Computational tools for microwave imaging – some finite element aspects

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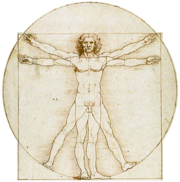
Physical methods to investigate the ...

human – *medical imaging*	earth – *geophysics*
▸ ultrasound	▸ seismics
▸ ultrawideband (UWB) microwave backscattering	▸ ground penetrating radar (GPR)
▸ microwave tomography	▸ controlled source electromagnetics (CSEM)
▸ electrical impedance tomography (EIT)	▸ direct current (DC) resitivity
▸ magnetic resonance imaging (MRI)	▸ nuclear magnetic resonance (NMR)



Microwave imaging of the ...

	human	earth
length scale	0.1–1 cm	1–100 m
electrical conductivity	0.01–1 S/m	0.01–1 S/m
relative electrical permittivity	1–81	1–81
interrogation frequency	0.1–10 GHz	0.1–100 Hz
skindepth	3.6–36 cm	0.036–3.6 km
wavelength	0.33–242 cm	0.32–32 km

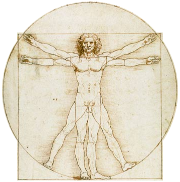
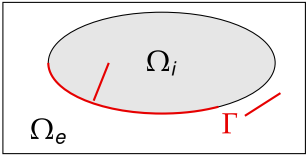


Image reconstruction as an optimisation problem

Consider a physical experiment that relates a constitutive parameter m, defined on $\Omega = \Omega_i \cup \Omega_e$, to a physical observable $d(m)$ on $\Gamma \subset \Omega$. Given observed data d^{obs} on $\Gamma \subset \Omega$ and m on Ω_e, we seek to reconstruct the spatial distribution of m within Ω_i such that $d(m)$ matches d^{obs}.



$$m^* = \arg\min_m \phi(m)$$

$$\phi(m) = \frac{1}{2}\left\|d(m) - d^{obs}\right\|^2 + \alpha R(m)$$

1. forward problem $d(m)$
2. regularisation operator $R(m)$
3. optimisation of $\phi(m)$
4. choice of α

1. Forward problem $d(m)$

$$d(m) = Qu = QA(m)^{-1}f$$

f … sources
$A(m)$ … forward operator
u … physical state in Ω
Q … observation operator $\Omega \to \Gamma$

Here: microwave tomography

- $A(m)$ is FE discretisation of time-harmonic Maxwell's equations
- m electrical permittivity ε and conductivity σ

Forward operator $A(m)$ discretises

$$\int_\Omega \nabla \times \mathbf{\Phi} \cdot \mu^{-1} \nabla \times \boldsymbol{E} \, dV - \int_\Omega \mathbf{\Phi} \cdot i\omega(\sigma - i\omega\varepsilon)\boldsymbol{E} \, dV = \int_\Omega \mathbf{\Phi} \cdot i\omega \boldsymbol{j_s} \, dV$$

FEM: $\Omega = \cup_k \Omega_k$; $\boldsymbol{E}(\boldsymbol{x}) \to \sum_j u_j \mathbf{\Phi}_j(\boldsymbol{x})$; $\mathbf{\Phi} \to \mathbf{\Phi}_i$
Assumption: μ, ε, σ elementwise constant

$$Au = f$$

$$A_{i,j} = \sum_k \mu_k^{-1} \int_{\Omega_k} \nabla \times \mathbf{\Phi}_i \cdot \nabla \times \mathbf{\Phi}_j \, dV - \sum_k i\omega(\sigma_k - i\omega\varepsilon_k) \int_{\Omega_k} \mathbf{\Phi}_i \cdot \mathbf{\Phi}_j \, dV$$

$$f_i = \int_\Omega \mathbf{\Phi}_i \cdot i\omega \boldsymbol{j_s} \, dV$$

Note:

$$A_{i,j} = \sum_k \mu_k^{-1} \int_{\Omega_k} \nabla \times \mathbf{\Phi}_i \cdot \nabla \times \mathbf{\Phi}_j \, dV - \sum_k i\omega(\sigma - i\omega\varepsilon) \int_{\Omega_k} \mathbf{\Phi}_i \cdot \mathbf{\Phi}_j \, dV$$

