Mathematical Modelling and Simulations

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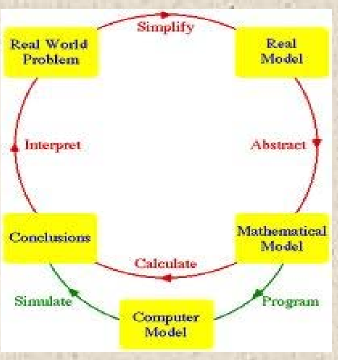
CAIMS

Special Session on

Mathematics Education

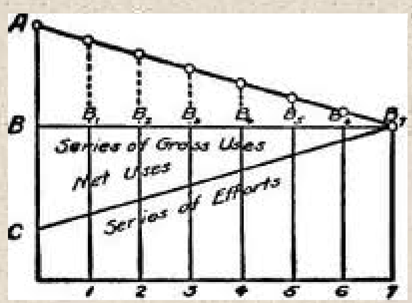


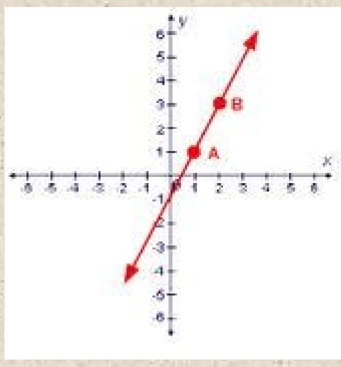
Mathematical Modelling

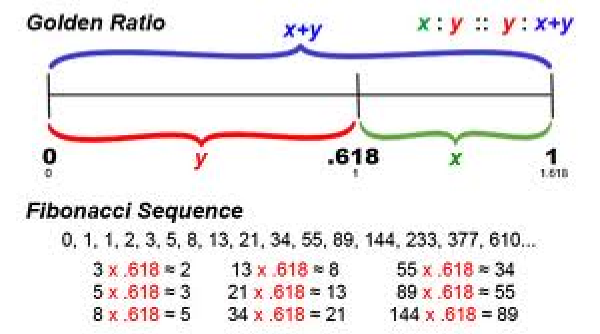
The use of mathematics to

Mathematical Modelling

- Show patterns existing in real world phenomena
  - Sequences, series, proportionality, equations can are used to represent and describe the real world
  - Differences, sums, means, medians
  - Fibonacci numbers, golden ratio
  - Geometric series
  - Equation of a line
  - Linear or non-linear equations
  - etc

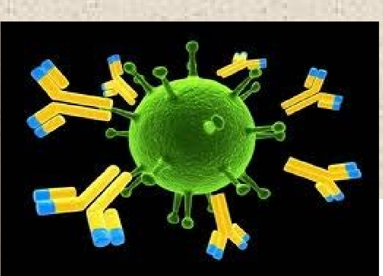


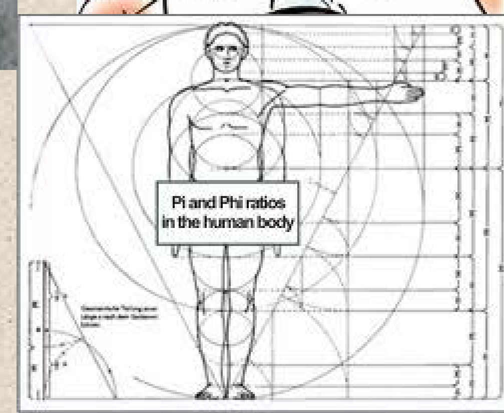




Mathematical Modelling

- Investigate questions about the observed world
  - Biology, Chemistry, Ecology, Economics, Engineering, Geography, Physics
  - Health, Immunology, Epidemiology, Vaccine and Drug development

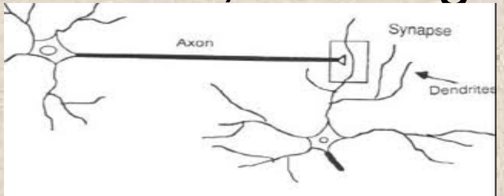


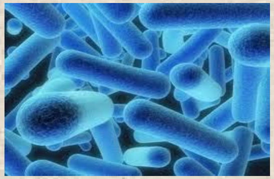


Mathematical Modelling

- Explain real-world phenomena
  - Population growth
  - Spread of disease
  - Effects of change in climate on environmental variables
  - Supply and demand

  - Hazard rate – warranties
  - Interest rates
  - Price of lottery tickets, gambling
  - How the brain works
  - Relationship between distance, velocity, acceleration...
  - War, marriage, bullying, hardcore fan





Mathematical Modelling

- Test ideas
  - What happens if we
    - Vaccinate or give drug therapy?
    - Decrease supply?
    - Decrease demand?
    - Change the environment/living conditions?
    - Use a different manufacturing material?
    - Change amount needed for down payment on a home?
    - Change the price of a lottery ticket?
    - Change the size of the jackpot?

Mathematical Modelling

- Make predictions about the real world
  - Size of an epidemic depends on the pathogen traits
  - Insect and Animal populations are affected by climate change
  - Gas prices will continue to go up even if demand decreases considerably
  - Sustain a fishery, farm, resource
  - If I make a light bulb out of a new material, I can make the warranty longer, and charge more for the bulb
  - Interest rates will go up or down depending on the market and the 'feeling' about the market
  - The optimal price of a lottery ticket depends on the size of the minimum and maximum jackpots – need to know this to make a new lottery
  - Models of the brain can be used to make better computers
  - Who is the faster runner, over different distances?
  - Win the war?, Get a divorce?, Reduce bullying in school yard?
  - What do we need to change the allegiance of a fan? - baseball strike, a hockey team in Hamilton?

Mathematical Modelling

- Have Fun!!
  - Zombies
  - Bieber Fever
  - Hockey teams
  - Game of telephone
  - Pinball
  - Fractals – clouds, moutains, etc
  - Animated movies

Mathematical Modelling

- Help us better understand things
- Idealization of the real world
  - Never completely accurate
- But, we can get conclusions that are meaningful
- A mathematical model can help us understand a behaviour, and help us plan for the future

Mathematical Modelling

- Models simplify reality
- Use proportionality
  - Process x is proportional to process y
    - $x=ky$
  - Or…. the change in x is proportional to y
    - $dx/dt = ky$
    - $X(t) = ky+x(t-\Delta t)$ ** future value = present value + change
    - $\Delta a_n=a_1-a_0$

Mathematical Modelling

- Steps in modelling
  - Identify the problem
  - Make assumptions
    - Classify the variables
    - Determine interrelationships among variables selected for study
  - Simplification vs complexity
  - Solve or interpret the model
  - Do the results make sense? – verification
  - Implement the model – tell the world!
  - Maintain the model or increase its complexity

Mathematical Modelling

- A modeller undertakes experiments on mathematical representations of the real world
- There is no best model, only better models
- Challenge in mathematical modelling is

Not to produce the most comprehensive descriptive model

but

to produce the simplest possible model that incorporates the major features of the phenomenon of interest

(Howard Emmons)

