets as the basis. These are actually *finite elements*; which partly explains the terminology.
- The set of all partial functions from a countable set to itself ordered by inclusion of graphs.
- The set of all subprobability distributions on a finite set, ordered pointwise.
- A countable basis is given by all the distributions that assign rational weights to each point.
- Continuous domains arise whenever one is dealing with real numbers: probabilistic systems, real-time systems, computing with real numbers.

Algebraic domains

- One wants to relate the denotational semantics with the operational semantics; one needs to work with “syntactically representable elements” as a way of forging this connection.
- It usually happens that this connection is mediated by finite elements.
- A continuous domain in which all the basis elements are finite (not finite in number!) is called an **algebraic** domain.
- For the traditional semantic applications algebraic domains are very important. For more recent applications to real-time, hybrid and probabilistic systems continuous domains are necessary.
- Whence comes this name “algebraic”?
- The collection of finitely generated subgroups in the lattice of subgroups of a given group forms an algebraic dcpo. Many examples in algebra come from finitely generated meaning “finite”.

Function spaces

- What are general functions spaces?
- If D and E are dcpos then we define $[D \longrightarrow E]$ to be the poset of *continuous* functions from D to E with the following order

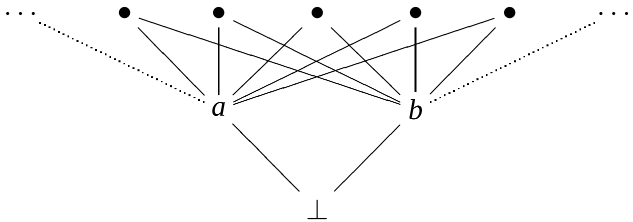
$$f \leq g \text{ iff } \forall d \in D,\ \ f(d) \leq_E g(d).$$

- It is not hard to show that $[D \longrightarrow E]$ is itself a dcpo.
- We can define $D \times E$ as $\{(x,\ y) | x \in D, y \in E\}$ with the order $(x,\ y) \leq (x',\ y')$ iff $x \leq_D x'$ *and* $y \leq_E y'$.
- If we define **Dcpo** to be the category with dcpos as objects and continuous functions as morphisms we get a cartesian closed category.

Algebraic functions spaces

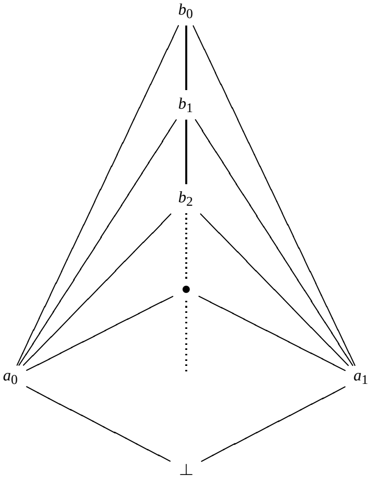
- If we adopt ω-algebraicity as a basic requirement for our domains we need to ensure that the function spaces are also ω-algebraic.
- However, we cannot take domains to be arbitrary ω-algebraic dcpos.
- There are three famous examples due to Gordon Plotkin of ω-algebraic dcpos D with $[D \longrightarrow D]$ not ω-algebraic.

Plotkin's first example

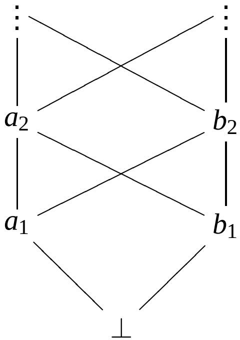


This is not too bad, it *is* algebraic but not ω-algebraic.

Plotkin's second example



Plotkin's third example



These last two are really terrible!

Scott domains

Definition

A pair of elements x, y in a dcpo are said to be **bounded** or **consistent** if there is some z such that $x, y \leq z$.

Definition

A **Scott domain** is an ω-algebraic dcpo such that every non-empty finite set of elements has a least upper bound.

They are also called bounded-complete dcpos or consistently-complete dcpos.

