FAMILIES OF POLYNOMIALS

Conference in Number Theory

Carleton University

June 28, 2011

Hugh L. Montgomery

University of Michigan

I. Moments of a Thue–Morse generating function

(joint with Christian Mauduit & Joël Rivat)

II. An Eulerian curiosity

I. Thue–Morse

$$n = \sum_i b_i 2^i \qquad (b_i = 0 \text{ or } 1)$$

$$s(n) = \sum_i b_i$$

A. O. Gelfond (1968): Show that $s(p)$ is odd for asymptotically half the primes.

sum of base b digits modulo m

I. Thue–Morse

$$n = \sum_i b_i 2^i \qquad (b_i = 0 \text{ or } 1)$$

$$s(n) = \sum_i b_i$$

A. O. Gelfond (1968): Show that $s(p)$ is odd for asymptotically half the primes.

sum of base b digits modulo m

settled by Mauduit & Rivat (2010)

I. Thue–Morse

$$n = \sum_i b_i 2^i \qquad (b_i = 0 \text{ or } 1)$$

$$s(n) = \sum_i b_i$$

A. O. Gelfond (1968): Show that $s(p)$ is odd for asymptotically half the primes.

sum of base b digits modulo m

settled by Mauduit & Rivat (2010)

$$\sum_{m=0}^{\infty} (-1)^{s(m)} z^m = \prod_{n=0}^{\infty} \left(1 - z^{2^n}\right) \qquad (|z| < 1)$$

I. Thue–Morse

$$n = \sum_i b_i 2^i \qquad (b_i = 0 \text{ or } 1)$$

$$s(n) = \sum_i b_i$$

A. O. Gelfond (1968): Show that $s(p)$ is odd for asymptotically half the primes.

sum of base b digits modulo m

settled by Mauduit & Rivat (2010)

$$\sum_{m=0}^{\infty} (-1)^{s(m)} z^m = \prod_{n=0}^{\infty} \left(1 - z^{2^n}\right) \qquad (|z| < 1)$$

$$T_N(x) = \sum_{0 \le m < 2^N} (-1)^{s(m)} e(mx) = \prod_{0 \le n < N} \left(1 - e(2^n x)\right)$$

$$e(\theta) = e^{2\pi i\theta}$$

Parseval: $\|T_N\|_2 = 2^{N/2}$

Parseval: $\|T_N\|_2 = 2^{N/2}$

Need:

$$\|T_N\|_\infty = O\big(2^{\beta N}\big) \qquad (\beta < 1)$$

$$\|T_N\|_1 = O\big(2^{\alpha N}\big) \qquad (\alpha < 1/2)$$

Parseval: $\|T_N\|_2 = 2^{N/2}$

Need:

$$\|T_N\|_\infty = O\big(2^{\beta N}\big) \qquad (\beta < 1)$$

$$\|T_N\|_1 = O\big(2^{\alpha N}\big) \qquad (\alpha < 1/2)$$

$$M_k(N) = \int_0^1 |T_N(x)|^{2k}\, dx$$

$M_1(N) = 2^N \qquad M_k(N) \in \mathbb{Z} \qquad 2^{kN} < M_k(N) < 2^{2kN}$

Parseval: $\|T_N\|_2 = 2^{N/2}$

Need:

$$\|T_N\|_\infty = O\big(2^{\beta N}\big) \qquad (\beta < 1)$$

$$\|T_N\|_1 = O\big(2^{\alpha N}\big) \qquad (\alpha < 1/2)$$

$$M_k(N) = \int_0^1 |T_N(x)|^{2k}\, dx$$

$M_1(N) = 2^N \qquad M_k(N) \in \mathbb{Z} \qquad 2^{kN} < M_k(N) < 2^{2kN}$

N	0	1	2	3	4	5	6
$M_2(N)$	1	6	28	152	752	3936	19904

Parseval: $\|T_N\|_2 = 2^{N/2}$

Need:

$$\|T_N\|_\infty = O\big(2^{\beta N}\big) \qquad (\beta < 1)$$

$$\|T_N\|_1 = O\big(2^{\alpha N}\big) \qquad (\alpha < 1/2)$$

$$M_k(N) = \int_0^1 |T_N(x)|^{2k}\, dx$$

$M_1(N) = 2^N \qquad M_k(N) \in \mathbb{Z} \qquad 2^{kN} < M_k(N) < 2^{2kN}$

N	0	1	2	3	4	5	6
$M_2(N)$	1	6	28	152	752	3936	19904

$$\begin{bmatrix} 1 & 6 & 28 \\ 6 & 28 & 152 \\ 28 & 152 & 752 \end{bmatrix}, \begin{bmatrix} 6 & 28 & 152 \\ 28 & 152 & 752 \\ 152 & 752 & 3936 \end{bmatrix}, \begin{bmatrix} 28 & 152 & 752 \\ 152 & 752 & 3936 \\ 752 & 3936 & 19904 \end{bmatrix}, \ldots$$