Mathematical Modelling

- Interesting tidbits
  - Similar modelling tools and techniques are used in different fields
  - Sometimes a complex process can have a very simple answer
    - Central limit theorem
  - Even a very simple model can produce meaningful conclusions
  - Models help those involved visualize what is happening
    - I say I am a mathematical immunologist – “ACK!!”
    - But after describing what I do please are amazed that they can understand my work

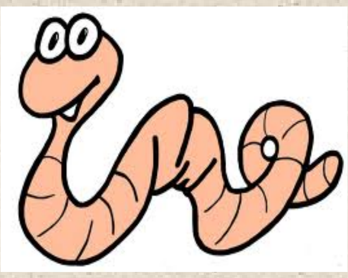
MI2 Group Projects

Multi-scale

In-Host	Between Host	Pop'n Level	Experimental
HIV			
HBV/HCV			
Measles	Measles	Measles	
Influenza	Influenza	Influenza	Influenza
TB			
Multiple Exposure	Multiple Exposure	Multiple Exposure	
		Herpes	

Activities

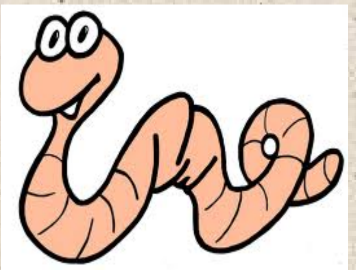
- How should a bird select worms?
  - A bird is searching a patch of lawn for some worms
  - Suppose that there are two types of worms
    - Big, fat, juicy ones (highly nutritious)
    - Long, thin, skinny ones (less nutritious)
  - Which worms should the bird eat?
    - Depends on
      - The number of worms of each type
      - Energy required to search and dig for each type
      - Number of calories gained from eating each type



Activities

- Type i = 1, 2
- E_i – energy gained from a worm of type i
- h_i – time to find worm of type i
- Suppose that type 1 is more profitable, then

$$\frac{E_1}{h_1} > \frac{E_2}{h_2}$$



Activities

- Type i = 1, 2
- Assume that a bird takes 1 minute of time to assess the state of the lawn before starting to eat
- L_i – number of worms of type i that the bird spots in the fasting minute
- u_i – number of worms of type i that the bird eats
- What is the total number of calories gained by the bird after a meal?
- What it is the total amount of time that the birds spends on this meal?

Activities

- Total number of calories

$$u_1 E_1 + u_2 E_2$$

- Total time devoted to meal

$$1 + u_1 h_1 + u_2 h_2$$

- Average rate of energy uptake

$$P = \frac{u_1 E_1 + u_2 E_2}{1 + u_1 h_1 + u_2 h_2}$$

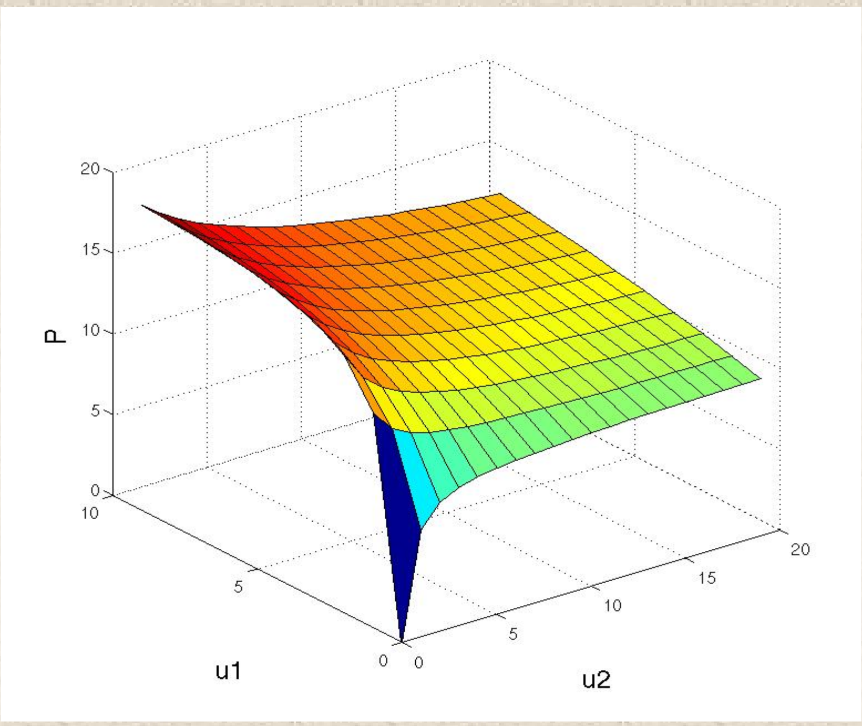
Activities

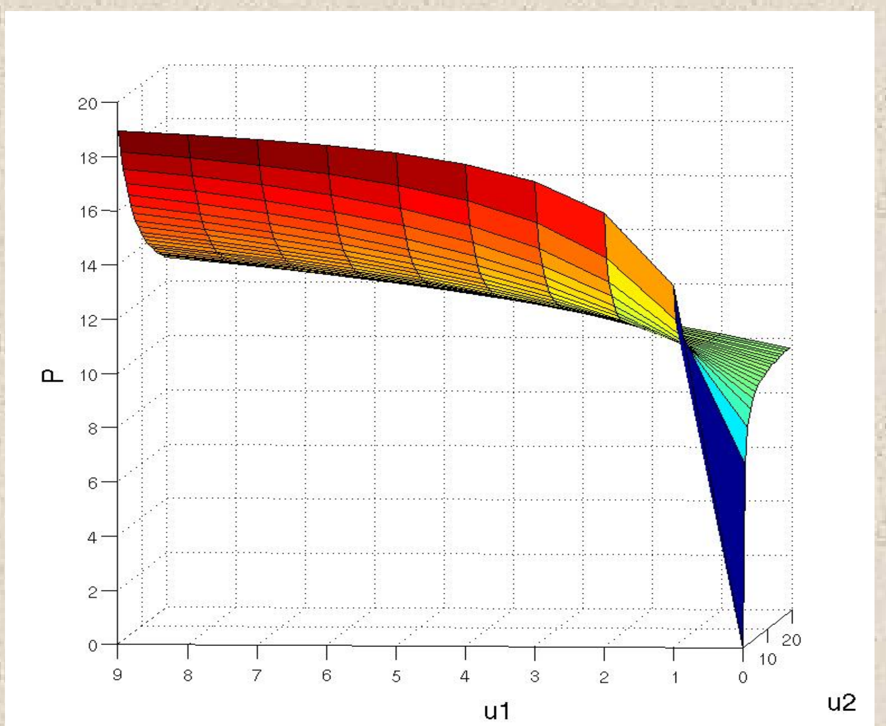
- Birds choose their prey so as to maximize the rate of energy uptake (this is what biologists suggest)
- The bird must choose a meal (u1, u2) so that P is maximized

$$0 < u_1 < L_1 \qquad 0 < u_2 < L_2$$

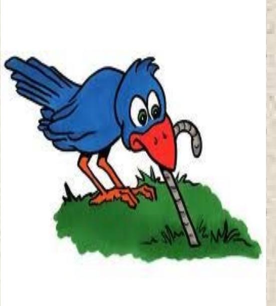
$$P = \frac{u_1 E_1 + u_2 E_2}{1 + u_1 h_1 + u_2 h_2}$$