Function spaces of Scott domains

- Easy to see that Plotkin's examples are all ruled out.
- Easy fact: if e_1, e_2 are compact and $e_1 \sqcup e_2$ exists, then it is also compact; hence, same is true for finite sets of compact elements.
- If D and E are Scott domains and the finite elements are denoted $\{d_i\}$ and $\{e_j\}$ respectively, then the following are compact elements of the function space

$$d_i \nearrow e_j(x) = \begin{cases} e_j, & \text{if } d_i \leq x; \\ \bot & \text{otherwise.} \end{cases}$$

- They are called step functions.
- Do reasonable sups of these things always exist?

Sups of step functions

- When should $d_1 \nearrow e_1$ and $d_2 \nearrow e_2$ be consistent?
- When d_1 and d_2 are consistent then e_1 and e_2 should be consistent.
- In that case $e = e_1 \sqcup e_2$ exists,
- *because of bounded completeness*!
- Then we can define

$$(d_1 \nearrow e_1 \sqcup d_2 \nearrow e_2)(x) = \begin{cases} e_1, & \text{if } d_1 \leq x \text{ but } d_2 \not\leq x; \\ e_2, & \text{if } d_2 \leq x \text{ but } d_1 \not\leq x; \\ e, & \text{if } d_1 \leq x \text{ and } d_2 \leq x; \\ \bot & \text{otherwise.} \end{cases}$$

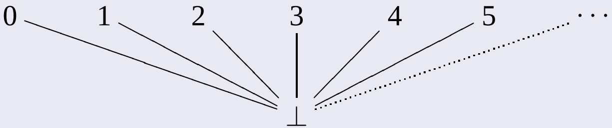
- Now we can get a basis for the function space by taking sups of all bounded (consistent) finite collections of step functions.
- The category of Scott domains is cartesian closed.

Is this the best one can do?

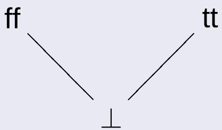
- Gordon Plotkin defined a larger category – the SFP domains – which ruled out his three examples and showed that this gives a CCC of ω-algebraic domains. He needed it for his work on powerdomains and nondeterministic computation.
- Mike Smyth showed that this is the *largest* CCC of ω-algebraic domains.
- Carl Gunter showed that the Scott domains are the largest *first-order axiomatizable* CCC of ω-algebraic domains.
- Achim Jung showed that there were exactly 4 maximal CCCs of algebraic domains.
- Why do we need more CCCs if Scott domains are good enough for PCF?
- We need them when we add new features – nondeterminism, probability – to the language and need to model them.

Basic domains for PCF

The "flat" domain of naturals: $\mathbb{N}_\perp$



The flat domain of booleans: $\mathcal{B}_\perp$



Denotational semantics of PCF I

- The ground types

$$[\![Nat]\!] = \mathbb{N}_\perp; \quad [\![Bool]\!] = \mathcal{B}_\perp.$$

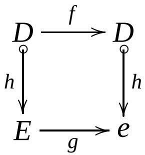
- The higher types

$$[\![\sigma \times \tau]\!] = [\![\sigma]\!] \times [\![\tau]\!]; \;\; [\![\sigma \to \tau]\!] = [[\![\sigma]\!] \to [\![\tau]\!]].$$

- The constants, pairing, projection, plus, equals and conditionals are interpreted the obvious way.
- The λ-calculus part is interpreted in the manner we have already indicated. We need to show various things are continuous.
- It only remains to explain *fix*.

Fix

- Given D a Scott domain (any dcpo with $\perp$ will do);
  define $\mathsf{fix}_D : [D \longrightarrow D] \longrightarrow D$ by
  $\mathsf{fix}_D(f) = \bigvee\{\perp, f(bot), \ldots, f^{(n)}(\perp), \ldots\}$.
- This is itself a continuous function.
- The family fix_D is the *unique* family satisfying the following uniformity condition. If h is strict ($h(\perp) = \perp$) and the diagram



  commutes, then $h(\mathsf{fix}_D(f)) = \mathsf{fix}_E(g)$.
- $$[\![fix(M)]\!] = \mathsf{fix}([\![M]\!]).$$

Subject reduction and soundness

Theorem

If $\Gamma \vdash M : \tau$ *is a valid typing judgment and* $M \xrightarrow{*} N$ *then* $\Gamma \vdash N : \tau$ *is a valid typing judgment.*

Theorem

If $\Gamma \vdash M : \tau$ *and* $M \xrightarrow{*} N$ *then* $[\![M]\!] = [\![N]\!]$.