Parseval: $\|T_N\|_2 = 2^{N/2}$

Need:

$$\|T_N\|_\infty = O\big(2^{\beta N}\big) \qquad (\beta < 1)$$

$$\|T_N\|_1 = O\big(2^{\alpha N}\big) \qquad (\alpha < 1/2)$$

$$M_k(N) = \int_0^1 |T_N(x)|^{2k}\, dx$$

$M_1(N) = 2^N \qquad M_k(N) \in \mathbb{Z} \qquad 2^{kN} < M_k(N) < 2^{2kN}$

N	0	1	2	3	4	5	6
$M_2(N)$	1	6	28	152	752	3936	19904

$$\begin{bmatrix} 1 & 6 & 28 \\ 6 & 28 & 152 \\ 28 & 152 & 752 \end{bmatrix}, \begin{bmatrix} 6 & 28 & 152 \\ 28 & 152 & 752 \\ 152 & 752 & 3936 \end{bmatrix}, \begin{bmatrix} 28 & 152 & 752 \\ 152 & 752 & 3936 \\ 752 & 3936 & 19904 \end{bmatrix}, \ldots$$

Conjecture $M_2(N) = 2M_2(N-1) + 16M_2(N-2)$

Parseval: $\|T_N\|_2 = 2^{N/2}$

Need:

$$\|T_N\|_\infty = O\big(2^{\beta N}\big) \qquad (\beta < 1)$$

$$\|T_N\|_1 = O\big(2^{\alpha N}\big) \qquad (\alpha < 1/2)$$

$$M_k(N) = \int_0^1 |T_N(x)|^{2k}\, dx$$

$M_1(N) = 2^N \qquad M_k(N) \in \mathbb{Z} \qquad 2^{kN} < M_k(N) < 2^{2kN}$

N	0	1	2	3	4	5	6
$M_2(N)$	1	6	28	152	752	3936	19904

$$\begin{bmatrix} 1 & 6 & 28 \\ 6 & 28 & 152 \\ 28 & 152 & 752 \end{bmatrix}, \begin{bmatrix} 6 & 28 & 152 \\ 28 & 152 & 752 \\ 152 & 752 & 3936 \end{bmatrix}, \begin{bmatrix} 28 & 152 & 752 \\ 152 & 752 & 3936 \\ 752 & 3936 & 19904 \end{bmatrix}, \ldots$$

Conjecture $M_2(N) = 2M_2(N-1) + 16M_2(N-2)$

Theorem $M_k(N)$ satisfies a linear recurrence of order k.

antecedent: Fouvry & Mauduit (1996)

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k\, f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

$$Q_k^N\, 1 = |T_N(x)|^{2k}$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k\, f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

$$Q_k^N\, 1 = |T_N(x)|^{2k}$$

$$M_k(N) = \int_0^1 Q_k^N\, 1\, dx = \langle Q_k^N\, 1, 1 \rangle$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k\, f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

$$Q_k^N\, 1 = |T_N(x)|^{2k}$$

$$M_k(N) = \int_0^1 Q_k^N\, 1\, dx = \langle Q_k^N\, 1, 1\rangle$$

$f, g \in L^2(\mathbb{T})$:

$$\langle Q_k\, f, g\rangle = \int_0^1 (Q_k\, f)(x)\overline{g(x)}\, dx$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k\, f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

$$Q_k^N\, 1 = |T_N(x)|^{2k}$$

$$M_k(N) = \int_0^1 Q_k^N\, 1\, dx = \langle Q_k^N\, 1, 1\rangle$$

$f, g \in L^2(\mathbb{T})$:

$$\begin{aligned} \langle Q_k\, f, g\rangle &= \int_0^1 (Q_k\, f)(x)\overline{g(x)}\, dx \\ &= \int_0^1 |2 \sin \pi x|^{2k} f(2x)\overline{g(x)}\, dx \end{aligned}$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k\, f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

$$Q_k^N\, 1 = |T_N(x)|^{2k}$$

$$M_k(N) = \int_0^1 Q_k^N\, 1\, dx = \langle Q_k^N\, 1, 1 \rangle$$

$f, g \in L^2(\mathbb{T})$:

$$\begin{aligned}
\langle Q_k\, f, g \rangle &= \int_0^1 (Q_k\, f)(x)\overline{g(x)}\, dx \\
&= \int_0^1 |2 \sin \pi x|^{2k} f(2x)\overline{g(x)}\, dx \\
&= \frac{1}{2} \int_0^2 |2 \sin \pi u/2|^{2k} f(u)\overline{g(u/2)}\, du
\end{aligned}$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k\, f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

$$Q_k^N\, 1 = |T_N(x)|^{2k}$$

$$M_k(N) = \int_0^1 Q_k^N\, 1\, dx = \langle Q_k^N\, 1, 1\rangle$$

$f, g \in L^2(\mathbb{T})$:

$$\begin{aligned}\langle Q_k\, f, g\rangle &= \int_0^1 (Q_k\, f)(x)\overline{g(x)}\, dx \\ &= \int_0^1 |2 \sin \pi x|^{2k} f(2x)\overline{g(x)}\, dx \\ &= \frac{1}{2}\int_0^2 |2 \sin \pi u/2|^{2k} f(u)\overline{g(u/2)}\, du\end{aligned}$$

$$= \frac{1}{2}\int_0^1 f(u)\big(|2 \sin \pi u/2|^{2k}\overline{g(u/2)} + |2 \cos \pi u/2|^{2k}\overline{g((u+1)/2)}\big)\, du$$

antecedent: Fouvry & Mauduit (1996)

