The Dichotomy for Conservative Constraint Satisfaction Problems Revisited

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Abstract—A central open question in the study of non-uniform constraint satisfaction problems (CSPs) is the dichotomy conjecture of Feder and Vardi stating that the CSP over a fixed constraint language is either NP-complete, or tractable. One of the main achievements in this direction is a result of Bulatov (LICS'03) confirming the dichotomy conjecture for conservative CSPs, that is, CSPs over constraint languages containing all unary relations. Unfortunately, the proof is very long and complicated, and therefore hard to understand even for a specialist. This paper provides a short and transparent proof.

Introduction

The constraint satisfaction problem (CSP) provides a common framework for many theoretical problems in computer science as well as for many real-life applications. An instance of the CSP consists of a number of variables and constraints imposed on them and the objective is to determine whether variables can be evaluated in such a way that all the constraints are met. The CSP can also be expressed as the problem of deciding whether a given conjunctive formula is satisfiable, or as the problem of deciding whether there exists a homomorphism between two relational structures.

The general CSP is NP-complete, however certain natural restrictions on the form of the constraints can ensure tractability. This paper deals with so called *non-uniform* CSP — the same decision problem as the ordinary CSP, but the set of allowed constraint relations is fixed. A central open problem in this area is the dichotomy conjecture of Feder and Vardi [1] stating that, for every finite, fixed set of constraint relations (a fixed *constraint language*), the CSP defined by it is NP-complete or solvable in polynomial time, i.e. the class of CSPs exhibits a dichotomy.

Most of recent progress toward the dichotomy conjecture has been made using the algebraic approach to the CSP [2], [3], [4]. The main achievements include the algorithm for CSPs with "Maltsev constraints" [5] (which was substantially simplified in [6] and generalized in [7], [8]), the characterization of CSPs solvable by local consistency methods [9], [10], the dichotomy theorem for CSPs over a three element domain [11] (which generalizes the Boolean CSP dichotomy theorem [12]) and the dichotomy theorem for conservative CSPs [13].

The last result proves the dichotomy conjecture of Feder and Vardi for the CSP over any template which contains all unary relations. In other words, this Bulatov's theorem proves the dichotomy for the CSPs, in which we can restrict the value of each variable to an arbitrary subset of the domain (that is why the conservative CSPs are sometimes called list CSPs, or, in homomorphism setting, list homomorphism problems). This result is of major importance in the area, but, unfortunately, the proof is very involved (the full paper has 80 pages and it has not yet been published), which makes the study of possible generalizations and further research harder.

This paper provides a new, shorter and more natural proof. It relies on techniques developed and successfully applied in [14], [15], [16], [9], [17], [18].

Related work

The complexity of list homomorphism problems has been studied by combinatorial methods, e.g., in [19], [20]. A structural distinction between tractable and NP-complete list homomorphism problem for digraphs was found in [21]. A finer complexity classification for the list homomorphism problem for graphs was given in [22]. The conservative case is also studied for different variants of the CSP, see, e.g., [23], [24].

Organization of the paper

In Section I we define the CSP and its non-uniform version. In Section II we introduce the necessary notions concerning algebras and the algebraic approach to the CSP. In Section III we collect all the necessary ingredients. One of them is a reduction to minimal absorbing subuniverses, details are provided in Section V. Also the core algebraic result is just stated in this section and its proof covers Section VI. In Section IV we formulate the algorithm for tractable conservative CSPs and prove its correctness.

I. CSP

An *n-ary relation* on a set A is a subset of the n-th cartesian power A^n of the set A.

Definition I.1. *An* instance of the constraint satisfaction problem *(CSP) is a triple $P = (V, A, C)$ with*

- *V a nonempty, finite set of* variables,
- *A a nonempty, finite* domain,
- *C a finite set of* constraints, *where each constraint is a pair $C = (\mathbf{x}, R)$ with*

- **x** *a tuple of distinct variables of length n, called the* scope *of C, and*
- *R an n-ary relation on A, called the* constraint relation *of C.*

The question is whether there exists a solution *to P, that is, a function $f : V \to A$ such that, for each constraint $C = (\mathbf{x}, R) \in \mathcal{C}$, the tuple $f(\mathbf{x})$ belongs to R.*

For purely technical reasons we have made a nonstandard assumption that the scope of a constraint contains distinct variables. This clearly does not change the complexity modulo polynomial-time reductions.

In the non-uniform CSP we fix a domain and a set of allowed constraints:

Definition I.2. *A* constraint language *Γ is a set of relations on a finite set A. The* constraint satisfaction problem over *Γ, denoted* $\mathrm{CSP}(\Gamma)$*, is the subclass of the CSP defined by the property that any constraint relation in any instance must belong to Γ.*

The following dichotomy conjecture was originally formulated in [1] only for finite constraint languages. The known results suggest that even the following stronger version might be true.

