Set Mean Estimation and Confidence Supersets using Oriented Distance Functions

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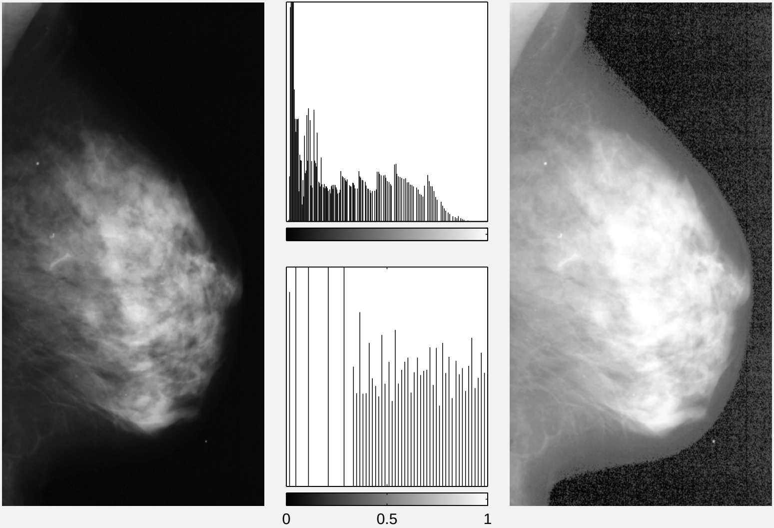
MOTIVATION



OUTLINE

1. distance functions and oriented distance functions (ODFs)
2. random closed sets (RCSs) and their expectation
   - selection expectation
   - Baddeley & Molchanov definition
   - ODF definition
3. properties of new definition
4. confidence regions/supersets
5. examples
   - sand grains
   - boundary reconstruction in a mammogram image

ORIENTED DISTANCE FUNCTION (ODF)

Fix $D \subset \mathbb{R}^d$, and let $d(x, y) = |x - y|$ denote Euclidean distance.

The distance function of $A \subset D$ such that $A \neq \emptyset$ is

$$d_A(x) = \inf_{y \in A} d(x, y) \quad \text{for } x \in D.$$

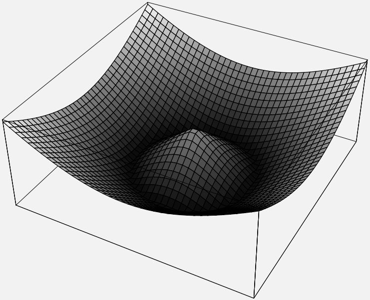
Note that $d_A(x) = d_C(x)$ iff $\overline{A} = \overline{C}$, and $A = \{x : d_A(x) = 0\}$.

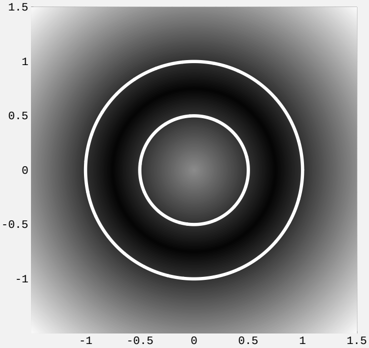
The oriented distance function of $A \subset D$ such that $\partial A \neq \emptyset$ is

$$b_A(x) = d_A(x) - d_{A^c}(x).$$

Here, $A = \{x : b_A(x) \leq 0\}$ and $\partial A = \{x : b_A(x) = 0\}$.