$$|1 - e(\theta)| = |2 \sin \pi\theta|$$

For $f \in L^2(\mathbb{T})$, put

$$(Q_k\, f)(x) = |2 \sin \pi x|^{2k} f(2x)$$

$$Q_k^N\, 1 = |T_N(x)|^{2k}$$

$$M_k(N) = \int_0^1 Q_k^N\, 1\, dx = \langle Q_k^N\, 1, 1\rangle$$

$f, g \in L^2(\mathbb{T})$:

$$\langle Q_k\, f, g\rangle = \int_0^1 (Q_k\, f)(x)\overline{g(x)}\, dx$$

$$= \int_0^1 |2 \sin \pi x|^{2k} f(2x)\overline{g(x)}\, dx$$

$$= \frac{1}{2}\int_0^2 |2 \sin \pi u/2|^{2k} f(u)\overline{g(u/2)}\, du$$

$$= \frac{1}{2}\int_0^1 f(u)\big(|2 \sin \pi u/2|^{2k}\overline{g(u/2)} + |2 \cos \pi u/2|^{2k}\overline{g((u+1)/2)}\big)\, du$$

$$= \langle f, P_k\; g\rangle$$

where

$$(P_k\, g)(x) = \frac{1}{2}\big(|2 \sin \pi x/2|^{2k} g(x/2) + |2 \cos \pi x/2|^{2k} g((x+1)/2)\big)$$

$$M_k(N) = \langle Q_k^N \, 1, 1 \rangle = \langle 1, P_k^N \, 1 \rangle$$

$$M_k(N) = \langle Q_k^N\, 1, 1\rangle = \langle 1, P_k^N\, 1\rangle$$

$E_k :=$ { even trig polys of period 1 and degree $< k$}

$$f \in E_k \implies P_k\, f \in E_k$$

$$M_k(N) = \langle Q_k^N\, 1, 1\rangle = \langle 1, P_k^N\, 1\rangle$$

$E_k :=$ { even trig polys of period 1 and degree $< k$}

$$f \in E_k \implies P_k\, f \in E_k$$

$$1,\ \sin^2 \pi x,\ \sin^4 \pi x,\ \ldots,\ \sin^{2(k-1)} \pi x$$

is a basis for E_k. $\widetilde{P}_k =$ restriction of P_k to E_k

$$M_k(N) = \langle Q_k^N\, 1, 1\rangle = \langle 1, P_k^N\, 1\rangle$$

$E_k :=$ { even trig polys of period 1 and degree $< k$}

$$f \in E_k \implies P_k\, f \in E_k$$

$$1,\ \sin^2 \pi x,\ \sin^4 \pi x,\ \ldots\ ,\ \sin^{2(k-1)} \pi x$$

is a basis for E_k. $\widetilde{P}_k$ = restriction of P_k to E_k

$$\widetilde{P_k}\, \sin^{2j} \pi x = \sum_{i=0}^{k-1} a_{ij} \sin^{2i} \pi x$$

where

$$a_{ij} = (-1)^i 2^{2k-2i-1} \binom{j+k-i}{i} \frac{j+k}{j+k-i} \qquad (0 \le i, j < k)$$

$$M_k(N) = \langle Q_k^N\, 1, 1\rangle = \langle 1, P_k^N\, 1\rangle$$

$E_k :=$ { even trig polys of period 1 and degree $< k$}

$$f \in E_k \implies P_k\, f \in E_k$$

$$1,\ \sin^2 \pi x,\ \sin^4 \pi x,\ \ldots,\ \sin^{2(k-1)} \pi x$$

is a basis for E_k. $\widetilde{P}_k$ = restriction of P_k to E_k

$$\widetilde{P_k}\, \sin^{2j} \pi x = \sum_{i=0}^{k-1} a_{ij} \sin^{2i} \pi x$$

where

$$a_{ij} = (-1)^i 2^{2k-2i-1} \binom{j+k-i}{i} \frac{j+k}{j+k-i} \qquad (0 \le i, j < k)$$

$$A_k = [a_{ij}]$$

$$p_k(z) = \det\,(zI - A_k) = z^k - c_{k-1,k} z^{k-1} - \cdots - c_{0,k} \ \in\ \mathbb{Z}[z]$$

$$M_k(N) = \langle Q_k^N\, 1, 1\rangle = \langle 1, P_k^N\, 1\rangle$$

$E_k :=$ { even trig polys of period 1 and degree $< k$}

$$f \in E_k \implies P_k\, f \in E_k$$

$$1,\ \sin^2 \pi x,\ \sin^4 \pi x,\ \ldots,\ \sin^{2(k-1)} \pi x$$

is a basis for E_k. $\widetilde{P}_k =$ restriction of P_k to E_k

$$\widetilde{P_k}\, \sin^{2j} \pi x = \sum_{i=0}^{k-1} a_{ij} \sin^{2i} \pi x$$

where

$$a_{ij} = (-1)^i 2^{2k-2i-1} \binom{j+k-i}{i} \frac{j+k}{j+k-i} \qquad (0 \le i, j < k)$$

$$A_k = [a_{ij}]$$

$$p_k(z) = \det\,(zI - A_k) = z^k - c_{k-1,k} z^{k-1} - \cdots - c_{0,k} \ \in\ \mathbb{Z}[z]$$