Contexts

- A **context** in PCF is essentially a term with a “hole” in it into which another term of the appropriate type can be plugged in.
- For example $\lambda x.\langle 2, x[\cdot]\rangle$. If we put a term of the right type in the hole, we will get a PCF term.
- A semantics is *compositional* if $[\![M]\!] = [\![N]\!]$ implies that for all contexts $C[\cdot]$ (of the right type) $[\![C[M]]\!] = [\![C[N]]\!]$.
- The denotational semantics of PCF based on domains (the standard model) is compositional.
- If $C[\cdot]$ is such that $C[M]$ is of ground type, we say C is a ground context.

Observations

- We cannot test terms of all types for equality, only ground types.
- We can observe a ground term by seeing to what value it reduces.
- We write $M \Downarrow m$ if the term $M : Nat$ eventually reduces to the number m.
- What can we observe about higher type terms?
- We say M, N are **observationally equivalent** if for all *ground* contexts $C[\cdot]$ for M and N, $C[M] \Downarrow v$ if and only if $C[N] \Downarrow v$; we write $M \equiv_{obs} N$.
- We write $M \Downarrow \perp$ to mean $\forall v. \neg(M \Downarrow v)$.
- We would like our denotational semantics to be a good guide to observational equivalence.

Adequacy

Definition

We say a semantics is **adequate** if

$$[\![M]\!] = [\![N]\!] \Rightarrow M \equiv_{obs} N.$$

This is equivalent to

Theorem

$$[\![M]\!] = [\![v]\!] \Leftrightarrow M \Downarrow v.$$

Proof sketch

Assume $[\![M]\!] = [\![N]\!]$ and the proposition holds. Let $C[\cdot]$ be a ground context and v a value such that $C[M] \Downarrow v$. Thus $[\![C[M]]\!] = [\![C[v]]\!] = [\![C[N]]\!]$, where we have used compositionality. Thus, $C[N] \Downarrow v$.

The grand theorem of PCF

Theorem

The denotational semantics of PCF is adequate.

Reasoning with higher-type languages

- How can we reason about higher type languages?
- We use both the term structure and the type structure.
- Terms of simple structure – like variables – can have arbitrarily complicated types.
- Therefore the induction arguments are not just nicely nested.
- Furthermore, we have to deal with substitutions into open terms.
- The main technique uses *logical relations* invented by Tait in 1967 to prove strong normalization of simply-typed λ-calculus.
- We will illustrate logical relations with the proof of adequacy.
- For simplicity, I will forget about products.

The computability predicate

- If $M : Nat$ is *closed* it is said to be **computable** if $[\![M]\!] = [\![v]\!]$ implies $M \Downarrow v$.
- If $M : \tau \longrightarrow \tau'$ is closed it is computable if, for every closed *computable* term $N : \tau$, $MN : \tau'$ is computable.
- If M has free variables $\{x_1, \ldots, x_k\}$ then it is computable if for every substitution $M[N_1/x_1, \ldots, N_k/x_k]$ of closed computable terms for the free variables we get a computable term.
- We call such a substitution computable.
- We write σ for a substitution and $\sigma[M]$ for the term resulting from the substitution.

The Proof I

- We claim every PCF term is computable. Induction on structure of terms and types.
- $M = x : \tau$; a computable substitution will certainly produce a computable term.
- Cases where M is a conditional or $plus$ are easy structural induction cases.

The Proof II

- $M = \lambda x.Q : \tau_1 \longrightarrow \tau_2$. Let σ be a computable substitution and let $\vec{T}$ be a sequence of closed computable terms such that $\sigma[M]\vec{T}$ is of ground type and that $[\![\sigma[M]\vec{T}]\!] = [\![v]\!]$.
- $\sigma[M]\vec{T} = \sigma[\lambda x.Q]T_1T_2\ldots T_k = (\lambda x.\sigma[Q])T_1T_2\ldots T_k$
- $= (\sigma[Q][T_1/x_1])T_2\ldots T_k$.
- Now the term $(\sigma[Q][T_1/x_1]) = Q[T_1/x_1, S_1/y_1, S_2/y_2, \ldots]$ is just another substitution instance of Q by a computable substitution σ'. Hence, by the induction hypothesis it is computable.
- Thus $[\![\sigma[M]]\!]\vec{T} = [\![\sigma'[Q]T_2\ldots T_k]\!] = [\![v]\!]$ implies that $\sigma'[Q]T_2\ldots T_k \Downarrow v$.
- Hence $\sigma[M]\vec{T} \Downarrow v$.
- One can prove the application case with similar arguments.