leads to a sparse derivative matrix $\nabla_m(A(m)u^{fix})$ since

$$\frac{\partial A_{i,j}}{\partial \mu_k} = -\mu_k^{-2} \int_{\Omega_k} \nabla \times \mathbf{\Phi}_i \cdot \nabla \times \mathbf{\Phi}_j \, dV$$

$$\frac{\partial A_{i,j}}{\partial \sigma_k} = -i\omega \int_{\Omega_k} \mathbf{\Phi}_i \cdot \mathbf{\Phi}_j \, dV$$

$$\frac{\partial A_{i,j}}{\partial \varepsilon_k} = -\omega^2 \int_{\Omega_k} \mathbf{\Phi}_i \cdot \mathbf{\Phi}_j \, dV$$

2. Regularisation operator $R(m)$

Primal-dual finite element formulation for $\|\nabla(m - m_{ref})\|^2$ with elementwise constant m,

$$\min_m \max_{\boldsymbol{\omega}} \frac{1}{2} \left\| d(m) - d^{obs} \right\|^2 + \alpha R(m, \boldsymbol{\omega})$$

where $\boldsymbol{\omega} \in H(\mathrm{div}; \Omega)$ and

$$R(m, \boldsymbol{\omega}) = -\frac{1}{2}\|\boldsymbol{\omega}\|^2 - (m - m_{ref}, \nabla \cdot \boldsymbol{\omega})$$

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RT_0 discretisation of ω

$$R(m, w) = -\frac{1}{2} w^\top M_w w - (m - m_{ref})^\top D w$$

$$\nabla_w R(m, w) = 0$$

$$w = -M_w^{-1} D^\top (m - m_{ref})$$

$$R(m) = \frac{1}{2}(m - m_{ref})^\top D M_w^{-1} D^\top (m - m_{ref})$$

3. Optimisation of $\phi(m)$

$$\phi(m) = \frac{1}{2}\Big((d(m) - d^{obs})^\top (d(m) - d^{obs}) \\ + \alpha(m - m_{ref})^\top D M_w^{-1} D^\top (m - m_{ref})\Big)$$

Necessary condition $g(m) := \nabla_m \phi(m) = 0$.

Solve by Gauss-Newton method.
Given an initial model m_0, iterate for $\ell = 0, 1, \ldots$

$$m_{\ell+1} = m_\ell + \gamma_\ell \delta m_\ell$$

where δm_ℓ solves

$$\big(J(m_\ell)^\top J(m_\ell) + \alpha D M_w^{-1} D^\top\big)\delta m_\ell = -g(m_\ell)$$

Predicted data

$$d(m_\ell) = Qu_\ell = QA(m_\ell)^{-1}f$$

Gradient

$$g(m_\ell) = J(m_\ell)^\top (d(m_\ell) - d^{obs}) + \alpha D M_w^{-1} D^\top (m_\ell - m_{ref})$$

Sensitivity matrix

$$J(m_\ell) = -Q^\top A(m_\ell)^{-1} G(m_\ell)$$

where $G(m_\ell) = \nabla_{m_\ell}(A(m_\ell)u_\ell^{fix})$.

If $\big(J(m_\ell)^\top J(m_\ell) + \alpha D M_w^{-1} D^\top\big)\delta m_\ell = -g(m_\ell)$ is solved iteratively, the matrix-vector products involving $J(m_\ell)$ and $J(m_\ell)^\top$ only require operations with sparse matrices Q, A and G.

4. Choice of α

Iterated Tikhonov regularisation:

Set $m_{ref} = m_\ell$ at Gauss-Newton step ℓ and $\gamma_\ell = 1$.

$$\left(J(m_\ell)^\top J(m_\ell) + \alpha D M_w^{-1} D^\top\right)(m_{\ell+1} - m_\ell) = \\ -J(m_\ell)^\top (d(m_\ell) - d^{obs})$$