Conjecture I.3. *For every constraint language Γ, $\mathrm{CSP}(\Gamma)$ is either tractable, or NP-complete.*

Our main theorem, first proved by Bulatov [13], confirms the dichotomy conjecture for conservative CSPs:

Definition I.4. *A constraint language Γ on A is called* conservative*, if Γ contains all unary relations on A (i.e., all subsets of A).*

Theorem I.5. *For every conservative constraint language Γ, $\mathrm{CSP}(\Gamma)$ is either tractable, or NP-complete.*

II. ALGEBRA AND CSP

A. Algebraic preliminaries

An *n*-ary operation on a set A is a mapping $f : A^n \to A$. An operation f is called *cyclic*, if $n \geq 2$ and $f(a_1, a_2, \ldots, a_n) = f(a_2, a_3, \ldots, a_n, a_1)$ for any $a_1, a_2, \ldots, a_n \in A$. A ternary operation m is called *Maltsev*, if $f(a, a, b) = f(b, a, a) = b$ for any $a, b \in A$.

A *signature* is a finite set of symbols with natural numbers (the *arities*) assigned to them. An *algebra* of a signature Σ is a pair $\mathbf{A} = (A, (t^{\mathbf{A}})_{t \in \Sigma})$, where A is a set, called the *universe* of $\mathbf{A}$, and $t^{\mathbf{A}}$ is an operation on A of arity $\mathrm{ar}(t)$. We use a boldface letter to denote an algebra and the same letter in the plain type to denote its universe. We omit the superscripts of operations as the algebra is always clear from the context.

A *term operation* of $\mathbf{A}$ is an operation which can be obtained from operations in $\mathbf{A}$ using composition and the projection operations. The set of all term operations of $\mathbf{A}$ is denoted by $\mathrm{Clo}(\mathbf{A})$.

There are three fundamental operations on algebras of a fixed signature Σ: forming subalgebras, factoralgebras and products.

A subset B of the universe of an algebra $\mathbf{A}$ is called a *subuniverse*, if it is closed under all operations (equivalently term operations) of $\mathbf{A}$. Given a subuniverse B of $\mathbf{A}$ we can form the algebra $\mathbf{B}$ by restricting all the operations of $\mathbf{A}$ to the set B. In this situation we say that $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ and we write $B \leq \mathbf{A}$ or $\mathbf{B} \leq \mathbf{A}$. We call the subuniverse B (or the subalgebra $\mathbf{B}$) *proper* if $\emptyset \neq B \neq A$.

We define the *product* of algebras $\mathbf{A}_1, \ldots, \mathbf{A}_n$ to be the algebra with the universe equal to $A_1 \times \cdots \times A_n$ and with operations computed coordinatewise. The product of n copies of an algebra $\mathbf{A}$ is denoted by $\mathbf{A}^n$.

An equivalence relation $\sim$ on the universe of an algebra $\mathbf{A}$ is a *congruence*, if it is a subalgebra of $\mathbf{A}^2$. The corresponding *factor algebra* $\mathbf{A}/\sim$ has, as its universe, the set of $\sim$-blocks and operations are defined using (arbitrary chosen) representatives. Every algebra $\mathbf{A}$ has two *trivial congruences*: the diagonal congruence $\sim = \{(a, a) : a \in A\}$ and the full congruence $\sim = A \times A$. A congruence is *proper*, if it is not equal to the full congruence. A congruence is *maximal*, if the only coarser congruence of $\mathbf{A}$ is the full congruence.

For a finite algebra $\mathbf{A}$ the class of all factor algebras of subalgebras of finite powers of $\mathbf{A}$ will be denoted by $V_{\mathrm{fin}}(\mathbf{A})$.

An operation $f : A^n \to A$ is *idempotent*, if $f(a, a, \ldots, a) = a$ for any $a \in A$. An operation $f : A^n \to A$ is *conservative*, if $f(a_1, a_2, \ldots, a_n) \in \{a_1, a_2, \ldots, a_n\}$ for any $a_1, a_2, \ldots, a_n \in A$. An algebra is *idempotent* (resp. *conservative*), if all operations of $\mathbf{A}$ are idempotent (resp. conservative). In other words, an algebra is idempotent (resp. conservative), if all one-element subsets of A (resp. all subsets of A) are subuniverses of $\mathbf{A}$.