The Proof III: Fix sketched

- Here we need another theorem: approximation.
- Imagine the recursion unwound to some depth and then wherever fix occurs we replace it with $\perp$.
- The collection of partial unwindings are the *syntactic approximants*.
- We can show that the denotational semantics of the syntactic approximants give a directed set with least upper bound the meaning of the original term.
- We can show that if any of the approximants applied to closed computable terms converges to v then so does the original term.
- We prove by induction on the depth of the unwinding that the unwindings are computable.
- Putting all this together we can complete the argument.

A perfect match?

- We would like our denotational semantics to be a perfect match with observational equivalence.

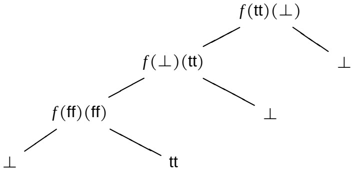
$$[\![M]\!] = [\![N]\!] \Leftrightarrow M \equiv_{obs} N.$$

- Unfortunately, it is not!
- Consider the function "parallel or" with the following table

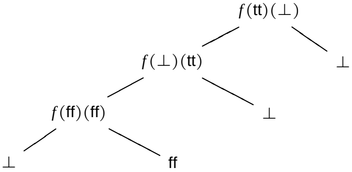
por	$\bot$	tt	ff
$\bot$	$\bot$	tt	$\bot$
tt	tt	tt	tt
ff	$\bot$	tt	ff

- This function cannot be defined in PCF; proved by Plotkin in 1977.
- This function is "listening in parallel" to two inputs and will use whichever one converges first.
- However, the operational semantics of PCF is sequential.

The problem with parallel or



Call this term T.



Call this term F.

Consider the terms $(\lambda f.T)por = \text{tt}$ and $(\lambda f.F)por = \text{ff}$. So it is definitely the case that $[\![T]\!] \neq [\![F]\!]$. However, no PCF definable term will ever see the difference.

What can be done about this?

- Add parallel or to the language or some other parallel construct.
- Various extended languages were shown to have fully abstract domain models.
- Key step in proving full abstraction: all the finite elements are definable.
- Construct a fully abstract model from the syntax: Milner 1977.
- All fully abstract models are isomprphic, so the question is one of presenting a fully abstract model in an insightful way. The domain model gives insight into the nature of computation that is not just mimicking the operational semantics.
- Try to characterize sequential computation mathematically.

Stable domain theory: Berry

- Berry introduced a new restriction and a new order on functions – the stable order – and introduced stronger finiteness conditions.
- In Scott domains a finite (i.e. compact) element can be above infinitely many elements! This does not happen in stable domain theory.
- PCF can be given an adequate semantics with stable domains.
- Parallel or does not appear in stable domains.
- Unfortunately, other more complicated examples can be given – discovered by Berry himself – that show that full abstraction fails.

Need a more intensional view

- Berry and Curien started the study of sequential *algorithms* on concrete data structures.
- Girard invented linear logic in the mid 1980s and this made a huge impact on the semantics community by making resource sensitivity an integral part of logic and proof theory.
- Abramsky and Jagadeesan developed full completeness results for linear logic based on dialogue games.
- Abramsky, Jagadeesan and Malacaria and simultaneously and independently Hyland and Ong and also independently Nickau developed fully abstract games models for PCF.
- O'Hearn and Riecke gave domain theoretic fully abstract models but they were also based on intensional ideas.

Loader's result

- Ralph Loader showed that observational equivalence of even finitary PCF is undecidable.
- This means that no fully abstract model can be effectively presented.

The Game Universe

- The basic idea is to model data types as dialogue games and programs as strategies: there is no notion of winning or losing.
- Remarkably different programming paradigms appear as different restrictions on allowed strategies.
- Two important restrictions needed for modelling PCF are called *innocence* and *bracketing*. Loosening these restrictions yields fully abstract models of extensions of PCF!