Activities





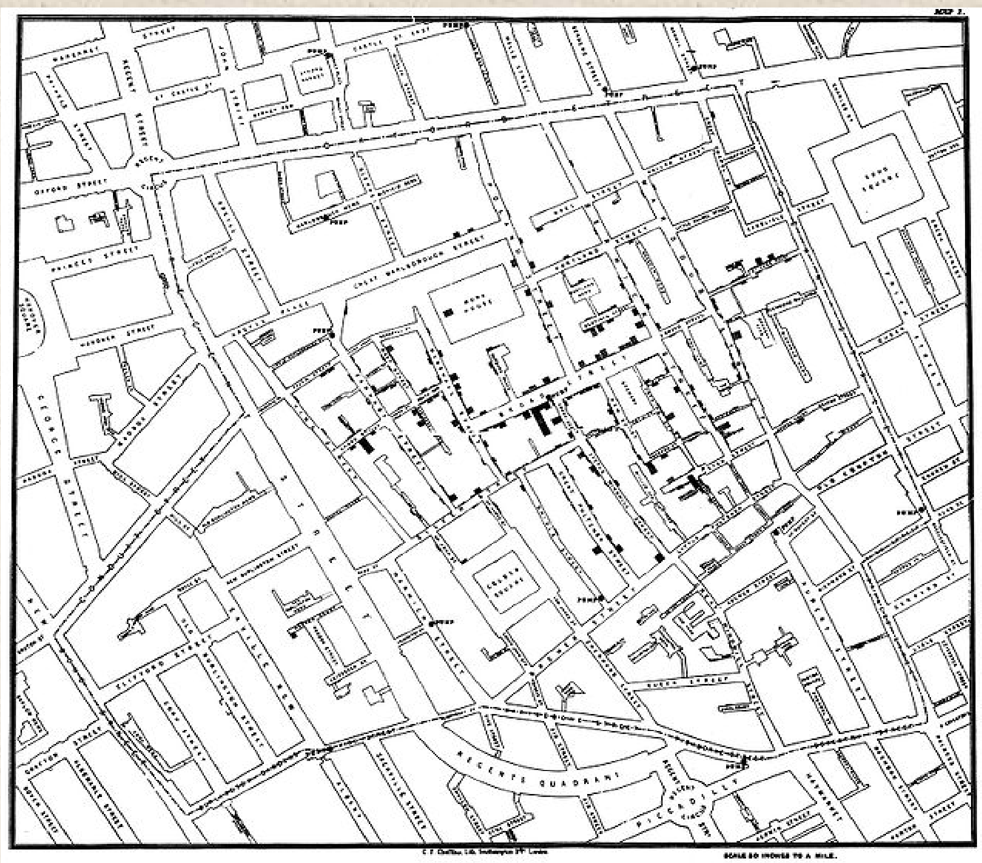
$$P = \frac{u_1 E_1 + u_2 E_2}{1 + u_1 h_1 + u_2 h_2}$$



Some history of epidemiology

John Snow (15 March 1813 – 16 June 1858) was a British physician and a leader in the adoption of anesthesia and medical hygiene.

He is considered to be one of the fathers of epidemiology, because of his work in tracing the source of a cholera outbreak in Soho, England, in 1854.



Original map by Dr. John Snow showing the clusters of cholera cases London epidemic of 1854 .

Centred around Broad St pump.

Used statistics to show connection between the quality of the source of water and cholera cases.

Some history of epidemiology

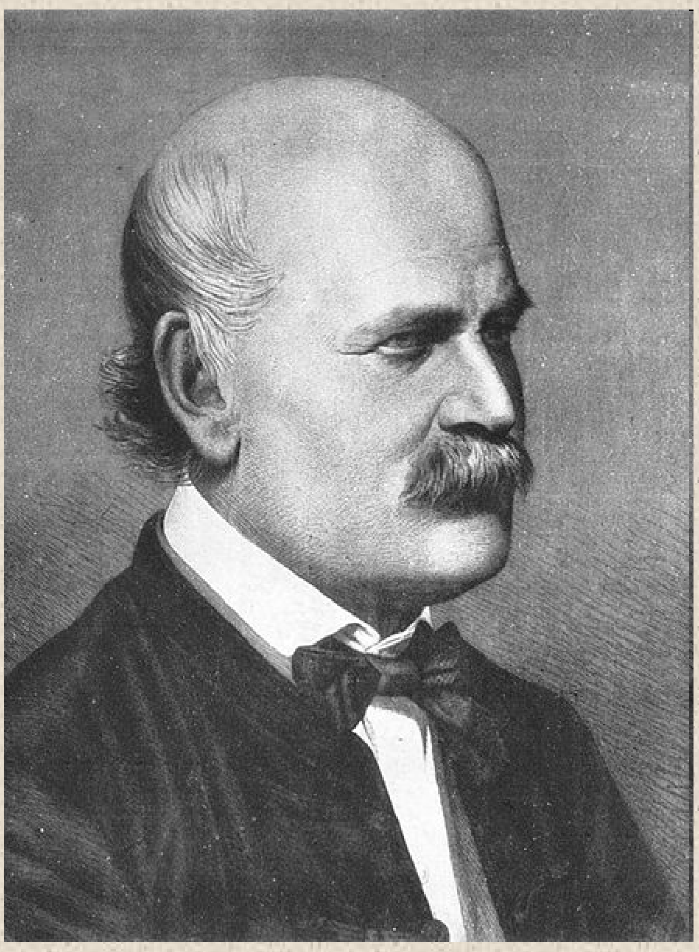
Companies delivering water from sewage-polluted parts of Thames had increased cholera incidence.

Snow's study was a major event in the history of public health, and can be regarded as the founding event of the science of epidemiology.