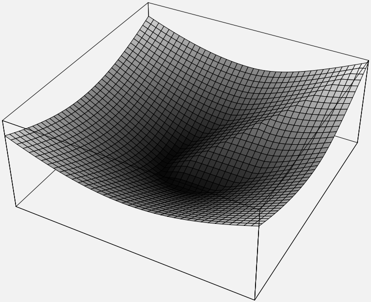
ODF OF A DONUT

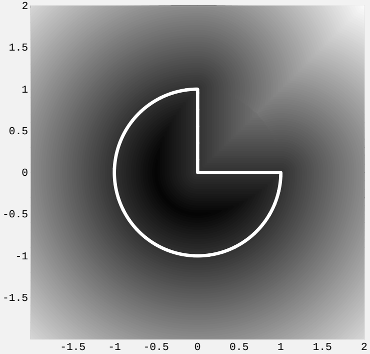




$$\text{donut} = \{x \in \mathbb{R}^2 : 0.5 \leq |x| \leq 1\}$$

ODF OF PACMAN





RANDOM CLOSED SET (RCS)

Let $\mathcal{F}$ be the family of closed subsets of $\mathbb{R}^d$ and $\mathcal{K}$ be the family of all compact subsets of $\mathbb{R}^d$. Consider a probability space given by the triple $(\Omega, \mathcal{A}, P)$.

Definition

A random closed set is the mapping $A : \Omega \mapsto \mathcal{F}$, such that for every compact set K

$$\{\omega : A(\omega) \cap K \neq \emptyset\} \in \mathcal{A}.$$

- Foundations laid by Choquet (1950s), Matheron (1975)
- Modern review by Molchanov (2005)
- As $\mathcal{F}$ is nonlinear, there is no natural way to define the expectation of a set.

SET EXPECTATION

Examples: Let $\mathbf{A} = \{x : |x| \leq \Theta\}$ and $\mathbf{B} = \{\xi\}$.

1. selection (Aumann) expectation
   - most studied
   - depends on structure of $(\Omega, \mathcal{A}, P)$
   - gives convex answer
   - $\mathrm{E}_A[\mathbf{A}] = \{x : |x| \leq \mathrm{E}[\Theta]\}$ and $\mathrm{E}_A[\mathbf{B}] = \{\mathrm{E}[\xi]\}$
2. Vorobe'ev expectation
   - most intuitive in terms of image analysis
   - $\mathrm{E}_V[\mathbf{A}] = \{x : |x| \leq \sqrt{\mathrm{E}[\Theta^2]}\}$ and $\mathrm{E}_V[\mathbf{B}] = \emptyset$
3. Baddeley & Molchanov definition
   - depends on significant user input (choice of two metrics)
   - complicated to calculate
   - $\mathrm{E}_{BM}[\mathbf{A}] = \{x : |x| \leq \mathrm{E}[\Theta]\}$ and $\mathrm{E}_{BM}[\mathbf{B}] = \{\mathrm{E}[\xi]\}$

OUR DEFINITION

Definition

Suppose that A is a random closed set such that $\partial A \neq \emptyset$ a.s. and $E|b_A(x_0)| < \infty$ for some $x_0 \in D$, then

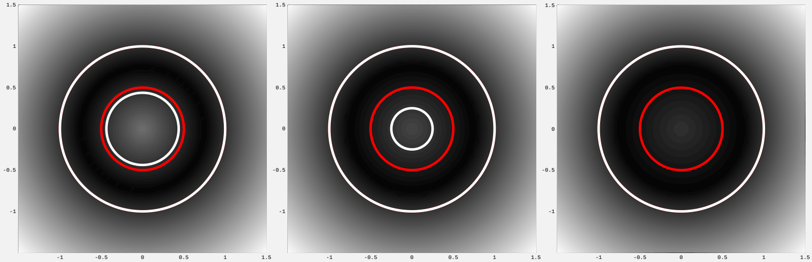
$$\mathrm{E}[\mathbf{A}] = \{x : \mathrm{E}[b_{\mathbf{A}}(x)] \leq 0\}$$
$$\Gamma[\mathbf{A}] = \{x : \mathrm{E}[b_{\mathbf{A}}(x)] = 0\}$$

