Conductivity imaging in the presence of perfectly conducting and insulating inclusions from one interior measurement

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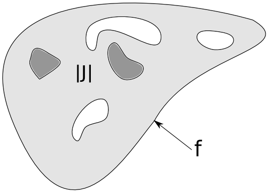
The problem: conductivity imaging

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with connected Lipschitz boundary. The goal is to determine

- isotropic conductivity σ
- the shape and location of the perfectly conducting and insulating inclusions

from one measurement of the magnitude of the current density field $|J|$ generated inside Ω while imposing the voltage f at $\partial\Omega$.



Mathematical Model

Let U, V be open subsets of Ω with $\overline{U} \subset \Omega$, $\overline{V} \subset \Omega$, $\overline{U} \cap \overline{V} = \emptyset$, and the boundaries ∂U, ∂V are piecewise $C^{1,\alpha}$. Also let $\sigma_1 \in L^\infty(U)$, and $\sigma \in L^\infty(\Omega \setminus \overline{U \cup V})$ be bounded away from zero. For $k > 0$ consider the conductivity problem

$$\begin{cases} \nabla \cdot [(\chi_U(k\sigma_1 - \sigma) + \sigma)\nabla u] = 0, & \text{in } \Omega \setminus \overline{V} \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial V, \\ u\big|_{\partial\Omega} = f. \end{cases} \tag{1}$$

The perfectly conducting inclusions occur in the limiting case $k \to \infty$.

The limiting equation

The limiting solution is the unique solution to the problem:

$$
\begin{cases}
\nabla \cdot \sigma \nabla u_0 = 0, & \text{in}\, \Omega \setminus \overline{U \cup V}, \\
\nabla u_0 = 0, & \text{in} \;\; U, \\
u_0|_+ = u_0|_-, & \text{on} \;\; \partial(U \cup V), \\
\int_{\partial U_j} \sigma \frac{\partial u_0}{\partial \nu}|_+ ds = 0, & j = 1, 2, ..., \\
\frac{\partial u_0}{\partial \nu}|_+ = 0, & \text{on}\, \partial V, \\
u_0|_{\partial \Omega} = f,
\end{cases}
\tag{2}
$$

where $U = \cup_{j=1}^{\infty} U_j$ is the partition in open connected components.

The Inverse Problem

Is it possible to uniquely determine the open sets U and V and the conductivity σ on $\Omega \setminus \overline{U \cup V}$ from the knowledge of $(f, |J|)$?

We prove that the answer is **yes**, under some mild assumptions. Indeed we will indentify u_σ as the unique minimizer of the functional

$$F(u) = \int_\Omega |J||\nabla u|,$$

over

$$A = \{u \in W^{1,1}(\Omega) : \quad u = f \quad \text{on} \quad \partial\Omega\}.$$

Singular Inclusions and failure of the Ohm's low

- For $\sigma \in C^\alpha$ with $\alpha < 1$ the non-trivial solutions of the elliptic equation
$$\nabla.(\sigma \nabla u) = 0 \;\; \text{in} \;\; \Omega \setminus \overline{U \cup V}$$
may be constant on an open set $W \subset \Omega \setminus \overline{U \cup V}$ and consequently $|J| \equiv 0$ in W. We call such regions W singular inclusions.
- Ohm's law is not valid inside perfectly conducting inclusions. In particular the current inside perfectly conducting inclusions U is not necessarily zero while $\nabla u \equiv 0$ in U.

The limiting equation

The limiting solution is the unique solution to the problem:

$$\begin{cases} \nabla \cdot \sigma \nabla u_0 = 0, & \text{in}\, \Omega \setminus \overline{U \cup V}, \\ \nabla u_0 = 0, & \text{in} \quad U, \\ u_0|_+ = u_0|_-, & \text{on} \quad \partial(U \cup V), \\ \int_{\partial U_j} \sigma \frac{\partial u_0}{\partial \nu}|_+ ds = 0, & j = 1, 2, ..., \\ \frac{\partial u_0}{\partial \nu}|_+ = 0, & \text{on}\, \partial V, \\ u_0|_{\partial \Omega} = f, \end{cases} \tag{3}$$

where $U = \cup_{j=1}^{\infty} U_j$ is the partition in open connected components.