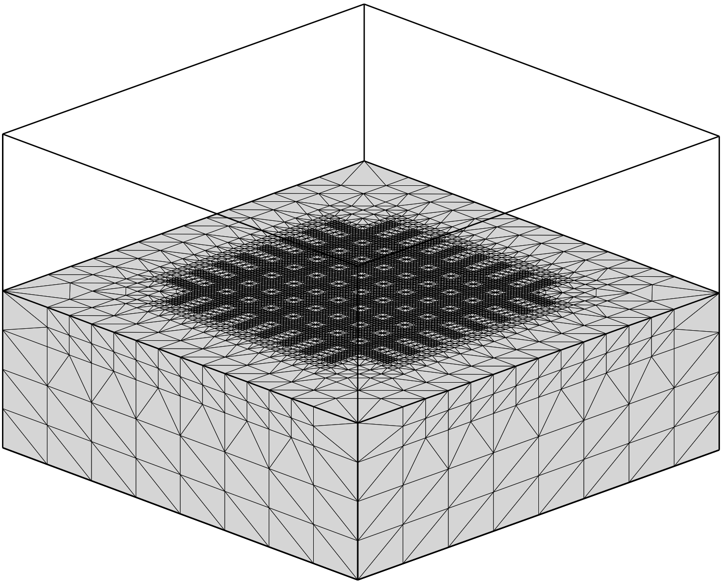
Let $DM_w^{-1}D^\top = C^\top C$, $\tilde{m} = Cm$, $\tilde{J}(\tilde{m}) = J(m)C^{-1}$ and rename $\alpha = 1/\delta t$. Then

$$\frac{1}{\delta t}(\tilde{m}_{\ell+1} - \tilde{m}_\ell) = \\ -\tilde{J}(\tilde{m}_\ell)^\top \left(\tilde{J}(\tilde{m}_\ell)(\tilde{m}_{\ell+1} - \tilde{m}_\ell) + d(\tilde{m}_\ell) - d^{obs}\right)$$

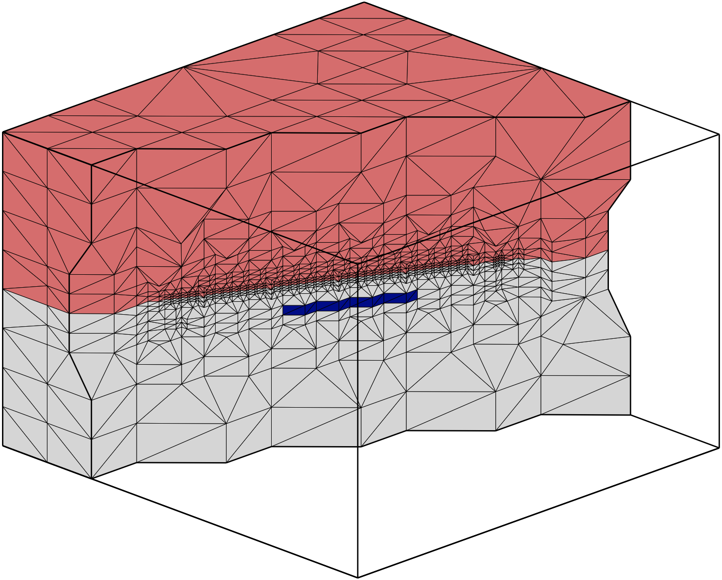
$\Rightarrow$ *semi-implicit time stepping*