B. Algebraic approach

An operation $f : A^n \to A$ is *compatible* with a relation $R \subseteq A^m$ if the tuple

$$(f(a_1^1, a_1^2, \ldots, a_1^n), f(a_2^1, a_2^2, \ldots, a_2^n), \ldots, f(a_m^1, a_m^2, \ldots, a_m^n))$$

belongs to R whenever $(a_1^i, a_2^i, \ldots, a_m^i) \in R$ for all $i \leq n$.

An operation compatible with all relations in a constraint language Γ is a *polymorphism* of Γ. The set A together with all polymorphisms of Γ is the *algebra of polymorphisms of Γ*, it is denoted $\mathrm{Pol}\,\Gamma$, or often just $\mathbf{A}$ (we formally define the signature of $\mathbf{A}$ to be identical with the set of its operations). Note that every relation in Γ is a subalgebra of a power of $\mathbf{A}$. The set of all subalgebras of powers of $\mathbf{A}$ is denoted by $\mathrm{Inv}\,\mathbf{A}$.

In the following discussion we assume, for simplicity, that Γ contains all singleton unary relations (it is known that CSP can be reduced to CSP over such a constraint language). Observe that in such a case the algebra $\mathbf{A}$ is idempotent. Moreover, if Γ is conservative, then $\mathbf{A}$ is conservative as well.

Already the first results on the algebraic approach to CSP [2], [3], [4] show that $\mathbf{A}$ fully determines the computational complexity of $\mathrm{CSP}(\Gamma)$, at least for finite constraint languages. Moreover, a borderline between tractable and NP-complete CSPs was conjectured in terms of the algebra of polymorphisms: if there exists a two-element factor algebra

of a subalgebra of $\mathbf{A}$ whose every operation is a projection, then $\mathrm{CSP}(\Gamma)$ is NP-complete, otherwise $\mathrm{CSP}(\Gamma)$ is tractable. The hardness part of this algebraic dichotomy conjecture is known [3], [4]:

Theorem II.1. *Let Γ be a constraint language containing all singleton unary relations, and let $\mathbf{A} = \mathrm{Pol}\,\Gamma$. If $\mathbf{A}$ has a subalgebra with a two-element factor algebra whose every operation is a projection, then $\mathrm{CSP}(\Gamma)$ is NP-complete.*

The algebras, which satisfy this necessary (and conjecturally sufficient) condition for tractability, are called *Taylor* algebras, that is, $\mathbf{A}$ is Taylor if no two-element factor algebra of a subalgebra $\mathbf{A}$ has projection operations only. We will use the following characterization of Taylor algebras from [17], [18] although the characterization in terms of weak near-unanimity operations [25] would suffice for our purposes.

Theorem II.2. *Let $\mathbf{A}$ be a finite idempotent algebra and let $p > |A|$ be a prime number. The following are equivalent.*

- *$\mathbf{A}$ is a Taylor algebra.*
- *$\mathbf{A}$ has a cyclic term operation of arity p.*

In view of Theorem II.1, the dichotomy for conservative CSPs will follow when we prove:

Theorem II.3. *Let $\mathbf{A}$ be a finite conservative Taylor algebra. Then $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$ is tractable.*

A polynomial time algorithm for solving $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$, where $\mathbf{A}$ is a finite conservative Taylor algebra, is presented in Section IV.

III. Ingredients

The building blocks of our algorithm are the (k, l)-minimality algorithm (Subsection III-B), a reduction to minimal absorbing subuniverses (Subsection III-D) and the algorithm for Maltsev instances (Subsection III-F). Subsection III-A and Subsection III-C cover necessary notation. The main new algebraic tool for proving correctness is stated in Subsection III-E.

A. Projections and restrictions

Tuples are denoted by boldface letters and their elements are indexed from 1, for instance $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. For an n-tuple $\mathbf{a}$ and a tuple $\mathbf{k} = (k_1, \ldots, k_m)$ of elements of $\{1, 2, \ldots, n\}$ we define the *projection of $\mathbf{a}$ to $\mathbf{k}$* by

$$\mathbf{a}|_{\mathbf{k}} = (a_{k_1}, a_{k_2}, \ldots, a_{k_m}).$$

For a subset $K \subseteq \{1, 2, \ldots, n\}$ we put $\mathbf{a}|_K = \mathbf{a}|_{\mathbf{k}}$, where $\mathbf{k}$ is the list of elements of K in the ascending order.