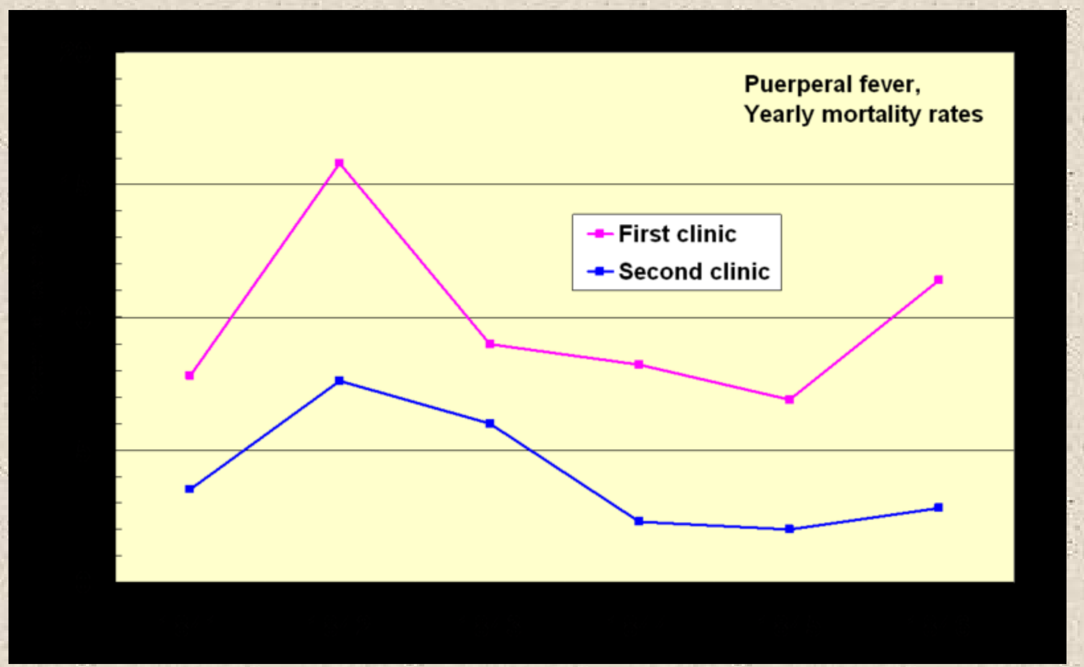
Some history of epidemiology

Ignaz Philipp Semmelweis (July 1, 1818 – August 13, 1865), was a Hungarian physician called the *"saviour of mothers"*

Discovered, by 1847, that cases of puerperal fever, also known as childbed fever could be drastically cut by hand washing.



Some history of epidemiology



Puerperal fever yearly mortality rates for the First and Second Clinic at the Vienna General Hospital 1841-1846. The First Clinic evidently has the larger mortality rate.

Semmelweis was severely troubled and literally sickened that his First Clinic had a much higher mortality rate due to puerperal fever than the Second Clinic.

It "made me so miserable that life seemed worthless".

The two clinics used almost the same techniques, and Semmelweis started a meticulous work eliminating all possible differences, even including religious practices. The only major difference was the individuals who worked there. The First Clinic was the teaching service for medical students, while the Second Clinic had been selected in 1841 for the instruction of midwives only.

Some history of epidemiology

Semmelweis' discovery took place at the Vienna General Hospital — this image from its inauguration in 1784.

The *Second Clinic's* rate was considerably lower, averaging less than 4%.

This fact was known outside the hospital.

The two clinics admitted on alternate days but women begged to be admitted to the Second Clinic due to the bad reputation of the First Clinic

Some women even preferred to give birth in the streets, pretending to have given sudden birth *en route* to the hospital (a practice known as *street births*), which meant they would still qualify for the child care benefits.

Semmelweis was puzzled that puerperal fever was rare amongst women giving street births.

Some history of epidemiology

He excluded "overcrowding" as a cause because the Second Clinic was always more crowded as stated above but the mortality was lower.

He eliminated climate as a cause because the climate was not different.

The breakthrough for Ignaz Semmelweis occurred in 1847 following the death of his good friend who had been accidentally poked with a students scalpel while performing a postmortem examination.

- Autopsy showed a pathological situation similar to that of the women who were dying from puerperal fever. Semmelweis immediately proposed a connection between cadaveric contamination and puerperal fever.

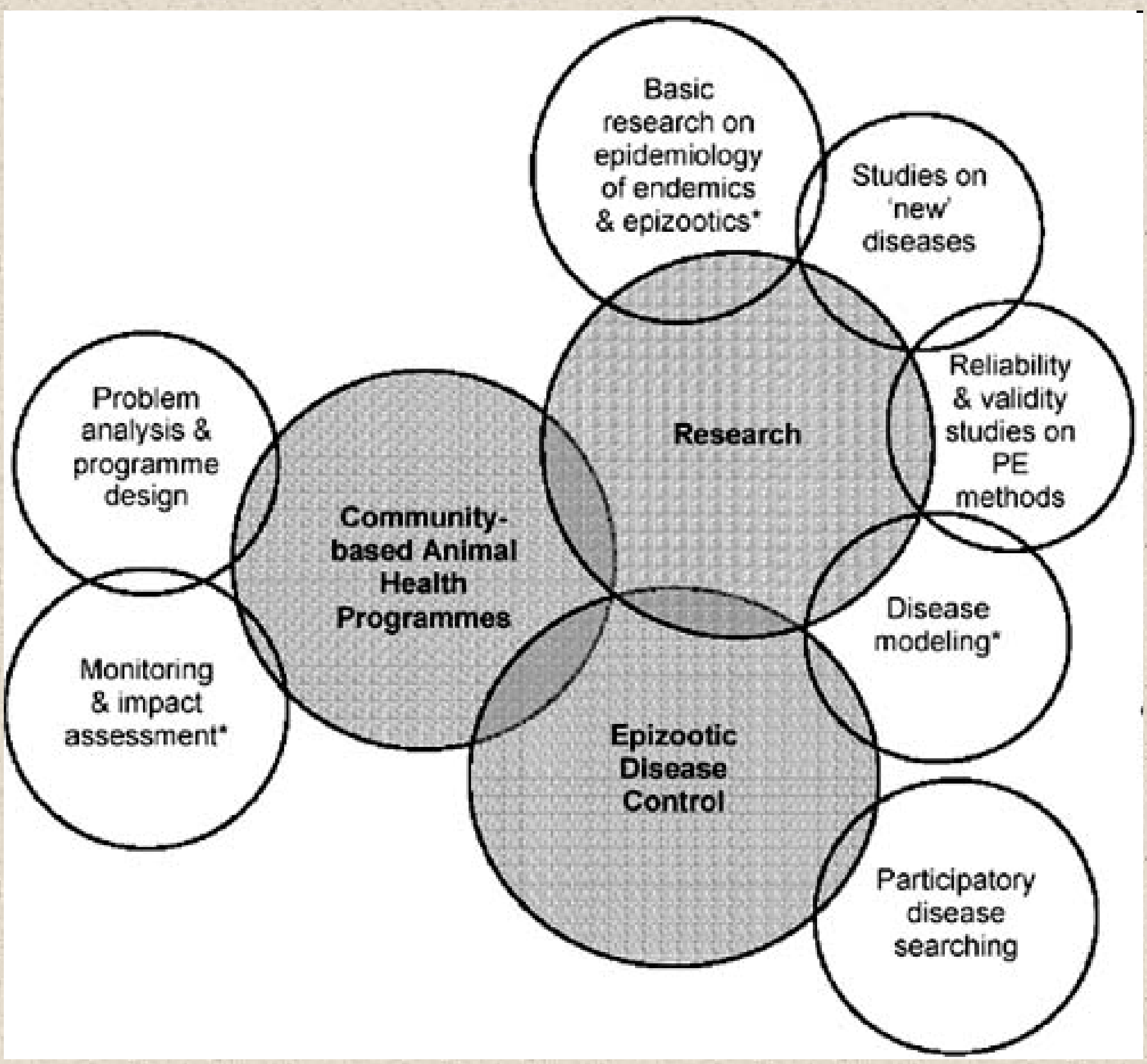
He concluded that he and the medical students carried "cadaverous particles" on their hands from the autopsy room to the patients they examined.

This explained why the student midwives in the Second Clinic who were not engaged in autopsies and had no contact with corpses experienced a much lower mortality rate.

The germ theory of disease had not yet been developed at the time. Thus, Semmelweis concluded that some unknown "cadaverous material" caused childbed fever.