- simple and intuitive, no user input
- algorithms for distance functions and level sets easily available (eg. MATLAB)
- includes definition for boundary
- $\mathrm{E}[\mathbf{A}] = \{x : |x| \leq \mathrm{E}[\Theta]\}$ and $\mathrm{E}[\mathbf{B}] = \emptyset$

EXAMPLE: MISSING TIMBIT

The RCS **A** is equal to

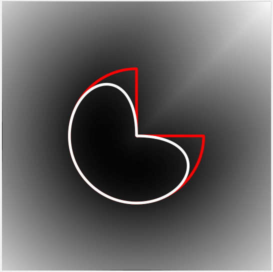
$$\begin{array}{lll} \text{a circle:} & \{x : |x| \leq 1\} & \text{with probability } 1 - p \\ \text{a donut:} & \{x : 0.5 \leq |x| \leq 1\} & \text{with probability } p \end{array}$$



$p = 0.9$ $p = 0.75$ $p < 2/3$

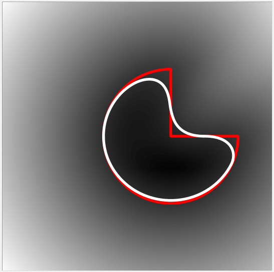
$\Gamma[\mathbf{A}]$ is shown in white.

EXAMPLE: PACMAN



- (white) mean pacman with uniform radius
- (red) pacman with mean radius

- (white) mean pacman with uniform NE shift
- (red) pacman with mean NE shift



- (white) mean pacman with uniform E shift
- (red) pacman with mean E shift

Properties

- some basics:
  - $\mathrm{E}[\mathbf{A}]$ is closed
  - $\partial\,\mathrm{E}[\mathbf{A}] \subset \Gamma[\mathbf{A}]$
  - if $\mathbf{A} \subset \mathbf{B}$ a.s. then $\mathrm{E}[\mathbf{A}] \subset \mathrm{E}[\mathbf{B}]$
  - if $\mathbf{A} = A$ a.s. then $\mathrm{E}[\mathbf{A}] = A$ and $\Gamma[\mathbf{A}] = \partial A$
  - if $\partial\mathbf{A} = B$ a.s. then $B \subset \Gamma[\mathbf{A}]$
- preservation of shape:
  - (translation) $\mathrm{E}[a + \mathbf{A}] = a + \mathrm{E}[\mathbf{A}]$
  - (dilation) $\mathrm{E}[\alpha\mathbf{A}] = \alpha\,\mathrm{E}[\mathbf{A}]$ for $\alpha \neq 0$
  - (equivariant w.r.t. orthogonal transformations) $\mathrm{E}[g\mathbf{A}] = g\,\mathrm{E}[\mathbf{A}]$, for $g(x) = \Lambda x + a$ and $\Lambda \in O(d)$
  - If $\mathbf{A}$ is convex a.s. then $\mathrm{E}[\mathbf{A}]$ is convex.
- preservation of smoothness:
  - If $\mathrm{E}[b_{\mathbf{A}}(x)]$ is smooth then $\Gamma[\mathbf{A}]$ is smooth.

ESTIMATION

Suppose that we observe the random sets $A_1, \ldots, A_n$ under IID sampling. Define $\bar{b}_n(x) = \sum_{i=1}^n b_{A_i}(x)/n$, and

$$\overline{A}_n = \{x : \bar{b}_n(x) \leq 0\} \quad \text{and} \quad \overline{\Gamma}_n = \{x : \bar{b}_n(x) = 0\}$$