$$M_k(N) = c_{k-1,k} M_k(N-1) + \cdots + c_{0,k} M_k(N-k) \qquad (N \ge k)$$

$$M_k(N) = \langle Q_k^N\, 1, 1\rangle = \langle 1, P_k^N\, 1\rangle$$

$E_k :=$ { even trig polys of period 1 and degree $< k$}

$$f \in E_k \implies P_k\, f \in E_k$$

$$1,\ \sin^2 \pi x,\ \sin^4 \pi x,\ \ldots\ ,\ \sin^{2(k-1)} \pi x$$

is a basis for E_k. $\widetilde{P}_k =$ restriction of P_k to E_k

$$\widetilde{P_k}\, \sin^{2j} \pi x = \sum_{i=0}^{k-1} a_{ij} \sin^{2i} \pi x$$

where

$$a_{ij} = (-1)^i 2^{2k-2i-1} \binom{j+k-i}{i} \frac{j+k}{j+k-i} \qquad (0 \le i, j < k)$$

$$A_k = [a_{ij}]$$

$$p_k(z) = \det\, (zI - A_k) = z^k - c_{k-1,k} z^{k-1} - \cdots - c_{0,k} \ \in\ \mathbb{Z}[z]$$

$$M_k(N) = c_{k-1,k} M_k(N-1) + \cdots + c_{0,k} M_k(N-k) \qquad (N \ge k)$$

$$\det\, A_k = \varepsilon_k 2^{k^2} \qquad (\varepsilon_k = \pm 1)$$

$$p_1(z) = z - 2$$

$$p_2(z) = z^2 - 2z - 16$$

$$p_3(z) = z^3 - 10z^2 - 96z + 512$$

$$p_4(z) = z^4 - 26z^3 - 880z^2 + 8704z + 65536$$

$$p_5(z) = z^5 - 82z^4 - 7104z^3 + 232448z^2 + 4325376z - 33554432$$

$$p_6(z) = z^6 - 242z^5 - 62416z^4 + 5463040z^3 + 340262912z^2 - 7851737088z - 68719476736$$

II. Eulerian

$$f_1(x) = \begin{cases} 1 & (-1/2 < x < 1/2), \\ 1/2 & (x = \pm 1/2), \\ 0 & (|x| > 1/2) \end{cases}$$

II. Eulerian

$$f_1(x) = \begin{cases} 1 & (-1/2 < x < 1/2), \\ 1/2 & (x = \pm 1/2), \\ 0 & (|x| > 1/2) \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

II. Eulerian

$$f_1(x) = \begin{cases} 1 & (-1/2 < x < 1/2), \\ 1/2 & (x = \pm 1/2), \\ 0 & (|x| > 1/2) \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$f_n(x) = f_1 * f_1 * \cdots * f_1$$

II. Eulerian

$$f_1(x) = \begin{cases} 1 & (-1/2 < x < 1/2), \\ 1/2 & (x = \pm 1/2), \\ 0 & (|x| > 1/2) \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$f_n(x) = f_1 * f_1 * \cdots * f_1$$

$$= \frac{1}{(n-1)!} \sum_{0 \le j \le n/2 - x} (-1)^j \binom{n}{j} (n/2 - j - x)^{n-1} \qquad (n > 1)$$

Simpson (1757), Lagrange (1770)

II. Eulerian

$$f_1(x) = \begin{cases} 1 & (-1/2 < x < 1/2), \\ 1/2 & (x = \pm 1/2), \\ 0 & (|x| > 1/2) \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$f_n(x) = f_1 * f_1 * \cdots * f_1$$

$$= \frac{1}{(n-1)!} \sum_{0 \le j \le n/2 - x} (-1)^j \binom{n}{j} (n/2 - j - x)^{n-1} \qquad (n > 1)$$

Simpson (1757), Lagrange (1770)

Transition points: $-n/2, -n/2+1, \ldots, n/2-1, n/2$

Euler (1750): Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = \#\text{ perms on } \{1, 2, \ldots, n\} \text{ with exactly } k \text{ rises}$$

Euler (1750): Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = \#\text{ perms on } \{1, 2, \ldots, n\} \text{ with exactly } k \text{ rises}$$

$$\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$$

Euler (1750): Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = \#\text{ perms on } \{1, 2, \ldots, n\} \text{ with exactly } k \text{ rises}$$

$$\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$$

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$$

Euler (1750): Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = \#\text{ perms on } \{1, 2, \ldots, n\} \text{ with exactly } k \text{ rises}$$

$$\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$$

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$$

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

Euler (1750): Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = \# \text{ perms on } \{1, 2, \ldots, n\} \text{ with exactly } k \text{ rises}$$

$$\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$$

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$$

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j+1)^n$$

Euler (1750): Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = \#\text{ perms on } \{1,2,\ldots,n\} \text{ with exactly } k \text{ rises}$$

$$\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$$

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$$

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j+1)^n$$

$$f_n(-n/2+k) = \frac{\left\langle {n-1 \atop k-1} \right\rangle}{(n-1)!}$$

Euler (1750): Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = \#\text{ perms on } \{1,2,\ldots,n\} \text{ with exactly } k \text{ rises}$$

