How noncommutative is noncommutative topological entropy?
or
On searching for classical subsystems of quantum evolutions

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(partly based on joint work with Joachim Zacharias, Wojtek Szymański and Jeong Hee Hong)

IMPAN

Fields Institute, 9 December 2010

Generalities about entropy

Dynamical entropy of a classical dynamical system is a certain number in $[0, \infty]$ describing the 'mixing' behaviour of the system.

Initially studied mainly for **measurable** dynamical systems on probability spaces and defined in terms of the growth of the randomness of partitions induced by the studied evolution, it was soon introduced also in compact **topological** dynamics, where one replaces partitions with finite covers and studies the growth of the cardinality of minimal subcovers.

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Classical topological entropy via spanning sets

(X, d) - compact metric space, $T : X \to X$ - continuous map.

A finite set $F \subset\subset X$ is called (n, ϵ)-**spanning** for T if

$$\forall_{x \in X} \exists_{f \in F} \quad d(T^k x, T^k f) \leq \epsilon \text{ for } k = 0, \ldots, n.$$

Compactness of X implies that finite (n, ϵ)-spanning sets exist. Put

$$s_{n,\epsilon}(T) = \min\{\text{card} F : F - (n, \epsilon) - \text{spanning for } T\}$$

Definition (Bowen, 1971)

The topological entropy of T, $h_{\text{top}}(T)$, is defined by:

$$h_{\text{top}}(T) = \sup_{\epsilon > 0} \lim_{n \to \infty} \frac{1}{n} \log(s_{n,\epsilon}(T))$$

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What is entropy good for?

Topological and measure entropy have been intensely studied for the last 60 years. They can be used for

- recognising chaotic behaviour;
- classifying dynamical systems (for certain classes of dynamical systems, in particular for Bernoulli shifts, measure entropy is a complete invariant).

Algebraic point of view

If X is compact, and $T : X \to X$ is continuous, the map

$$\alpha_T(f) = f \circ T, \;\; f \in C(X)$$

defines a unital *-homomorphism of the C^*-algebra $C(X)$. Of course all unital *-homomorphisms of $C(X)$ are of this type.

Problem

Let A be a unital C^*-algebra, $\alpha : A \to A$ an automorphism of A (a unital *-homomorphism, completely positive map, etc.). How to define the 'topological entropy' of α?

There were many examples of looking at this questions for the measure entropy, with the most successful definition given by Connes, Narnhofer and Thirring (after earlier work by Connes and Størmer) – resulting invariant is called the **CNT entropy**.

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Approximating triples

Fix a C^*-algebra A. M_n denotes the algebra of n by n complex matrices.

Write $(\phi, \psi, M_n) \in CPA(\mathrm{A})$ if $\phi : M_n \to \mathrm{A}$, $\psi : \mathrm{A} \to M_n$ are unital and *completely positive*.

For $\Omega \subset\subset \mathrm{A}$ and $\epsilon > 0$ the notation $(\phi, \psi, M_n) \in CPA(\mathrm{A}, \Omega, \epsilon)$ means that $(\phi, \psi, M_n) \in CPA(\mathrm{A})$ and

$$\forall_{a \in \Omega} \|\phi \circ \psi(a) - a\| < \epsilon.$$

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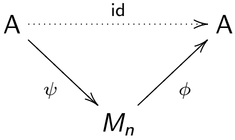

Voiculescu's definition

If A is **nuclear**, $CPA(A, \Omega, \epsilon) \neq \emptyset$ for all Ω, ϵ. Set

$$\mathrm{rcp}(\Omega, \epsilon) := \min\{n \in \mathbb{N} : (\phi, \psi, M_n) \in CPA(A, \Omega, \varepsilon)\}.$$

Definition (Voiculescu, 1995)

Let A be nuclear and let $\alpha : A \to A$ be a unital *-homomorphism. The *topological (approximation, Voiculescu) entropy* of θ is defined by the formula:

$$\mathrm{ht}\,\alpha = \sup_{\epsilon > 0,\, \Omega \subset\subset A} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{rcp}(\Omega^{(n)}, \epsilon).$$

Here $\Omega^{(n)} = \bigcup_{j=0}^{n} \alpha^j(\Omega)$.

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Basic properties of ht α

- if $A = C(X)$ for a compact X, $T : X \to X$ is a homeomorphism, then $\mathrm{ht}\,\alpha_T = h_{\mathrm{top}}(T)$
- the entropy ht has a natural modification/extension due to Brown allowing for considering transformations of exact (and not only nuclear) C^*-algebras
- ht is monotone under passing to invariant subalgebras (but it is not clear what happens to quotients!)
- the idea of 'approximation entropies' can be successfully applied outside the C^*-world - see the work of Kerr and Li on entropy of Banach space isomorphisms.

