Electrical Impedance Tomography: 3D reconstructions using scattering transforms

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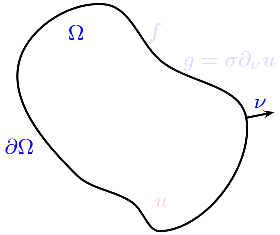
Potential equation and Calderón problem

- 3D domain Ω, boundary $\partial\Omega$, outward normal ν
- conductivity σ (real valued)



- voltage f on $\partial\Omega$

$\Rightarrow$ potential u:

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \quad \Omega,$$
$$u = f \quad \text{on} \quad \partial\Omega.$$

$\Rightarrow$ current $g = \sigma\partial_\nu u$ on $\partial\Omega$

Dirichlet-to-Neuman map (voltage-to-current map) $\Lambda_\sigma : f \mapsto g$.

Calderón Problem: Is σ uniquely determined by Λ_σ and does there exist an algorithm to compute σ from Λ_σ?

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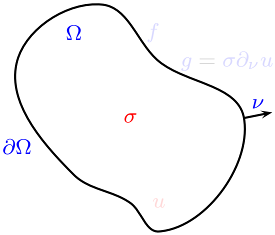

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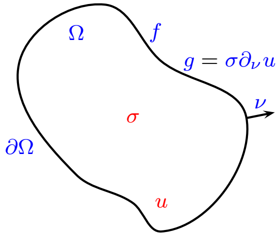

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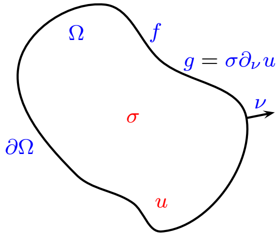

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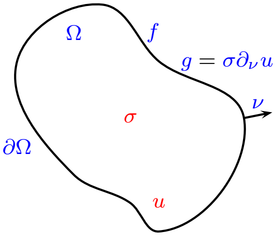

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Short and incomplete history

1980 Calderón: Problem posed, uniqueness for linearized problem, and linear, approximate reconstruction algorithm.

• 2D

1996 Nachman: Uniqueness and reconstruction for $W^{2,p}(\Omega)$ conductivities.
1997 Liu: Stabilty for $W^{2,p}(\Omega)$ conductivities.
1997 Brown-Torres: Uniqueness for $W^{1,p}(\Omega)$ conductivities.
2001 Barceló-Barceló-Ruiz: Stability for $C^{1+\varepsilon}$ conductivities.
2001 Knudsen-Tamasan: Reconstruction for $C^{1+\varepsilon}$ conductivities.
2005 Astala-Päivärinta: Uniqueness and reconstruction for $L^{\infty}(\Omega)$ conductivities.
2009 Knudsen-Lassas-Mueller-Siltanen: Regularized $\overline{\partial}$-method.
2010 Clop-Faraco-Ruiz: Stability for discontinuous conductivities.

• 3D

1987 Sylvester and Uhlmann: Uniqueness for smooth conductivities. Implicit reconstruction algorithm.
1987-88 Novikov, Nachman-Sylvester-Uhlmann, Nachman: Uniqueness for conductivities with 2 derivatives and explicit high frequency reconstruction algorithm. Multidimensional $\overline{\partial}$-bar equation.
1990 Alessandrini: Stability.
2003 Brown-Torres, Päivärinta-Panchenko-Uhlmann: Uniqueness for conductivities with $3/2$ derivatives.
2006 Cornean-Knudsen-Siltanen: Low frequency reconstruction algorithm.
2010 Bikowski-Knudsen-Mueller: Numerical implementation of simplified reconstruction algorithm.
2011 Delbary-Hansen-Knudsen: Implementation of more accurate numerical reconstruction

Outline

- Reconstruction algorithm
- Simplifications
- Implementation
- Numerical results
- Conclusion and outlook

Assumptions

- $\Omega = B(0,1)$, unit ball in $\mathbb{R}^3$ (to simplify implementation).

- $\sigma \in C^\infty(\overline{\Omega})$ (can be less regular).

- $\sigma = 1$ in the neighborhood of $\partial\Omega$ (to simplify implementation).