$$\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$$

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$$

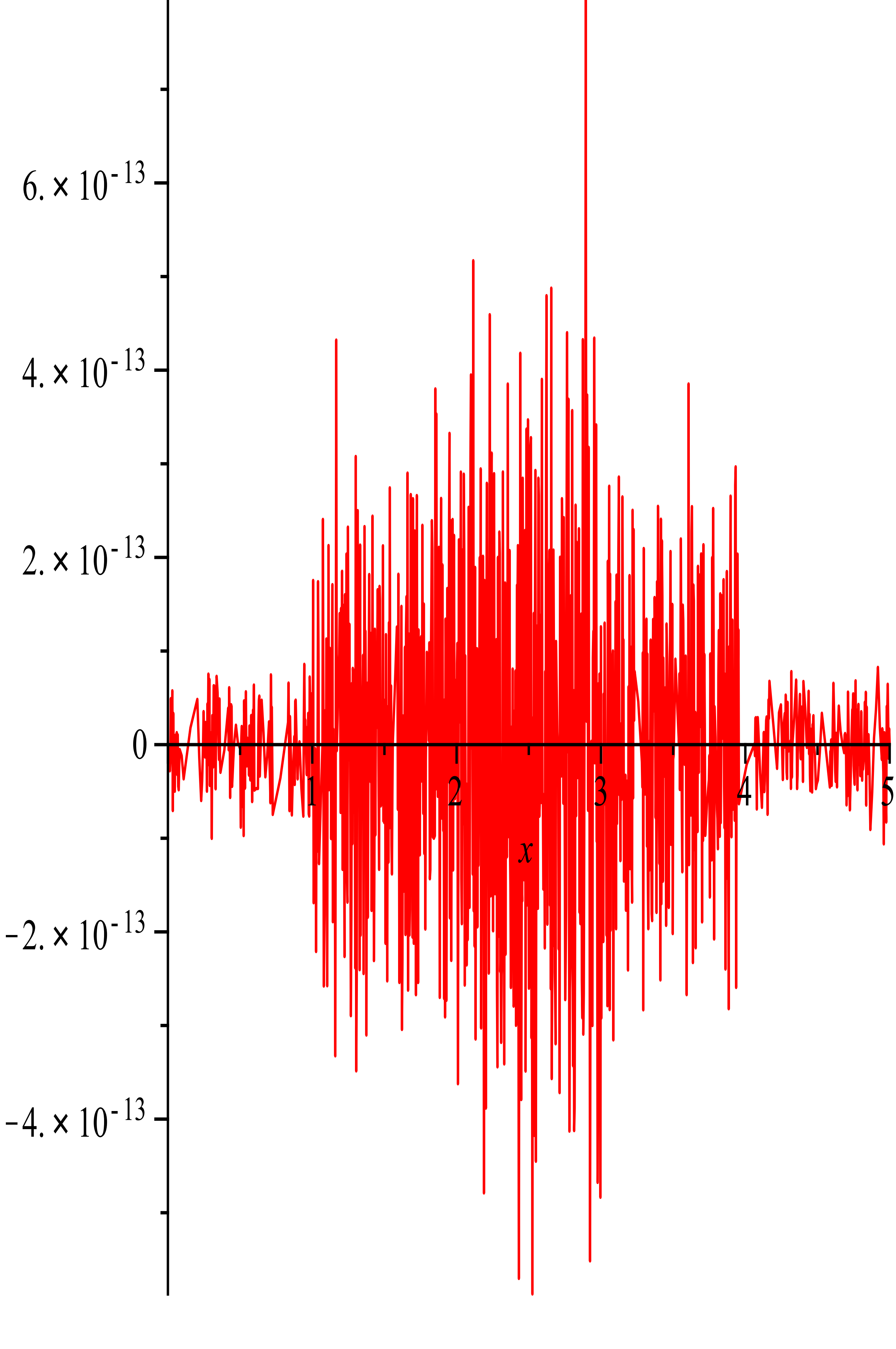
$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 4 & & 1 & & \\ & 1 & & 11 & & 11 & & 1 & \\ 1 & & 26 & & 66 & & 26 & & 1 \end{array}$$

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j+1)^n$$

$$f_n(-n/2+k) = \frac{\left\langle {n-1 \atop k-1} \right\rangle}{(n-1)!}$$

$$f_{n+1}(-(n+1)/2+k) = \frac{n+1-k}{n} f_n(-n/2+k-1) + \frac{k}{n} f_n(-n/2+k)$$

$(k = 0, 1, \ldots, n)$



6. × 10^-13
4. × 10^-13
2. × 10^-13
0
-2. × 10^-13
-4. × 10^-13
1
2
3
4
5
x

$$f_{n+1}(-(n+1)/2+k) = \frac{n+1-k}{n} f_n(-n/2+k-1) + \frac{k}{n} f_n(-n/2+k)$$

$$f_{n+1}(-(n{+}1)/2{+}k) = \frac{n+1-k}{n} f_n(-n/2{+}k{-}1) {+} \frac{k}{n} f_n(-n/2{+}k)$$

$$f_{n+1}(x) = \frac{1}{n}\left(\frac{n+1}{2}{-}x\right) f_n(x{-}1/2) {+} \frac{1}{n}\left(\frac{n+1}{2}{+}x\right) f_n(x{+}1/2)$$

$$f_{n+1}(-(n+1)/2+k) = \frac{n+1-k}{n} f_n(-n/2+k-1) + \frac{k}{n} f_n(-n/2+k)$$