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As stated above, in the case of a commutative dynamical system (i.e. a continuous map $T : X \to X$), the Voiculescu entropy of the map α_T coincides with $h_{\rm top}(T)$. Actually

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is easy, but no direct proof of

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Voiculescu's proof requires using the CNT entropy, classical measure entropy and the classical variational principle. The problem is related to understanding precisely the nature of matrix approximations on $C(X)$.

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Strategy for computing Voiculescu entropy

Given $\alpha : \mathsf{A} \to \mathsf{A}$

(i) find explicit approximations showing that $\mathrm{ht}\,\alpha \leq M$

(ii) find an invariant subalgebra $\mathsf{B} \subset \mathsf{A}$ and try to prove $\mathrm{ht}\,\alpha|_{\mathsf{B}} \geq M$

(iii) if B is commutative, we can use the classical entropy to compute $\mathrm{ht}\,\alpha|_{\mathsf{B}}$.

There is also a 'geometric' way of deducing that the Voiculescu entropy is positive, due to David Kerr and Hanfeng Li and inspired by the local geometry of Banach spaces (one needs to look for long orbits yielding isomorphic copies of l^1 inside A); their work suggests the need for 'local spectral commutativity'.

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Non-zero entropy vs commutative subsystems

There are other indications that 'high noncommutativity' $\approx$ zero Voiculescu entropy:

- Haagerup and Størmer showed that occurrence of maximal (CNT type) entropy for a system of subalgebras is related to existence of suitable maximally abelian subalgebras;
- Størmer proved that free shifts (i.e. 'very noncommutative' systems) have 0-entropy.

Question

Are there any pairs (A, α) such that $\mathrm{ht}\,\alpha$ is strictly bigger than

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Bitstream shifts

Let $S \subset \mathbb{N}$ (think of it as a sequence of 0s and 1s).
Consider a unital *-algebra generated by elements $(u_i)_{i\in\mathbb{Z}}$ satisfying the following conditions:

$$u_i = u_i^*, \quad u_i^2 = 1, \quad (u_i \text{ are selfadjoint unitaries}),$$

$$u_i u_j = u_j u_i (-1)^{\chi_S(|i-j|)}, \qquad i, j \in \mathbb{Z}.$$

A_S – universal C^*-completion of the *-algebra introduced above.

The transformation given by

$$\sigma(u_i) = u_{i+1}, i \in \mathbb{Z}$$

extends in a unique way to a *-preserving unital automorphism of A_S.

The pair (σ, α_S) is called a noncommutative **bitstream (or binary) shift** associated to S (and was introduced and first studied by Powers).

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- A_S is always nuclear (it can be realised as a twisted group C^*-algebra of a group $\prod_{i=1}^{\infty} \mathbb{Z}_2$);
- A_S admits a tracial state τ;
- τ is σ-invariant: $\tau \circ \sigma = \tau$.

Moreover if S is 'sufficiently chaotic' we also have

- τ is a *unique* σ-invariant state on A_S;
- the CNT entropy $h_\tau(\sigma) = 0$;
- the CNT entropy $h_{\tau\otimes\tau}(\sigma \otimes \sigma) = \log 2$.

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- τ is a *unique* σ-invariant state on A_S;
- the CNT entropy $h_\tau(\sigma) = 0$;
- the CNT entropy $h_{\tau\otimes\tau}(\sigma \otimes \sigma) = \log 2$.

Theorem (AS)

Let $S \subset \mathbb{N}$ be 'sufficiently chaotic'. Then

$$\mathrm{ht}_c(\sigma) = 0 < \frac{\log 2}{2} \leq \mathrm{ht}(\sigma).$$

So the Voiculescu entropy *is* genuinely noncommutative. What other methods can we use to compute it? We will consider this problem for endomorphisms of Cuntz algebras.

The corresponding problem for the CNT entropy for an automorphism of the hyperfinite II_1 factor is open.

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Cuntz algebra

Fix $N \in \mathbb{N}$. Let $\mathcal{O}_N$ – **Cuntz algebra**. with generating isometries $S_1, \ldots, S_N$. Use μ to denote a $\{1, \ldots, N\}$-valued multiindex and let

$$S_\mu := S_{\mu_1} S_{\mu_2} \ldots S_{\mu_k},$$

$$|\mu| = \sum_{i=1}^{k} \mu_i.$$

$\mathcal{O}_N$ contains a **diagonal masa** (maximal abelian subalgebra) $\mathcal{C}_N := \overline{\text{Lin}}\{S_\mu S_\mu^*\}$, isomorphic to the algebra of continuous functions on a Cantor set (equivalently, a full Markov shift on N letters).