- He instituted a policy of using a solution of chlorinated lime for washing hands between autopsy work and the examination of patients and the mortality rate dropped a ten-fold order of magnitude, comparable to the Second Clinic's.

Epidemiology



Some history of epidemiology

Kermack and McKendrick

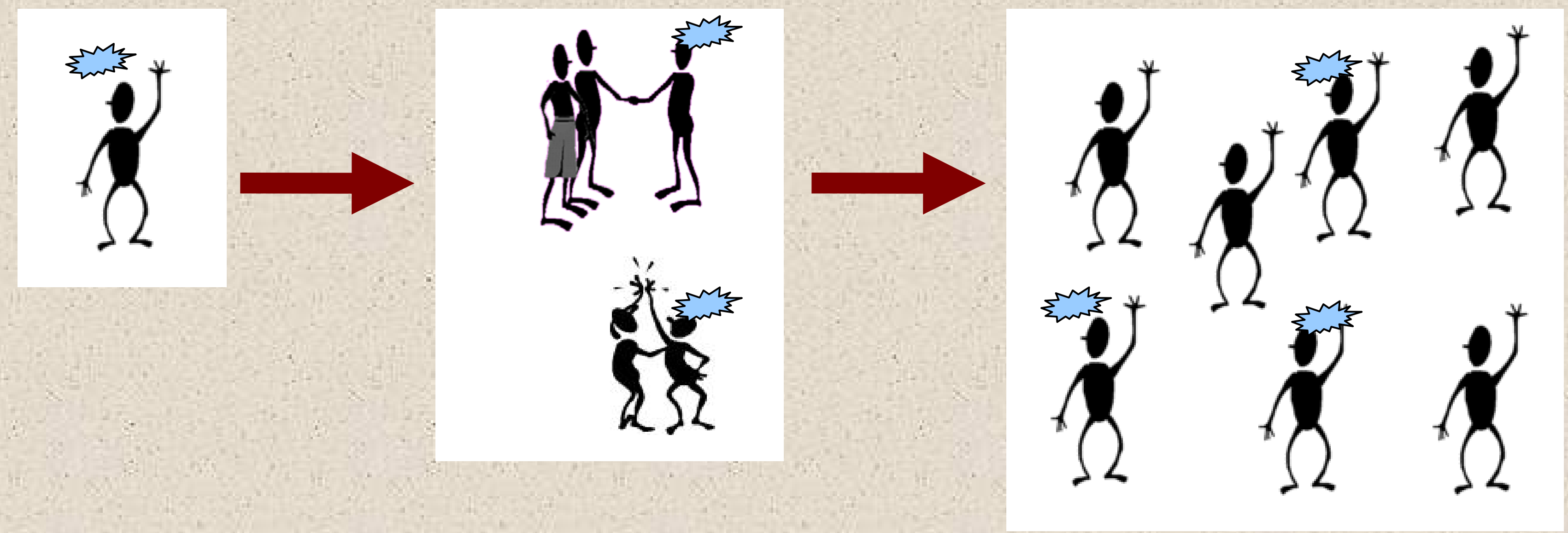
Developed mathematical tools to study disease spread.

Activities

- How can we measure disease spread?
  - Movie Trailer
  - http://www.imdb.com/title/tt1598778/

Activities

- How can we measure disease spread?
  - Basic reproductive ratio - R0
    - The number of new infecteds that one infected makes in a totally susceptible population



- R0 = 3

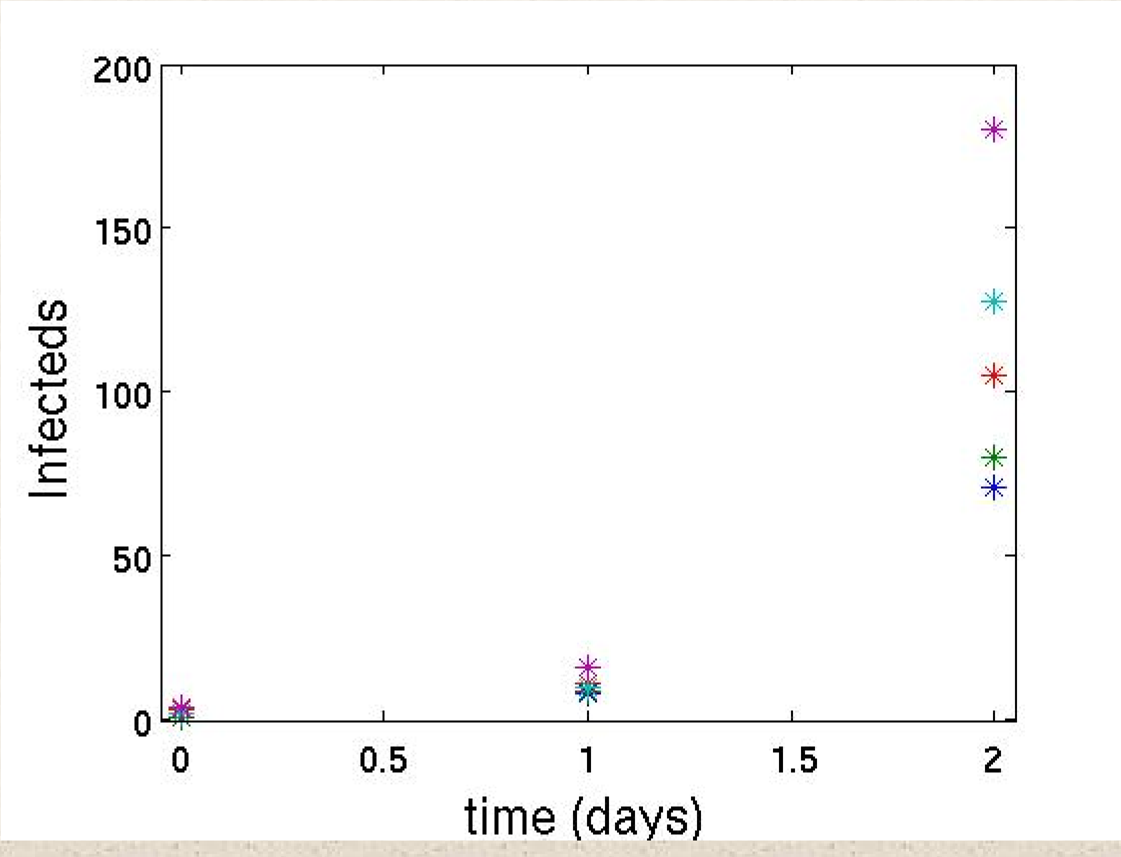
Activities

- How do we calculate this number?