  $\Rightarrow$ σ extended by 1 in $\mathbb{R}^3 \setminus \overline{\Omega}$

For $f \in H^{1/2}(\partial\Omega)$, if u is the solution to the potential equation

$$\begin{aligned} \nabla \cdot \sigma \nabla u &= 0 \quad \text{in} \quad \Omega, \\ u &= f \quad \text{on} \quad \partial\Omega, \end{aligned}$$

then $v = \sigma^{1/2} u$ is the solution to the Schrödinger equation

$$\begin{aligned} -\Delta v + q v &= 0 \quad \text{in} \quad \Omega, \\ v &= f \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $q = \dfrac{\Delta \sigma^{1/2}}{\sigma^{1/2}}$.

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CGO solutions ψ_ζ , $\zeta \in \mathbb{C}^3$, $\zeta \cdot \zeta = 0$

$$(-\Delta + q)\psi_\zeta = 0 \text{ in } \mathbb{R}^3,$$
$$\psi_\zeta(x) \approx e^{ix\cdot\zeta} \text{ for large } |x| \text{ or } |\zeta|.$$

Small or large $\zeta \Rightarrow$ existence and uniqueness of the solutions.

Reconstruction Algorithm
Scattering transform

$\xi \in \mathbb{R}^3$, $\zeta \in \mathbb{C}^3$, $\zeta^2 = (\xi + \zeta)^2 = 0$

$$\mathbf{t}(\xi, \zeta) = \int_{\mathbb{R}^3} q(x) e^{-ix\cdot(\xi+\zeta)} \psi_\zeta(x)\, dx.$$

$\psi_\zeta(x) \approx e^{ix\cdot\zeta}$ for large $\zeta \Rightarrow |\hat{q}(\xi) - \mathbf{t}(\xi, \zeta)| = \mathcal{O}\left(\frac{1}{|\zeta|}\right)$

How to compute $\mathbf{t}$ from Λ_σ?

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Green's formula in Ω

$$\mathbf{t}(\xi, \zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)} [(\Lambda_\sigma - \Lambda_1)\psi_\zeta](x)\, ds(x).$$

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Faddeev Green's function

$$G_\zeta(x) = \frac{e^{ix\cdot\zeta}}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{ix\cdot\xi}}{|\xi|^2+2\xi\cdot\zeta}\,d\xi\ ,\ x\in\mathbb{R}^3\setminus\{0\},$$

G_ζ fundamental solution to the Laplace equation

$$-\Delta G_\zeta = \delta_0 \ \text{ with }\ G_\zeta(x)\sim e^{ix\cdot\zeta}\ \text{ for large }\ |x|.$$

$$(-\Delta + q)\psi_\zeta = 0 \text{ in } \mathbb{R}^3,$$
$$\psi_\zeta(x) \approx e^{ix\cdot\zeta} \text{ for large } |x| \text{ or } |\zeta|.$$

Green's formula in $\mathbb{R}^3 \setminus \overline{\Omega}$ and condition at infinity

$$\psi_\zeta(x) + \int_{\partial\Omega} G_\zeta(x-y)[(\Lambda_\sigma - \Lambda_1)\psi_\zeta](y)\, ds(y) = e^{ix\cdot\zeta}, \; x \in \partial\Omega,$$

i.e.

$$\psi_\zeta(x) + [S_\zeta(\Lambda_\sigma - \Lambda_1)\psi_\zeta](x) = e^{ix\cdot\zeta}, \; x \in \partial\Omega,$$

with S_ζ single layer operator.

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Reconstruction Algorithm

$$\Lambda_\sigma \xrightarrow{1} \mathbf{t}(\xi,\zeta) \xrightarrow{2} q(x) \xrightarrow{3} \sigma(x)$$

1. Solve

$$\psi_\zeta(x) + [S_\zeta(\Lambda_\sigma - \Lambda_1)\psi_\zeta](x) = e^{ix\cdot\zeta}\,,\ x \in \partial\Omega$$

and compute

$$\mathbf{t}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}[(\Lambda_\sigma - \Lambda_1)\psi_\zeta](x)\,ds(x).$$

2. Compute q by Inverse Fourier transform and the limit

$$\lim_{|\zeta|\to\infty} \mathbf{t}(\xi,\zeta) = \hat{q}(\xi).$$

3. Solve

$$-\Delta\sigma^{1/2} + q\sigma^{1/2} = 0 \text{ in } \Omega,$$
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Linearization step by step

- **Step 1**: $\Lambda_\sigma \mapsto \mathbf{t}(\xi,\zeta)$ linearized around $\Lambda_\sigma = \Lambda_1 \Rightarrow$

$$\mathbf{t}(\xi,\zeta) \simeq \mathbf{t}^{\exp}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}[(\Lambda_\sigma - \Lambda_1)e^{iy\cdot\zeta}](x)\, ds(x).$$

- **Step 2**: $\mathbf{t}^{\exp}(\xi,\zeta) \simeq \hat{q}(\xi)$.
  $\hat{q} \mapsto q$ linear.