Admissibility ...

Definition 1 A pair of functions $(f, a) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$ is called *admissible* if the following conditions hold:
(i) There exist two disjoint open sets $U, V \subset \Omega$ (possibly empty) and a function $\sigma \in L^\infty(\Omega \setminus (U \cup V))$ bounded away from zero such that $\Omega \setminus (\overline{U \cup V})$ is connected and

$$\begin{cases} a = |\sigma \nabla u_\sigma| \text{ in } \Omega \setminus (\overline{U \cup V}), \\ a = 0 \text{ in } V, \end{cases}$$

where $u_\sigma \in H^1(\Omega)$ is the weak solution of (3).
(ii) The following holds

$$\inf_{u \in W^{1,1}(U)} \left(\int_U a|\nabla u| - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu}|_+ u \right) = 0, \tag{4}$$

where ν is the unit normal vector field on ∂U pointing outside U.

... admissibility

(iii) The set of zeroes of the function a outside $\overline{U}$ can be partitioned as follows

$$\{x \in \Omega : \; a(x) = 0\} \cap (\Omega \backslash \overline{U}) = V \cup \overline{W} \cup \Gamma, \tag{5}$$

where W is an open set (possibly empty) , Γ is a Lebesgue-negligible set, and $\overline{\Gamma}$ has empty interior.

We call σ a *generating conductivity* and u_σ the *corresponding potential*.

Physical data $(f, |J|)$ is admissible

Proposition 1:

Let $a \in L^\infty(\Omega)$ and U be an open subset of Ω. Then

- If $a \geq |J|$ in U for some J with $\nabla \cdot J \equiv 0$ in U and $J_- = \sigma \frac{\partial u_\sigma}{\partial \nu}|_+$ on ∂U, then the condition (4) in Definition 1 holds.
- If the the condition (4) in Definition 1 holds, then

$$\int_U \sigma \frac{\partial u_\sigma}{\partial \nu} = 0.$$

Unique determination ...

Theorem 1: Let $\Omega \subset R^n$, $n \geq 2$ be a domain with connected Lipschitz boundary and let $(f, |J|) \in C^{1,\alpha}(\partial\Omega) \times L^2(\Omega)$ be an admissible pair generated by some unknown $\sigma \in C^{\alpha}(\Omega \backslash (\overline{U \cup V}))$ conductivity, where U and V are open sets as described in Definition 1. Then the potential u_σ is a minimizer of the problem

$$u = \text{argmin}\{\int_\Omega |J||\nabla v| : v \in W^{1,1}(\Omega), \quad v|_{\partial\Omega} = f\}, \tag{6}$$

and if u is another minimizer of the above problem, then $u = u_\sigma$ in

$$\Omega \backslash \{x \in \Omega : \quad |J| = 0\}.$$

Moreover the set of zeros of $|J|$ and $|\nabla u_\sigma|$ can be decomposed as follows

$$\{x \in \Omega : \quad |J| = 0\} \cup \{x \in \Omega : \quad \nabla u_\sigma = 0\} =: Z \cup \Gamma,$$

where Z is an open set and Γ has measure zero and

$$Z = U \cup V \cup W.$$

Consequently $\sigma = \frac{|J|}{|\nabla u_\sigma|} \in L^\infty(\Omega \setminus \overline{Z})$ is the unique $C^\alpha(\Omega \setminus \overline{Z})$-conductivity outside Z for which $|J|$ is the magnitude of the current density while maintaining the voltage f at the boundary.

Determining type of the inclusions

Theorem 1 allows us to identify the potential $u = u_\sigma$ and the conductivity σ outside the open set $Z = U \cup V \cup W$.