Theorem

Suppose that $\mathrm{E}[\mathbf{A}]$ *satisfies* $\partial\,\mathrm{E}[\mathbf{A}^c] = \Gamma[\mathbf{A}]$ *then*

$$\overline{A}_n \to \mathrm{E}[\mathbf{A}] \quad a.s..$$

If in addition we have that $\partial\,\mathrm{E}[\mathbf{A}] = \Gamma[\mathbf{A}]$ *then*

$$\overline{\Gamma}_n \to \Gamma[\mathbf{A}] \quad a.s..$$

CONFIDENCE REGIONS/SUPERSETS

- Let $\mathbb{Z}_n(x) = \sqrt{n}\left(\bar{b}_n(x) - \mathrm{E}[b_{\mathbf{A}}(x)]\right)$ and assume that $\mathrm{E}[b_{\mathbf{A}}(x_0)^2] < \infty$ for some $x_0 \in D$ (compact).
- Then $\mathbb{Z}_n \Rightarrow \mathbb{Z}$, where $\mathbb{Z}$ is a smooth Gaussian field.
- Let q_1 and q_2 denote numbers such that $P(\sup_{x\in D} \mathbb{Z}(x) \leq q_1) = 0.95$ and $P(\sup_{x\in D} |\mathbb{Z}(x)| \leq q_2) = 0.95$.

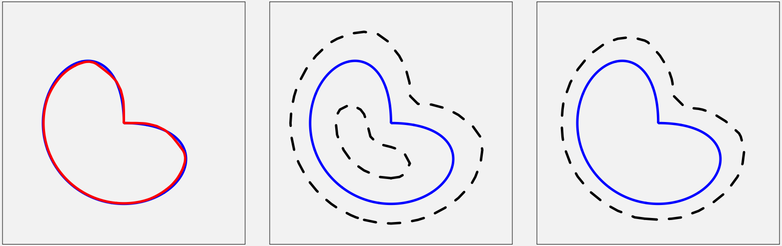
Definition

$$\left\{x : \bar{b}_m(x) \leq \frac{1}{\sqrt{m}}\, q_1\right\} \quad \text{and} \quad \left\{x : |\bar{b}_m(x)| \leq \frac{1}{\sqrt{m}}\, q_2\right\}$$

are 95% confidence regions for $\mathrm{E}[\mathbf{A}]$ and $\Gamma[\mathbf{A}]$.

CONFIDENCE REGIONS: PACMAN

Suppose the random model is pacman of random radius Θ where $\Theta \sim$Uniform$[0, 1]$. We observe 25 IID sets from this model.



LEFT: Estimated set (red) and the expected set (blue).

MID: Expected set boundary with 95% bootstrapped confidence interval.

RIGHT: Expected set with 95% bootstrapped confidence interval.

CONFIDENCE REGIONS: PROPERTIES

- Easy, visual way of describing variability around the mean
- CRs are conservative with a probability of at least 95% of capturing the expected set
- Quantiles are often intractable, but can be easily estimated via bootstrapping
- Immune to consistency conditions
- Allows for local changes in variability

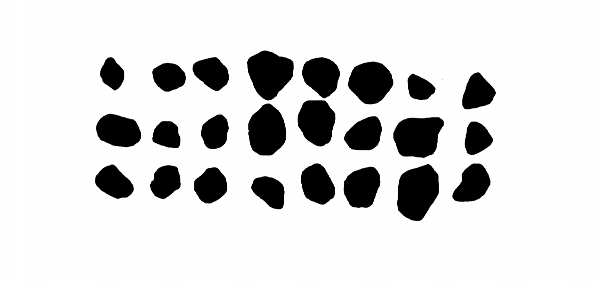
- EQUIVARIANCE PROPERTIES: let **C** denote the confidence region for E[**A**]. Then
  1. The confidence region for E[α**A**] is α**C**, for $\alpha \neq 0$.
  2. The confidence region for E[g**A**] is g**C**, where g is a rigid motion.