Example

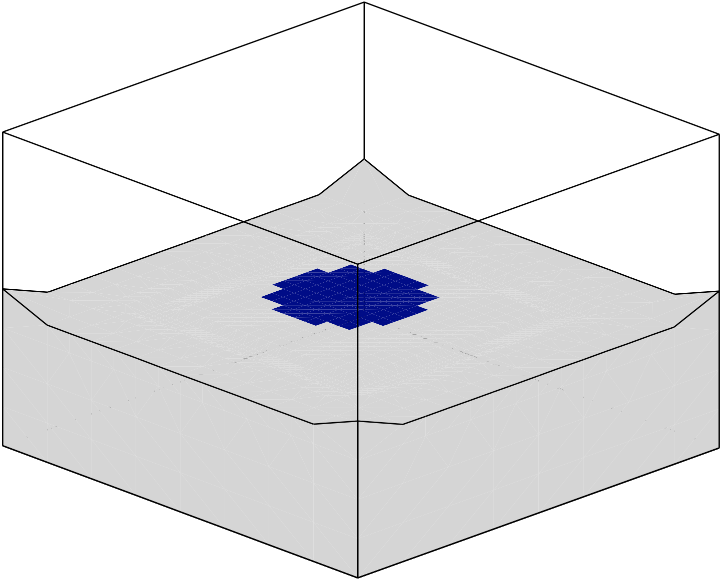
mesh and true model



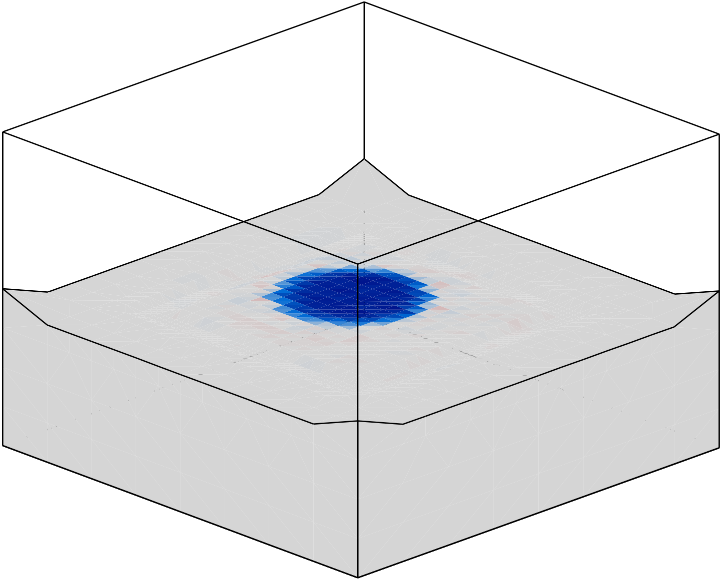
mesh and true model



true model



predicted model



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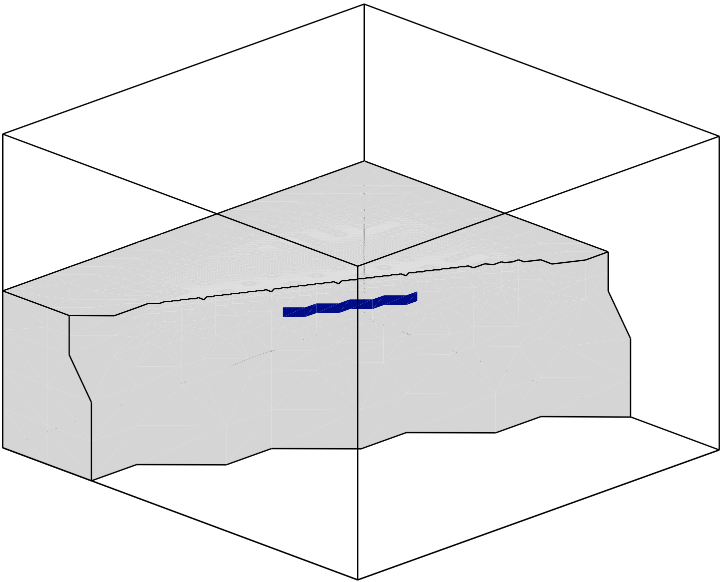

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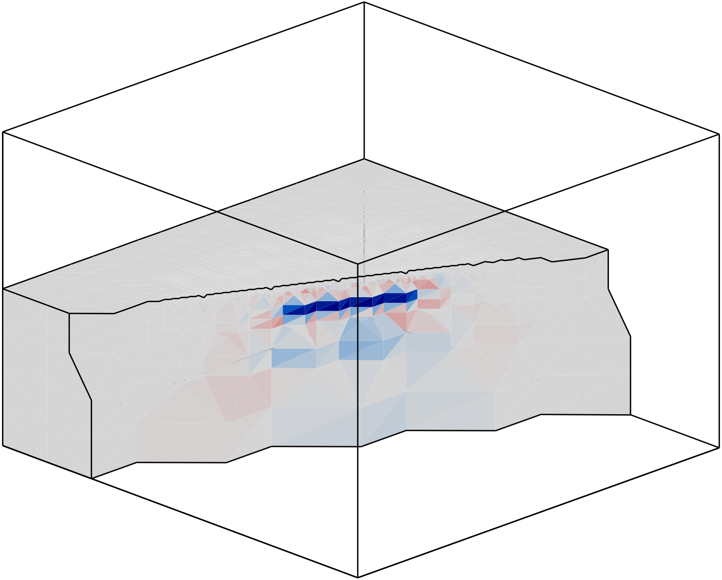
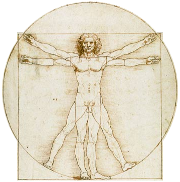


Road to the future . . .

high resolution and high contrast?



multi-modal imaging

joint inversion