- **Step 3**: $q \mapsto \sigma^{1/2} \mapsto \sigma$ linearized around $\sigma = 1 \Rightarrow$ Calderón's formula

$$\sigma^{\mathrm{app}}(x) = 1 - \frac{2}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\mathbf{t}^{\exp}(\xi,\zeta)}{|\xi|^2} e^{ix\cdot\xi}\, d\xi.$$

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$\mathbf{t}^{\exp}$ approximation

Compute $\mathbf{t}^{\exp}$ and use it in **Step 2** instead of $\mathbf{t}$.

$$\Lambda_\sigma \xrightarrow{1} \mathbf{t}^{\exp}(\xi,\zeta) \xrightarrow{2} q^{\exp}(x) \xrightarrow{3} \sigma^{\exp}(x)$$

$\mathbf{t}^0$ approximation

- Use the usual Green's function $G_0(x) = \dfrac{1}{4\pi|x|}$ instead of the Faddeev Green's function and solve

$$\psi_\zeta^0(x) + [S_0(\Lambda_\sigma - \Lambda_1)\psi_\zeta^0](x) = e^{ix\cdot\zeta}\ ,\ x \in \partial\Omega,$$

with S_0 usual single layer operator.

- Use ψ_ζ^0 to compute

$$\mathbf{t}^0(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}[(\Lambda_\sigma - \Lambda_1)\psi_\zeta^0](x)\, ds(x).$$

- Use $\mathbf{t}^0$ in **Step 2** instead of $\mathbf{t}$.

$$\Lambda_\sigma \xrightarrow{1} \mathbf{t}^0(\xi,\zeta) \xrightarrow{2} q^0(x) \xrightarrow{3} \sigma^0(x)$$

Implementation

- N: positive integer
- t_m: increasing $N+1$ zeros of the Legendre polynomial P_{N+1}
- $\theta_m = \arccos t_m$
- $\varphi_n = \pi n/(N+1)$
- $2(N+1)^2$ grid points on the unit sphere
  $m = 0 \dots N \ , \ n = 0 \dots 2N+1$

$$x_{m,n} = (\sin\theta_m \cos\varphi_n, \sin\theta_m \sin\varphi_n, \cos\theta_m) \, .$$

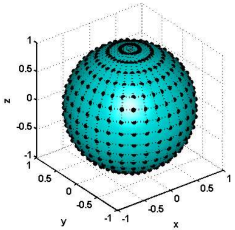

Figure: $N = 15 \rightarrow 512$ points

Implementation

- $\alpha_m = \dfrac{2(1-t_m^2)}{(N+1)^2[P_N(t_m)]^2}$: weights of the Gauß-Legendre quadrature rule of order $N+1$ on $[-1,1]$.

$\Rightarrow$ quadrature rule on the sphere (exact for spherical harmonics of degree less than or equal to $2N+1$)

$$\int_{\partial\Omega} \phi \, ds \simeq \frac{\pi}{N+1} \sum_{m=0}^{N} \sum_{n=0}^{2N+1} \alpha_m \phi(x_{m,n}) \ , \ \phi \in C^0(\partial\Omega).$$

Having $\Lambda_\sigma - \Lambda_1$, numerically compute

$$\mathbf{t}^{\exp}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}[(\Lambda_\sigma - \Lambda_1)e^{iy\cdot\zeta}](x)\,ds(x)$$

with previous quadrature rule.

• Boundary integral equation solved using a Nyström-like method: based on a quadrature rule to compute $S_0\phi$ for $\phi \in C^0(\partial\Omega)$

• Y_n^m are the eigenvectors of S_0

$$\frac{1}{4\pi}\int_{\partial\Omega}\frac{Y_n^m(y)}{|x-y|}\,ds(y)=\frac{Y_n^m(x)}{2n+1}\ ,\ x\in\partial\Omega.$$

- $L^2(\partial\Omega)$ orthogonal projection operator on the span of spherical harmonics of degree less than or equal to N

$$T_N\phi = \sum_{n=0}^{N} \sum_{m=-n}^{n} \langle \phi, Y_n^m \rangle Y_n^m \ , \ \phi \in L^2(\partial\Omega)$$