- If $\nabla u \equiv 0$ in O and $|J|(x) \neq 0$ for some $x \in O$, then O is a perfectly conducting inclusion.
- If $|J| \equiv 0$ in O and $u \not\equiv \textit{constant}$ on ∂O, then O is an insulating inclusion.
- If $J \equiv 0$ in O, $u = \textit{constant}$ on ∂O, and $|J|$ is not C^α at x for some $x \in O$, then O is either an insulating inclusion or a perfectly conducting inclusion.
- If $J \equiv 0$, $u = \textit{constant}$ on ∂O, and $|J| \in C^\alpha(\partial O)$, then the knowledge of the magnitude of the current $|J|$ (and even the full vector field J) is not enough to determine the type of the inclusion O.

A connection to weighted least gradient problems

Theorem 1 can also be applied independently to prove uniqueness of the minimizers of the weighted least gradient problem

$$u_0 = \text{argmin}\{\int_\Omega a|\nabla u|, \quad u \in W^{1,1}(\Omega), \quad \text{and} \quad u|_{\partial\Omega} = f\}, \tag{7}$$

$a \in L^\infty(\Omega)$.

Let $D = \{x \in R^2 : \quad x^2 + y^2 < 1\}$ be the unit disk and $f(x,y) = x^2 - y^2$. Consider the problem

$$u_0 = \operatorname{argmin}\{\int_D |\nabla u|, \quad u \in W^{1,1}(D), \quad \text{and} \quad u|_{\partial D} = f\}, \tag{8}$$

which corresponds to $a \equiv |J| \equiv 1$ in D. We show that $(1, x^2 - y^2)$ is an admissible pair.

let $U = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \times (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $V = \emptyset$. Define

$$\sigma = \begin{cases} \frac{1}{4|x|}, & \text{if} \quad |x| \geq \frac{1}{\sqrt{2}}, \quad |y| \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{4|y|}, & \text{if} \quad |x| \leq \frac{1}{\sqrt{2}}, \quad |y| \geq \frac{1}{\sqrt{2}}, \end{cases}$$

and

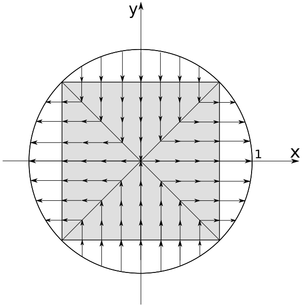
$$u_\sigma = \begin{cases} 2x^2 - 1, & \text{if} \quad |x| \geq \frac{1}{\sqrt{2}}, \quad |y| \leq \frac{1}{\sqrt{2}}, \\ 0, & \text{if} \quad (x, y) \in U, \\ 1 - 2y^2, & \text{if} \quad |x| \leq \frac{1}{\sqrt{2}}, \quad |y| \geq \frac{1}{\sqrt{2}}. \end{cases}$$

... Sternberg-Ziemer example

Define the vector field $J(x, y)$ in U as follows

$$J(x,y) = \begin{cases} -j, & \text{if} \quad y \geq |x|, \\ j, & \text{if} \quad -y \geq |x|, \\ i, & \text{if} \quad x > |y|, \\ -i, & \text{if} \quad -x > |y|, \end{cases}$$

Current density vector field for Sternberg -Ziemer example :



... Sternberg-Ziemer example

Let

$$U_0 = \{(x, y) \in U| \ \ |x| \neq |y|\} = T_1 \cup T_2 \cup T_3 \cup T_4,$$

where T_i, $1 \leq i \leq 4$, are the four disjoint triangles in the above figure. Then $|J| = 1$ in U, $J \in C^\infty(U_0)$ and we have

$$\begin{aligned} \int_U |\nabla u| - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u &\geq \int_{U_0} |J||\nabla u| - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u \\ &\geq \int_{U_0} J \cdot \nabla u - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u \\ &= \sum_{i=1}^{4} \int_{T_i} J \cdot \nabla u - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u \\ &= \int_{\partial U} J \cdot \nu u - \int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} u \\ &= 0, \end{aligned}$$

since $J \cdot \nu \equiv \sigma \frac{\partial u_\sigma}{\partial \nu}$ on ∂U. Thus and $(1, x^2 - y^2)$ is admissible.