EMPIRICAL COVERAGE PROBABILITIES

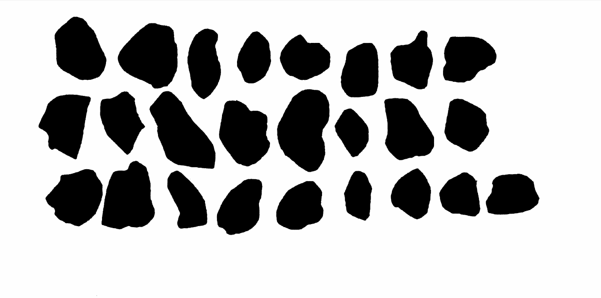
	$n = 25$		$n = 100$	
$100(1-\alpha)\%$	90%	95%	90%	95%
(A)	88.4/89.7	94.8/95.6	90.4/91.2	95.7/94.1
(B)	90.2/89.9	94.6/95.1	90.2/90.4	95.1/95.0
(C1)	90.1/91.2	94.4/95.1	91.6/91.1	95.3/95.3
(C2)	91.4/93.5	96.5/97.4	92.1/93.3	96.9/97.1
(D1)	92.0/91.0	96.3/95.7	91.8/91.7	96.2/95.7
(D2)	90.7/88.6	94.5/94.7	91.0/88.9	94.9/95.0

(A) $\mathbf{A} = [0,1]$ or $\{0,1\}$ with equal probability
(B) pacman with random radius
(C1)/(C2) union of two discs
(D1)/(D2) random ellipse

EXAMPLE: SAND GRAINS

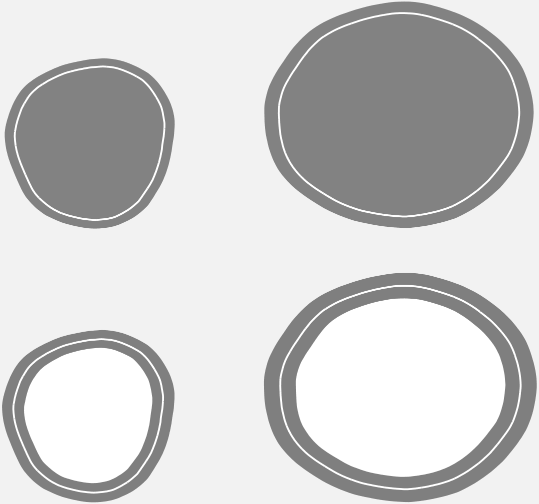


sand grains from the Baltic sea



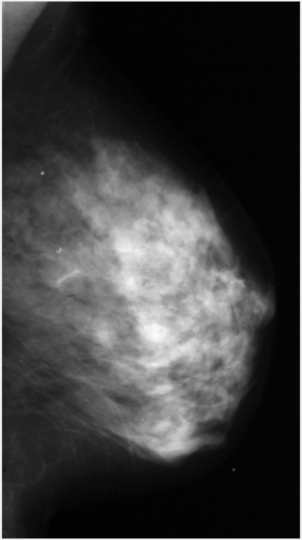
sand grains from the Zelenchuk river

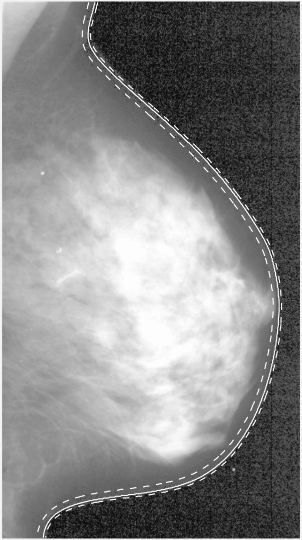
EXAMPLE: SAND GRAINS

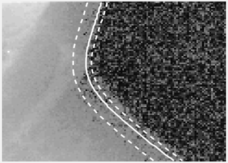


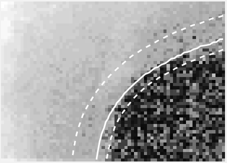
Zelenchuk river　　　　Baltic sea

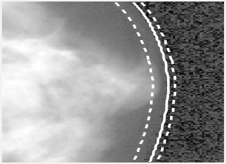
EXAMPLE: MAMMOGRAM











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