- Inner product approximated by quadrature rule

$\Rightarrow$ hyperinterpolation operator

$$L_N\phi = \frac{\pi}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} \sum_{k=0}^{N} \sum_{\ell=0}^{2N+1} \alpha_k \phi(x_{k\ell}) Y_n^{-m}(x_{k\ell}) Y_n^m \ , \ \phi \in C^0(\partial\Omega).$$

Implementation

Computing t^0

$\Rightarrow$ $S_0\phi$ approximated by $S_0 L_N \phi$

$$[S_0\phi](x) \simeq \frac{1}{4(N+1)} \sum_{n=0}^{N} \sum_{k=0}^{N} \sum_{\ell=0}^{2N+1} \alpha_k \phi(x_{k\ell}) P_n(x_{k\ell} \cdot x) \ , \ x \in \partial\Omega.$$

- Approximate the solution to

$$\psi_\zeta^0(x) + [S_0(\Lambda_\sigma - \Lambda_1)\psi_\zeta^0](x) = e^{ix\cdot\zeta} \ , \ x \in \partial\Omega,$$

by the solution to

$$[I + S_0 L_N (\Lambda_\sigma - \Lambda_1) L_N] \psi^N(x) = e^{ix\cdot\zeta} \ , \ x \in \partial\Omega.$$

(sums up to a finite dimensional linear system)

Implementation

Computing t^0

- Convergence rates: for any $s > 5/2$

$$\|\psi^N - \psi_\zeta^0\|_{H^s(\partial\Omega)} \leq \frac{C}{N^{s-5/2}} \|e^{ix\cdot\zeta}\|_{H^s(\partial\Omega)},$$

where C depends only on s.

- Having computed ψ_ζ^0, compute $\mathbf{t}^0$ using the quadrature rule.

- From the theory: $\lim_{|\zeta|\to\infty} \mathbf{t}(\xi,\zeta) = \hat{q}(\xi)$.
  Not true anymore for $\mathbf{t}^{\exp}$ or $\mathbf{t}^0$: divergence when $|\zeta| \to \infty$.

- Moreover: $e^{ix\cdot\zeta} \Rightarrow$ exponentially growing terms $\Rightarrow$ numerical instabilities.

$\Rightarrow$ ζ chosen of minimal norm.

Inverse Fourier Transform

- Computed using a FFT.
- $\hat{q}(\xi)$ computed on an equidistant mesh in a box $[-\xi_{\max}, \xi_{\max}]^3$

$\Rightarrow$ $q(x)$ on an equidistant x grid in $[-1, 1]^3$.

- Number N^3 of points in grid must satisfy

$$\xi_{\max} = N\frac{\pi}{2}.$$

$\Rightarrow$ Upper limit for the resolution in the x-mesh in terms of the mesh-size $h = \pi/\xi_{\max}$.

Solving the Schrödinger equation

- $q(x)$ interpolated on a tetrahedron mesh of the unit ball.

- Schrödinger equation solved using a FEM code of order 1.

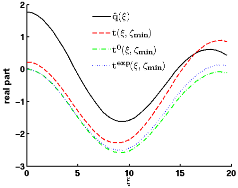

Figure: Approximations of $\hat{q}$.

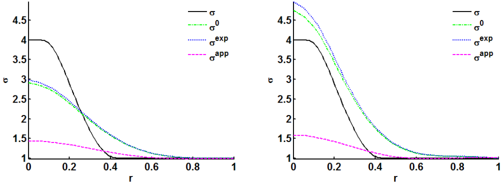

Figure: Reconstructions of σ with truncation: left $\xi_{\max} = 8$, right $\xi_{\max} = 9$.

Numerical examples

Radially symmetric conductivity: noisy data

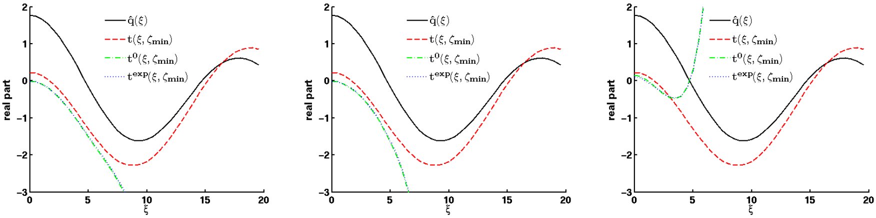

Figure: Approximations of $\hat{q}$ in case of noise in the data: left 0.1% noise, middle 1% noise, and right 5% noise.