Day	Toronto	Montreal	Vancouver	Halifax	Winnipeg
0	1	1	3	2	4
1	8	9	11	10	16
2	71	80	105	128	180

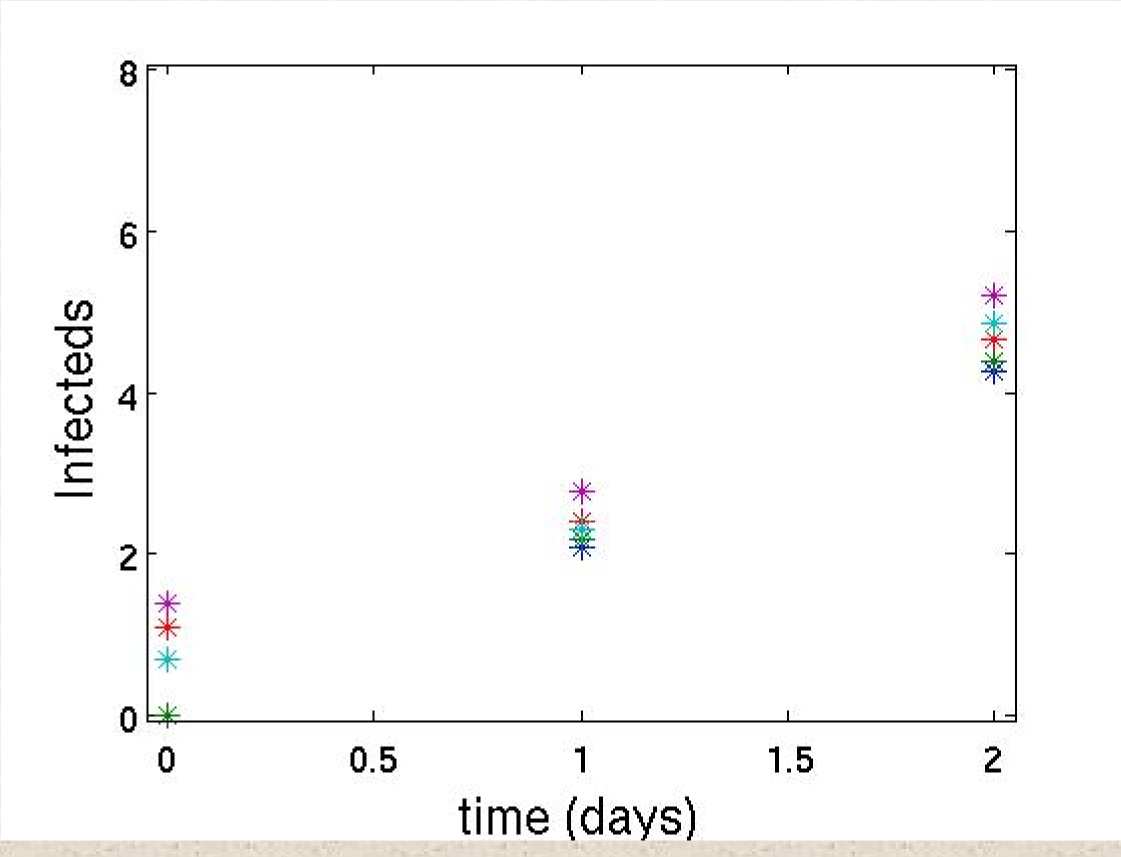
Activities

- Exponential growth - y = b*exp(r0*t)