A proposition

Let $\Omega \subset R^n$, $n \geq 2$ be a domain and $(f, |J|) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$. Then

1. Assume $(f, |J|)$ is admissible, say generated by some conductivity $\sigma \in L^\infty(\Omega \backslash (\overline{U \cup V}))$ where U and V is described in Definition 1 and u_0 is the corresponding voltage potential. Then u_0 is a minimizer for $\int_\Omega a|\nabla u|$ over
$$A := \{u \in W^{1,1}(\Omega) : \quad u|_\Omega = f\}. \tag{9}$$

2. Assume that the set of zeros of $a = |J|$ can be decomposed as follows
$$\{x \in \Omega : \quad a(x) = 0\} = V \cup \Gamma_1,$$
where V is an open set and Γ_1 has measure zero. Suppose u_0 is a minimizer for $\int_\Omega a|\nabla u|$ in over A and the set of zeroes of $|\nabla u_0|$ can be decomposed as follows
$$\{x \in \Omega \setminus V : |\nabla u_0| = 0\} = \overline{U} \cup \Gamma_2,$$
where U is an open set and $\overline{U \cup V} \subset \Omega$, and Γ_2 has measure zero. If $U \cap V = \emptyset$ and $|J|/|\nabla u_0| \in L^\infty(\Omega \backslash (\overline{U \cup V}))$, then $(f, |J|)$ is admissible.

Sketch of the uniqueness proof ...

- By our assumptions $|J| > 0$ a.e. in $\Omega \setminus \overline{U \cup V \cup W}$ which yields $|\nabla u_0| > 0$ a.e. on $\Omega \setminus \overline{U \cup V \cup W}$. Since $U \cup W$ is a disjoint union of countably many connected open sets and u_0 is constant on every connected open subset of $U \cup W$, the set

$$\Theta := \{u_0(x) : \quad x \in \overline{U \cup W}\}$$

is countable.

- Without loss of generality we can assume $u_0 \geq 0$ in $\overline{\Omega}$. Then

$$\begin{aligned} F(u_1) &= \int_{\Omega \setminus \overline{U \cup V \cup W}} \sigma |\nabla u_0|.|\nabla u_1| dx \geq \int_{\Omega \setminus \overline{U \cup V \cup W}} \sigma |\nabla u_0 . \nabla u_1| dx \\ &\geq \int_{\Omega \setminus \overline{U \cup V \cup W}} \sigma \nabla u_0 . \nabla u_1 = \int_{\partial\Omega} \sigma_0 \frac{\partial u_0}{\partial \nu} u_1 ds = \int_{\partial\Omega} \sigma_0 \frac{\partial u_0}{\partial \nu} f ds \\ &= F(u_0), \end{aligned}$$

where ν is the outer normal to the boundary of Ω.

- Consequently

$$\frac{\nabla u_0(x)}{|\nabla u_0(x)|} = \frac{\nabla u_1(x)}{|\nabla u_1(x)|} \tag{10}$$

  a.e. on

$$(\Omega \setminus \overline{U \cup V \cup W}) \cap \{x \in \Omega : |\nabla u_1| \neq 0\}.$$

- Let $E_t = \{x \in \Omega \setminus \overline{U \cup V \cup W} : u_0(x) > t\}$. Since Θ is countable, for a.e. $t > 0$, $\partial E_t \cap \overline{(U \cup W)} = \emptyset$ (otherwise u_0 must be a constant). By the regularity result of De Giorgi we conclude that $\partial E_t \cap \Omega \setminus \overline{V}$ is a C^1-hypersurface for almost all $t > 0$.

- By (10) we can show u_1 is constant on every C^1 connected component of $\partial E_t \cap (\Omega \backslash \overline{V})$.
- Finally we show that every connected component Σ_t of ∂E_t intersects $\partial\Omega$ and therefore $u_1 = u_2$.

Thank You