$$f_{n+1}(x) = \frac{1}{n}\left(\frac{n+1}{2} - x\right) f_n(x-1/2) + \frac{1}{n}\left(\frac{n+1}{2} + x\right) f_n(x+1/2)$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1
\end{array}$$

$$f_{n+1}(-(n+1)/2+k) = \frac{n+1-k}{n} f_n(-n/2+k-1) + \frac{k}{n} f_n(-n/2+k)$$

$$f_{n+1}(x) = \frac{1}{n}\left(\frac{n+1}{2} - x\right) f_n(x-1/2) + \frac{1}{n}\left(\frac{n+1}{2} + x\right) f_n(x+1/2)$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 4 & & 1 & & \\ & 1 & & 11 & & 11 & & 1 & \\ 1 & & 26 & & 66 & & 26 & & 1 \end{array}$$

$$1$$

$$1+x \quad 1-x$$

$$\tfrac{9}{4} + 3x + x^2 \quad \tfrac{3}{2} - 2x^2 \quad \tfrac{9}{4} - 3x + x^2$$

$$8 + 12x + 6x^2 + x^3 \quad 4 - 6x - 3x^3 \quad 4 - 6x^2 + 3x^3 \quad 8 - 12x + 6x^2 - x^3$$

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e(-tx)\,dx$$

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e(-tx)\, dx$$

$$\widehat{f_1}(t) = \frac{\sin \pi t}{\pi t}$$

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e(-tx)\, dx$$

$$\widehat{f_1}(t) = \frac{\sin \pi t}{\pi t}$$

$$\widehat{f_n}(t) = \left(\frac{\sin \pi t}{\pi t} \right)^n$$

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e(-tx)\, dx$$

$$\widehat{f_1}(t) = \frac{\sin \pi t}{\pi t}$$

$$\widehat{f_n}(t) = \left(\frac{\sin \pi t}{\pi t}\right)^n$$

$$f_n(x) = \int_{\mathbb{R}} \left(\frac{\sin \pi t}{\pi t}\right)^n e(xt)\, dt$$

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) e(-tx)\, dx$$

$$\widehat{f_1}(t) = \frac{\sin \pi t}{\pi t}$$

$$\widehat{f_n}(t) = \left(\frac{\sin \pi t}{\pi t}\right)^n$$

$$f_n(x) = \int_{\mathbb{R}} \left(\frac{\sin \pi t}{\pi t}\right)^n e(xt)\, dt$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi t}\right)^2 dt = 1$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi t}\right)^{n+1} e(xt)\, dt$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi t}\right)^{n+1} e(xt)\, dt$$

$$\int t^{-n-1}\, dt = -\, \frac{t^{-n}}{n}$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$\int t^{-n-1}\, dt = -\frac{t^{-n}}{n}$$

$$\int_{\varepsilon}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$\int t^{-n-1}\, dt = -\,\frac{t^{-n}}{n}$$

$$\int_{\varepsilon}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$= \Big[\frac{-1}{n\pi^{n+1}t^{n}}(\sin \pi t)^{n+1} e(xt)\Big|_{\varepsilon}^{\infty}$$

$$+\frac{1}{n\pi^{n+1}} \int_{\varepsilon}^{\infty} t^{-n}\big((n+1)(\sin \pi t)^{n}(\cos \pi t)\pi + (\sin \pi t)^{n+1} 2\pi i x\big) e(xt)\, dt$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$\int t^{-n-1}\, dt = -\,\frac{t^{-n}}{n}$$

$$\int_{\varepsilon}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$= \Big[\frac{-1}{n\pi^{n+1} t^{n}} (\sin \pi t)^{n+1} e(xt)\Big|_{\varepsilon}^{\infty}$$

$$+\frac{1}{n\pi^{n+1}} \int_{\varepsilon}^{\infty} t^{-n} \big((n+1)(\sin \pi t)^{n}(\cos \pi t)\pi + (\sin \pi t)^{n+1} 2\pi i x\big) e(xt)\, dt$$

$$f_{n+1}(x) = \frac{1}{n} \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n} \big((n+1)\cos \pi t + 2ix \sin \pi t\big) e(xt)\, dt$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$\int t^{-n-1}\, dt = -\frac{t^{-n}}{n}$$

$$\int_{\varepsilon}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$= \Big[\frac{-1}{n\pi^{n+1}t^n}(\sin \pi t)^{n+1} e(xt)\Big|_{\varepsilon}^{\infty}$$

$$+\frac{1}{n\pi^{n+1}} \int_{\varepsilon}^{\infty} t^{-n} \big((n+1)(\sin \pi t)^n (\cos \pi t)\pi + (\sin \pi t)^{n+1} 2\pi i x\big) e(xt)\, dt$$

$$f_{n+1}(x) = \frac{1}{n} \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n} \big((n+1)\cos \pi t + 2ix \sin \pi t\big) e(xt)\, dt$$

$$= \frac{1}{n}\Big(\frac{n+1}{2} + x\Big) \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n} e((x+1/2)t)\, dt$$

$$+ \frac{1}{n}\Big(\frac{n+1}{2} - x\Big) \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n} e((x-1/2)t)\, dt$$

$$f_{n+1}(x) = \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$\int t^{-n-1}\, dt = -\frac{t^{-n}}{n}$$

$$\int_{\varepsilon}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n+1} e(xt)\, dt$$