Activities

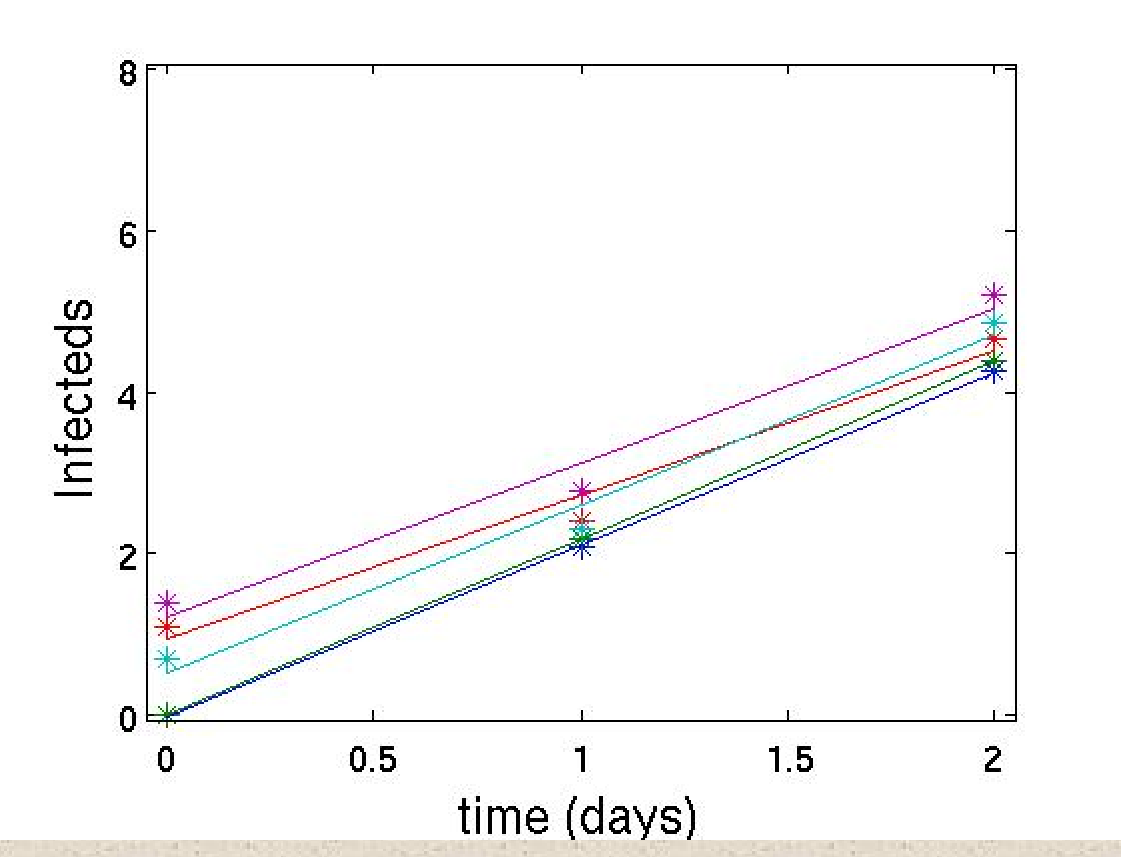
- Log the data   log(y) = log(b)+r0*t



Spreadsheet

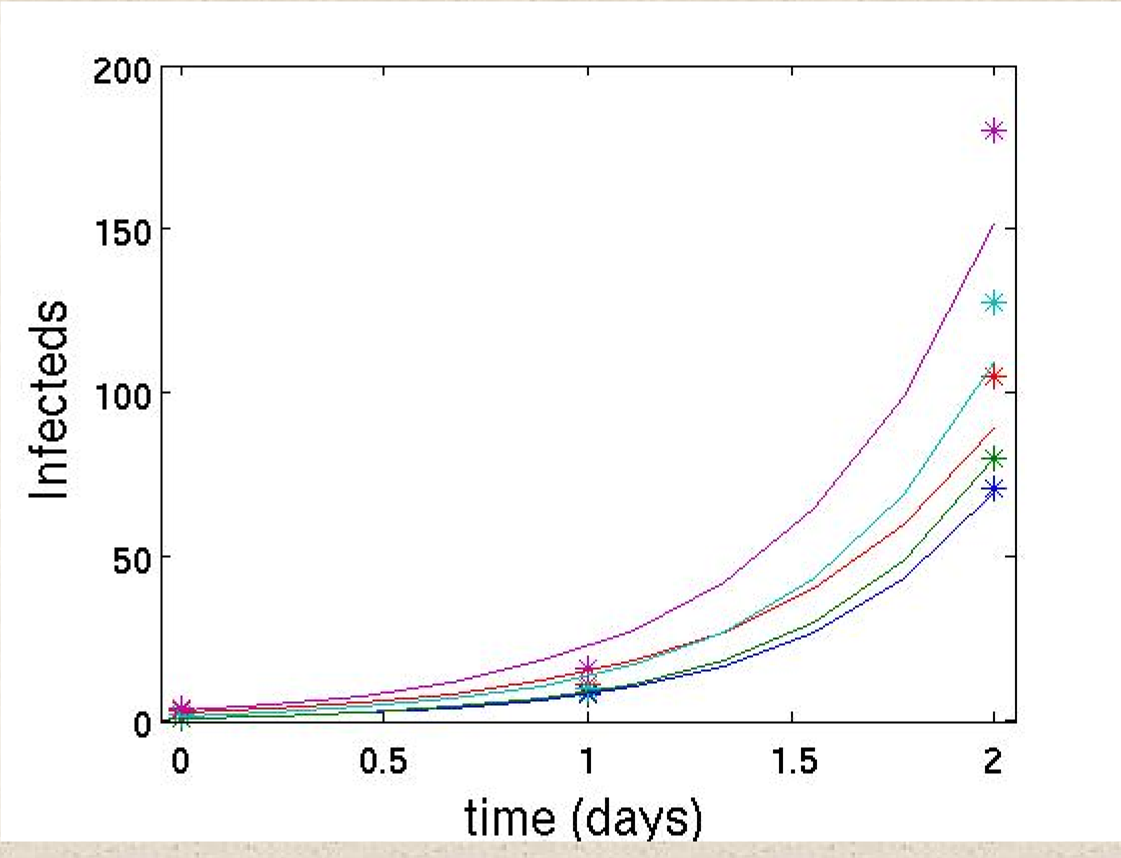
Activities

- Find slope and y-intercepts – lines of best fit



Activities

- Now we know growth rates r0



Activities

- How do we calculate this number?

Day	Toronto	Montreal	Vancouver	Halifax	Winnipeg
0	1	1	3	2	4
1	8	9	11	10	16
2	71	80	105	128	180
Slope (r0)	2.131	2.191	1.778	2.079	1.903
y-int	-0.017	0.002	0.939	0.536	1.214
R0					

Activities

- R0 from r0 $R_0 = 1 + r_o L$

Day	Toronto	Montreal	Vancouver	Halifax	Winnipeg
0	1	1	3	2	4
1	8	9	11	10	16
2	71	80	105	128	180
Slope (r0)	2.131	2.191	1.778	2.079	1.903
y-int	-0.017	0.002	0.939	0.536	1.214
R0 L=2	5.262	5.382	4.556	5.158	4.806

Activities

- R0 = 5.03, SD = 0.34

Day	Toronto	Montreal	Vancouver	Halifax	Winnipeg
0	1	1	3	2	4
1	8	9	11	10	16
2	71	80	105	128	180
Slope (r0)	2.131	2.191	1.778	2.079	1.903
y-int	-0.017	0.002	0.939	0.536	1.214
R0 L=2	5.262	5.382	4.556	5.158	4.806

Table of R0s

Disease	Transmission	R0
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diptheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza	Airborne droplet	2-3

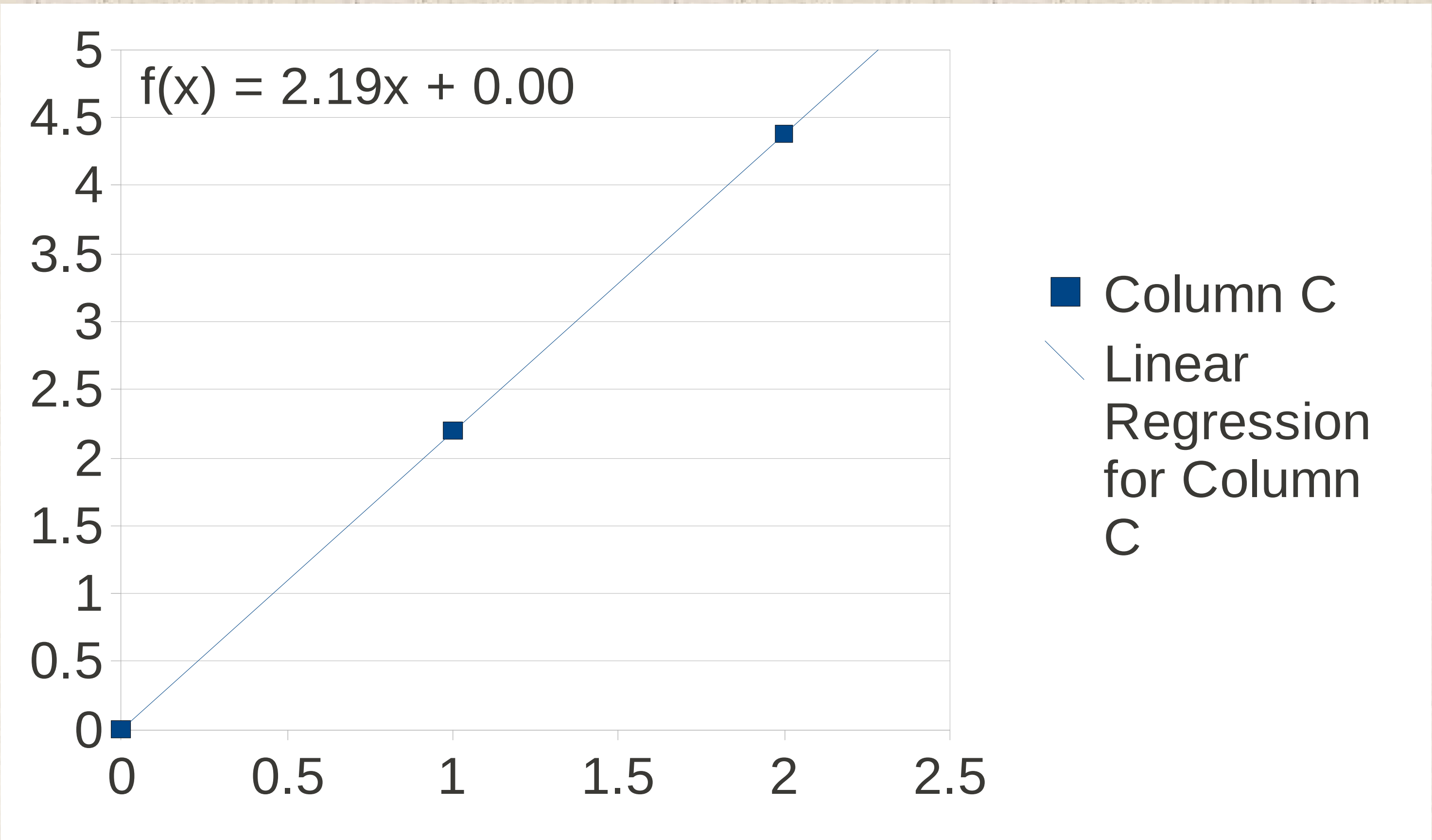
Activities

- We used a line of best fit for our calculations
- This can be done by eye in a classroom
- Can also use Least Squares
  - More advanced (includes derivatives)
  - Computer programming
- Can calculate error
  - Difference between line and data points

- These can be done easily using computer software programs

Activities

- Hamilton – fit TrendLine in spreadsheet



Activities

- What happens if R0 > 1?