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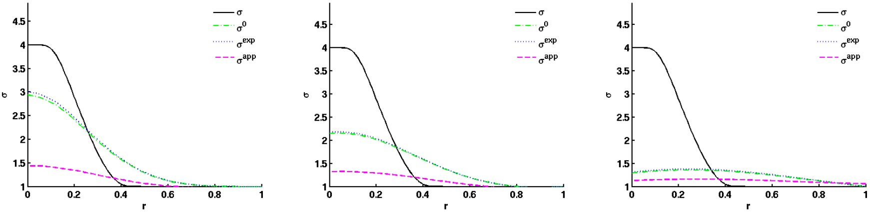

Figure: Reconstructions of σ with truncation: left 0.1% noise and $\xi_{\max} = 8$, middle 1% noise and $\xi_{\max} = 7$, and right 5% noise and $\xi_{\max} = 6$.

Truncation of the scattering transform $\Rightarrow$ non-linear regularization strategy.

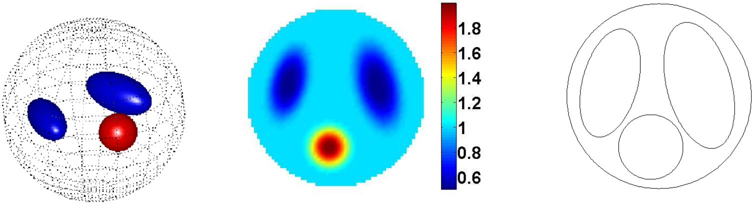

Figure: Left: 3D plot of phantom. Middle: profile of the conductivity σ in the (Oxy) plane. Right: support of σ-1.

Numerical examples

Non radially symmetric conductivity

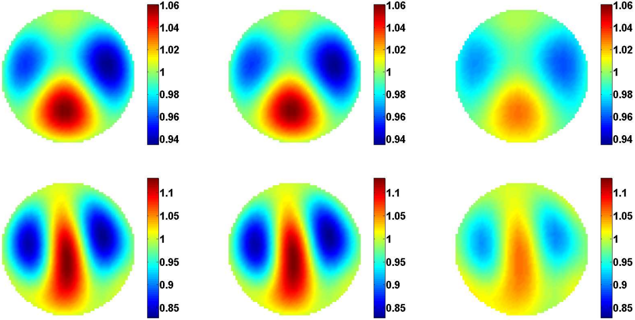

Figure: Upper row $\xi_{\max} = 6$, lower row $\xi_{\max} = 8$. Left σ^0, middle $\sigma^{\exp}$, and right σ^{app}.

Numerical examples

Non radially symmetric conductivity: noisy data

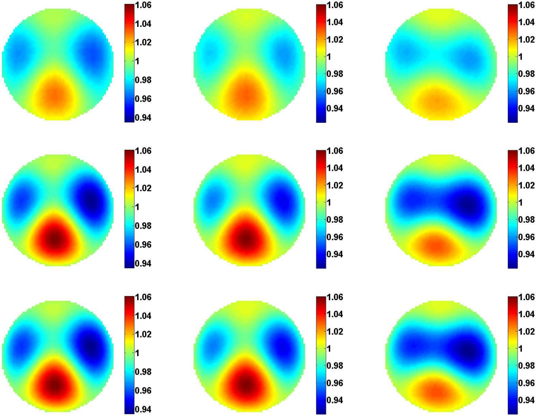

Figure: Reconstructions with different noise levels: left column 0%, middle column 0.1% and right column 1%. Upper row σ^{app}, middle row σ^{exp}, and lower row σ^{0}. Truncation is at $\xi_{\max} = 6$.

Conclusion and outlook

- + Three different numerical simplifications and implementations.
- + Fast reconstructions ($\sim$ 1 min).
- - Contrast not reliable.
- - $\mathbf{t}^0$ does not give better reconstructions.
- + Implementation for $\mathbf{t}^0 \Rightarrow$ implementation for $\mathbf{t}$ easily follows.
- + Implementation can be adapted to more general domains Ω.

- Study more complex 3D numerical examples.
- $\mathbf{t}$ implementation.
- More general domains Ω.
- Understanding spectral cut-off as regularization strategy.
- Limited data aperture.
- Real data.

Thank you

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