$$= \Big[\frac{-1}{n\pi^{n+1}t^{n}}(\sin \pi t)^{n+1} e(xt)\Big|_{\varepsilon}^{\infty}$$

$$+\frac{1}{n\pi^{n+1}} \int_{\varepsilon}^{\infty} t^{-n}\big((n+1)(\sin \pi t)^{n}(\cos \pi t)\pi + (\sin \pi t)^{n+1} 2\pi i x\big) e(xt)\, dt$$

$$f_{n+1}(x) = \frac{1}{n} \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n} \big((n+1)\cos \pi t + 2ix \sin \pi t\big) e(xt)\, dt$$

$$= \frac{1}{n}\Big(\frac{n+1}{2} + x\Big) \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n} e((x+1/2)t)\, dt$$

$$+ \frac{1}{n}\Big(\frac{n+1}{2} - x\Big) \int_{-\infty}^{\infty} \Big(\frac{\sin \pi t}{\pi t}\Big)^{n} e((x-1/2)t)\, dt$$

$$= \frac{1}{n}\Big(\frac{n+1}{2} + x\Big) f_n(x+1/2) + \frac{1}{n}\Big(\frac{n+1}{2} - x\Big) f_n(x-1/2)$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \big(1 + (n-1)x\big)E_{n-1}(x) + x(1-x)E'_{n-1}(x)$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \big(1 + (n-1)x\big) E_{n-1}(x) + x(1-x) E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \big(1 + (n-1)x\big) E_{n-1}(x) + x(1-x) E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_{n-1}(x_r) = 0 \qquad\qquad 0 > x_1 > x_2 > \cdots > x_{n-2}$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \bigl(1 + (n-1)x\bigr)E_{n-1}(x) + x(1-x)E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_{n-1}(x_r) = 0 \qquad\qquad 0 > x_1 > x_2 > \cdots > x_{n-2}$$

$$E'_{n-1}(x_1) > 0 \qquad E'_{n-1}(x_2) < 0 \qquad E'_{n-1}(x_3) > 0 \qquad \cdots$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \big(1 + (n-1)x\big) E_{n-1}(x) + x(1-x)E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_{n-1}(x_r) = 0 \qquad\qquad 0 > x_1 > x_2 > \cdots > x_{n-2}$$

$$E'_{n-1}(x_1) > 0 \qquad E'_{n-1}(x_2) < 0 \qquad E'_{n-1}(x_3) > 0 \qquad \cdots$$

$$E_n(x_1) < 0 \qquad E_n(x_2) > 0 \qquad E_n(x_3) < 0 \qquad \cdots$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \big(1 + (n-1)x\big) E_{n-1}(x) + x(1-x) E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_{n-1}(x_r) = 0 \qquad\qquad 0 > x_1 > x_2 > \cdots > x_{n-2}$$

$$E'_{n-1}(x_1) > 0 \qquad E'_{n-1}(x_2) < 0 \qquad E'_{n-1}(x_3) > 0 \qquad \cdots$$

$$E_n(x_1) < 0 \qquad E_n(x_2) > 0 \qquad E_n(x_3) < 0 \qquad \cdots$$

$E_n(x)$ has a zero in $(x_1, 0)$, in (x_2, x_1), $\ldots$, in (x_{n-2}, x_{n-3}), and in $(-\infty, x_{n-2})$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \bigl(1 + (n-1)x\bigr)E_{n-1}(x) + x(1-x)E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_{n-1}(x_r) = 0 \qquad 0 > x_1 > x_2 > \cdots > x_{n-2}$$

$$E'_{n-1}(x_1) > 0 \qquad E'_{n-1}(x_2) < 0 \qquad E'_{n-1}(x_3) > 0 \qquad \cdots$$

$$E_n(x_1) < 0 \qquad E_n(x_2) > 0 \qquad E_n(x_3) < 0 \qquad \cdots$$

$E_n(x)$ has a zero in $(x_1, 0)$, in (x_2, x_1), …, in (x_{n-2}, x_{n-3}), and in $(-\infty, x_{n-2})$

$$E_{n,\delta}(x) = n! \sum_{k=0}^{n} f_{n+1}(-(n+1)/2 + k + \delta)x^k$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \bigl(1 + (n-1)x\bigr) E_{n-1}(x) + x(1-x) E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_{n-1}(x_r) = 0 \qquad\qquad 0 > x_1 > x_2 > \cdots > x_{n-2}$$

$$E'_{n-1}(x_1) > 0 \qquad E'_{n-1}(x_2) < 0 \qquad E'_{n-1}(x_3) > 0 \qquad \cdots$$

$$E_n(x_1) < 0 \qquad E_n(x_2) > 0 \qquad E_n(x_3) < 0 \qquad \cdots$$

$E_n(x)$ has a zero in $(x_1, 0)$, in (x_2, x_1), $\ldots$, in (x_{n-2}, x_{n-3}), and in $(-\infty, x_{n-2})$

$$E_{n,\delta}(x) = n! \sum_{k=0}^{n} f_{n+1}(-(n+1)/2 + k + \delta) x^k$$

$$E_{n,1}(x) = E_n(x) \qquad\qquad E_{n,0}(x) = x E_n(x)$$

$$E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$$

$$E_n(x) = \bigl(1 + (n-1)x\bigr) E_{n-1}(x) + x(1-x)E'_{n-1}(x)$$

Frobenius (1910): Zeros of $E_n(x)$ are simple, negative real, and interlacing

$$E_{n-1}(x_r) = 0 \qquad\qquad 0 > x_1 > x_2 > \cdots > x_{n-2}$$

$$E'_{n-1}(x_1) > 0 \qquad E'_{n-1}(x_2) < 0 \qquad E'_{n-1}(x_3) > 0 \qquad \cdots$$