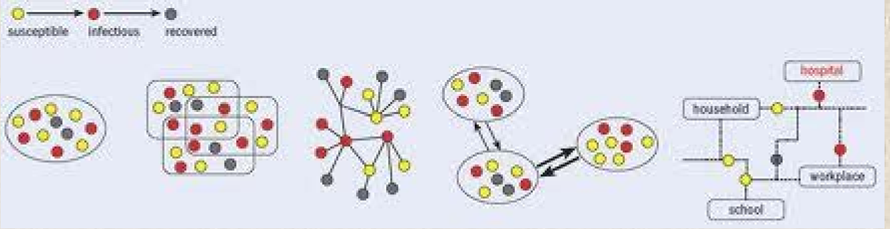
- What happens if R0 < 1?

- What happens if R0 = 1?

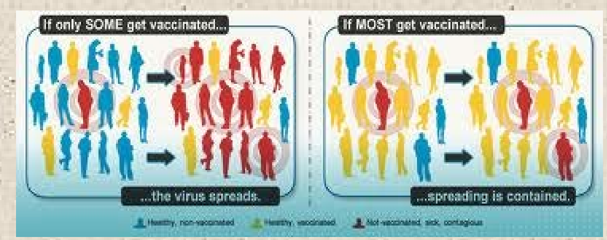
Activities

- What happens if R0 > 1?
- What happens if R0 < 1?
- What happens if R0 = 1?

- We want to reduce R0 < 1?
  - What should we do to do this?
    - Limit social contacts
    - Drug therapy
    - Vaccination
    - Wear face masks



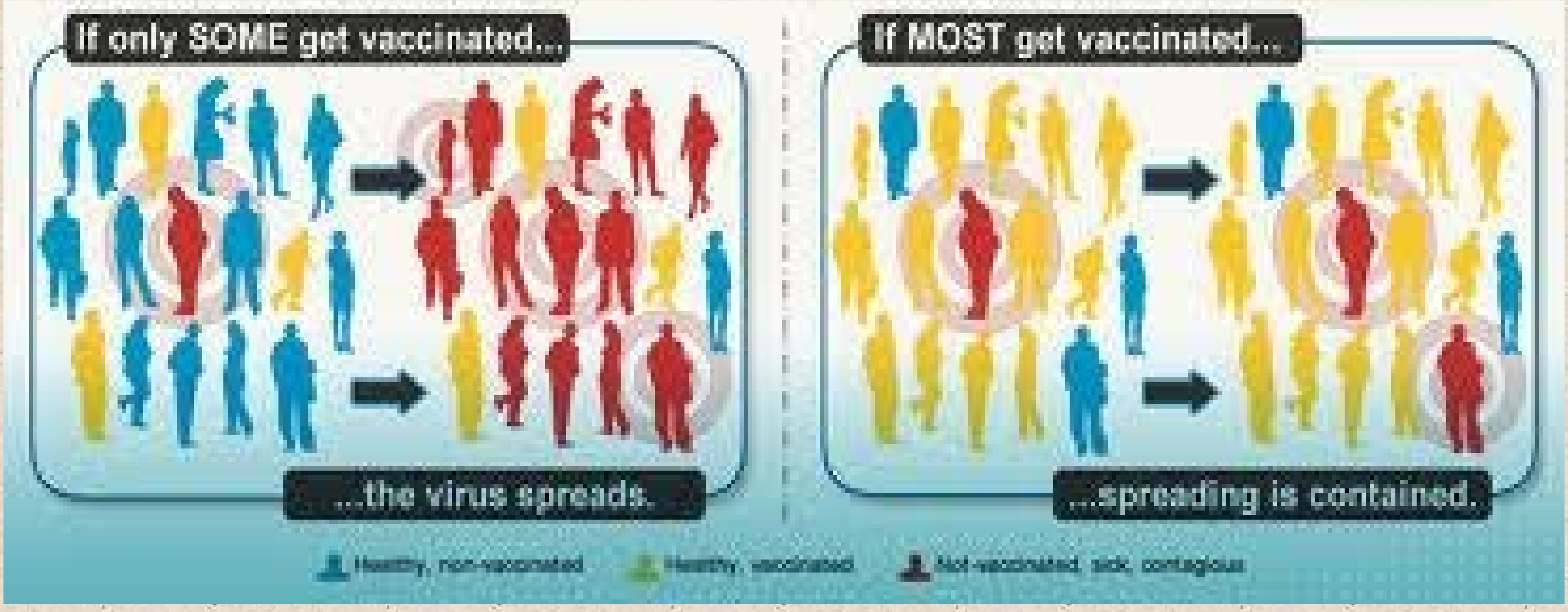
Activities



- Herd Immunity
  - Fraction of the population that has immunity that protects the whole population from infection
- Vaccination
  - Want to induce herd immunity
  - Want to decrease infections in the very susceptible, or those most likely to have adverse outcomes

- What fraction of the population do we need to vaccinate?

Activities



If only SOME get vaccinated...
...the virus spreads.
If MOST get vaccinated...
...spreading is contained.
Healthy, non-vaccinated
Healthy, vaccinated
Not-vaccinated, sick, contagious

Activities

- Calculate critical vaccination threshold

$$p_c = 1 - \frac{1}{R_0}$$

- R0 = 4.03, so pc = 0.75
- So we need to vaccinate 75% of the population

Table of R0s and pcs

Disease	Transmission	R0	Pc (%)
Measles	Airborne	12-18	92-94
Pertussis	Airborne droplet	12-17	92-95
Diptheria	Saliva	6-7	83-86
Smallpox	Social contact	5-7	80-86
Polio	Fecal-oral route	5-7	80-86
Rubella	Airborne droplet	5-7	80-86
Mumps	Airborne droplet	4-7	75-86
HIV/AIDS	Sexual contact	2-5	50-80
SARS	Airborne droplet	2-5	50-80
Influenza	Airborne droplet	1.5-3	33-67

Activities

- For childhood diseases, with life long immunity
- Calculate critical vaccination threshold
- L – expected life expectancy
- A – age of first infection
- Rq – control reproductive ratio
- Aq – age of first infection when control is in place

- Therefore, control policies increase the age of first infection!

$$R_0 = \frac{L}{A}$$

$$R_q = \frac{L}{A_q}$$

$$R_q = (1-q)R_0$$

$$A_q = \frac{A}{1-q}$$

Activities

- Network model
- ..\..\Desktop\network.exe

Activities

- Von Foerster doomsday equation

$$Population = \frac{1.79 \text{x} 10^{11}}{(2026.87 - t)^{0.99}}$$