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Natural inclusions

The inclusion $\mathcal{C}_N \subset \mathcal{O}_N$ can be viewed as a part of the

$$\mathcal{C}_N = \bigotimes_{n=1}^{\infty} D_N \subset \bigotimes_{n=1}^{\infty} M_N \subset \mathcal{O}_N.$$

By 'changing coordinates' in M_N and replacing diagonals D_N by $U^* D_N U$ ($U \in M_N$ - a unitary) we can construct other masas in $\mathcal{O}_N$. We will call them **standard masas**.

Canonical shift

Let $\Phi : \mathcal{O}_N \to \mathcal{O}_N$ be the **canonical shift** endomorphism:

$$\Phi(a) = \sum_{i=1}^{N} S_i a S_i^*, \quad a \in \mathcal{O}_N.$$

It leaves $\mathcal{F}_N$, $\mathcal{C}_N$ (and each other standard masa) invariant; on each standard masa it reduces to the classical full Markov shift.We have (as shown by Choda, see also Evans)

$$\mathrm{ht}\,\Phi = \log N = \mathrm{ht}\,\Phi|_{\mathcal{C}_N}.$$

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Endomorphisms of $\mathcal{O}_N$ vs unitaries in $\mathcal{O}_N$

Cuntz showed that there is a 1-1 correspondence between **unitaries** in $\mathcal{O}_N$ and **unital endomorphisms** of $\mathcal{O}_N$, given by formulas

$$\rho_U(S_i) = US_i, \quad i = 1, \ldots, N$$

and

$$U_\rho = \sum_{i=1}^{n} \rho(S_i)S_i^*.$$

In particular

$$U_\Phi = \sum_{i,j=1}^{N} S_i S_j S_i^* S_j^*.$$

This correspondence has been used in the recent intensive study of endomorphisms of $\mathcal{O}_N$ in the series of papers by Conti, Szymański and their collaborators.

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Computing Voiculescu entropy of endomorphisms of Cuntz algebras

From the entropy point of view we have the following result:

Theorem (AS + J.Zacharias)

If $U \in \mathcal{U}(\mathcal{O}_N)$, $U \in \mathrm{span}\{S_\mu S_\nu^* : |\mu| = |\nu| \leq k\}$, then

$$\mathrm{ht}\, \rho_U \leq (k-1) \log N.$$

It can happen that ρ_U leaves $\mathcal{C}_N$ invariant and $\mathrm{ht}\, \rho_U > \mathrm{ht}\, \rho_U|_{\mathcal{C}_N}$
(in fact in our example ρ_U looks like shift on some standard masas, and degenerates in others).

So it can happen that we are looking at a 'wrong' Cantor set in $\mathcal{O}_N$. But it can be even worse...

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Examples of endomorphisms which do not preserve any standard masa

Consider an endomorphism ρ' (one of the class studied by M. Izumi with relations to index theory):

$$\rho'(S_0) = \frac{1}{\sqrt{2}}(S_0 + S_1), \quad \rho'(S_1) = \frac{1}{\sqrt{2}}(S_0 S_0 S_0^* + S_1 S_1 S_1^* - S_1 S_0 S_0^* - S_0 S_1 S_1^*).$$

Proposition (AS)

$$\mathrm{ht}\,\rho' = \frac{1}{2}\log 2.$$

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The entropy computation from the last slide is in fact very easy, and uses the fact that ρ'^2 has a simple form. The following result however generalises and in a way explains last statement. It uses some basic index theory for subfactors and its connections with CNT entropy.

Theorem (AS)

Let V be an irreducible multiplicative unitary on $\mathrm{H} \otimes \mathrm{H}$, where H is an N-dimensional Hilbert space; view it as a matrix in $M_N \otimes M_N$ and further via the usual isomorphism $M_N \otimes M_N \subset \mathcal{O}_N$ as a unitary in $\mathcal{O}_N$. Let F be the flip unitary in $M_N \otimes M_N$. The topological entropy of the endomorphism of $\mathcal{O}_N$ associated with VF is equal to $\log N$.

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Conclusion and the general framework

The question of types of masas in a given von Neumann algebra (and, to a smaller extent, C^*-algebra) has been intensely studied for over 60 years, with a lot of progress and interest in the last 10 years mainly due to Sorin Popa and his collaborators. This line of investigation can be called

- the search for classical subspaces of quantum spaces

Very little is known about the existence and properties of invariant masas for a given automorphism (or endomorphism). In this talk we tried to argue that this is an important and natural question, which can be called

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