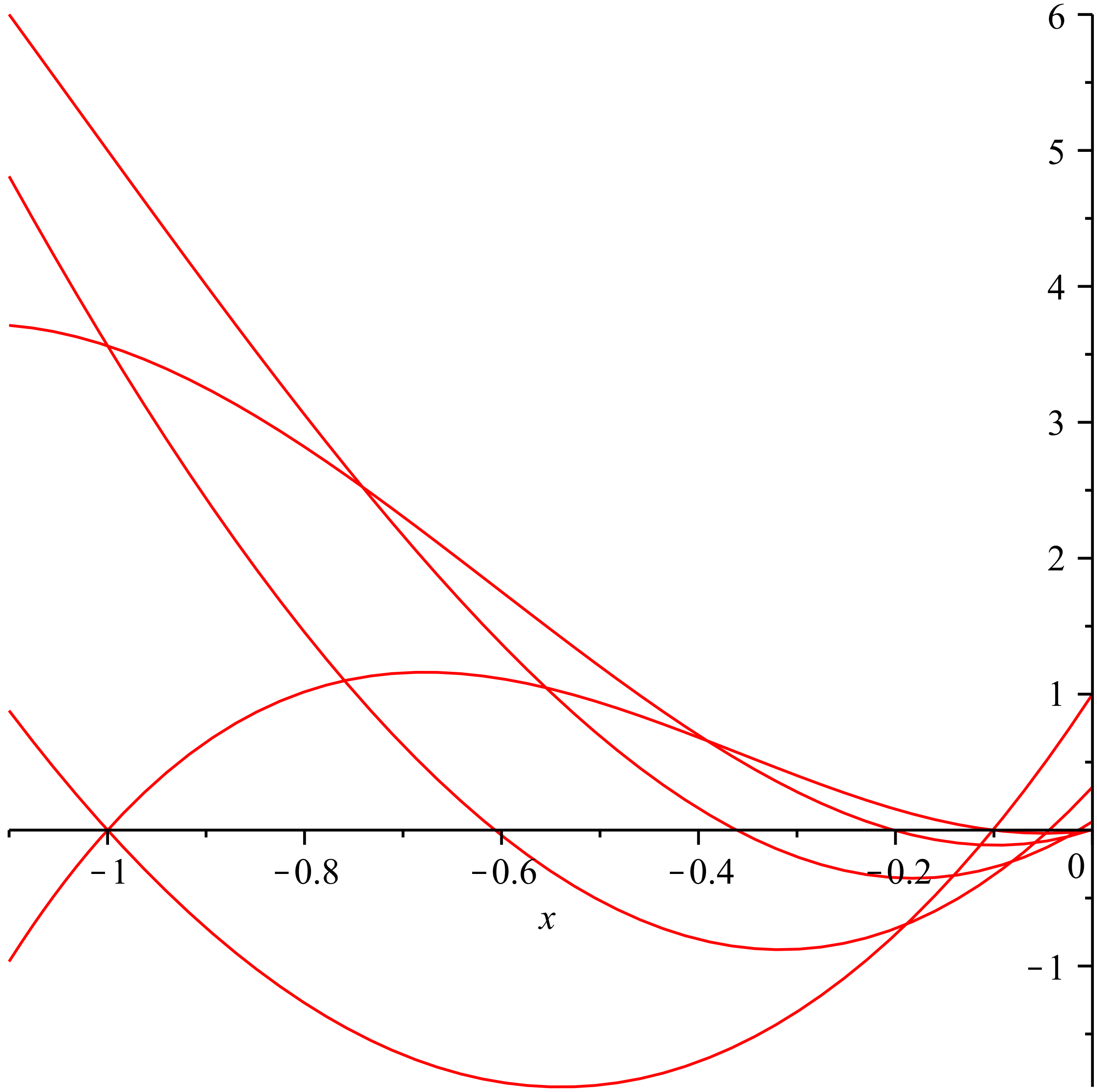
$$E_n(x_1) < 0 \qquad E_n(x_2) > 0 \qquad E_n(x_3) < 0 \qquad \cdots$$

$E_n(x)$ has a zero in $(x_1, 0)$, in (x_2, x_1), ..., in (x_{n-2}, x_{n-3}), and in $(-\infty, x_{n-2})$

$$E_{n,\delta}(x) = n! \sum_{k=0}^{n} f_{n+1}(-(n+1)/2 + k + \delta) x^k$$

$$E_{n,1}(x) = E_n(x) \qquad\qquad E_{n,0}(x) = xE_n(x)$$

$$E_{n,\delta}(x) = \bigl((n-\delta)x + \delta\bigr) E_{n-1,\delta}(x) + x(1-x)E'_{n-1,\delta}(x)$$



$E_{4,m/4}(x)$ for $m = 0, \ldots, 4$, $-1.1 \leq x \leq 0$