The *projection of a set $R \subseteq A_1 \times \ldots A_n$ to $\mathbf{k}$ (resp. K)* is defined by

$$R|_{\mathbf{k}} = \{\mathbf{a}|_{\mathbf{k}} : \mathbf{a} \in R\} \quad (\text{resp. } R|_K = \{\mathbf{a}|_K : \mathbf{a} \in R\}).$$

The set R is *subdirect* in $A_1 \times \cdots \times A_n$ (denoted by $R \subseteq_S A_1 \times \cdots \times A_n$), if $R|_{\{i\}} = A_i$ for all $i = 1, \ldots, n$. If, moreover, $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are algebras of the same signature and R is a subalgebra of their product, we write $R \leq_S \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$.

Let $P = (V, A, \mathcal{C})$ be an instance of the CSP. The *projection of a constraint $C = ((x_1, \ldots, x_n), R) \in \mathcal{C}$ to a tuple of variables $(x_{k_1}, \ldots, x_{k_m})$* is the relation

$$C|_{(x_{k_1}, \ldots, x_{k_m})} = \{(a_{k_1}, \ldots, a_{k_m}) : \mathbf{a} \in R\}.$$

Finally, we introduce two types of restrictions of a CSP instance. In the variable restriction we delete some of the variables and replace the constraints with appropriate projections, in the domain restriction we restrict the value of some of the variables to specified subsets of the domain.

The *variable restriction of P to a subset $W \subseteq V$* is the instance $P|_W = (W, A, \mathcal{C}')$, where $\mathcal{C}'$ is obtained from $\mathcal{C}$ by replacing each constraint $C = (\mathbf{x}, R) \in \mathcal{C}$ with $(\mathbf{x} \cap W, C|_{\mathbf{x} \cap W})$, where $\mathbf{x} \cap W$ is the subtuple of $\mathbf{x}$ formed by the variables belonging to W.

The *domain restriction of P to a system $\mathcal{E} = \{E_x : x \in W\}$* of subsets of A indexed by $W \subseteq V$ is the instance $P|_{\mathcal{E}} = (V, A, \mathcal{C}')$, where $\mathcal{C}'$ is obtained from $\mathcal{C}$ by replacing each constraint $C = ((x_1, \ldots, x_n), R) \in \mathcal{C}$ with $C' = ((x_1, \ldots, x_n), \{\mathbf{a} \in R : \forall i\ x_i \in W \Rightarrow a_i \in E_{x_i}\})$.

B. (k, l)-minimality

The first step in our algorithm will be to ensure a certain kind of local consistency. The following notion is the most convenient for our purposes.

Definition III.1. *Let $l \geq k > 0$ be natural numbers. An instance $P = (V, A, \mathcal{C})$ of the CSP is (k, l)-minimal, if:*

- *Every at most l-element tuple of distinct variables is the scope of some constraint in $\mathcal{C}$,*
- *For every tuple $\mathbf{x}$ of at most k variables and every pair of constraints C_1 and C_2 from $\mathcal{C}$ whose scopes contain all variables from $\mathbf{x}$, the projections of the constraints C_1 and C_2 to $\mathbf{x}$ are the same.*

A (k, k)-minimal instance is also called k-minimal.

For fixed k, l there is an obvious polynomial time algorithm for transforming an instance of the CSP to a (k, l)-minimal instance with the same set of solutions: First we add dummy constraints to ensure that the first condition is satisfied and then we gradually remove those tuples from the constraint relations which falsify the second condition (see [13] for a more detailed discussion). It is a folklore fact (which is in the literature often used without mentioning) that an instance of $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$ is in this way transformed to an instance of $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$, that is, the constraint relations of the new instance are still members of $\mathrm{Inv}\,\mathbf{A}$. See the discussion after Definition III.3 in [9], where an argument is given for a similar consistency notion.

If an instance P is (at least) 1-minimal, then, for each variable $x \in V$, there is a unique constraint whose scope is (x). We denote its constraint relation by S_x^P, i.e. $((x), S_x^P) \in \mathcal{C}$. Then the projection of any constraint whose scope contains x to (x) is equal to S_x^P. If, moreover, P is an instance of $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$, the set S_x^P is a subuniverse of $\mathbf{A}$ and we denote the corresponding subalgebra of $\mathbf{A}$ by $\mathbf{S}_x^P$.

If an instance is 2-minimal, then we have a unique constraint $((x, x'), S_{(x, x')}^P)$ for each pair of distinct variables $x, x' \in V$,

and $C|_{(x,x')} = S^P_{(x,x')}$ for any constraint C whose scope contains x and x'. We formally define $S^P_{(x,x)} = \{(a,a) : a \in S^P_x\}$.

$(2,3)$-minimal instances have the following useful property.

Lemma III.2. *Let P be a $(2,3)$-minimal instance and let $x, x', x'' \in V$. Then for any $(a, a') \in S^P_{(x,x')}$ there exists $a'' \in A$ such that $(a, a'') \in S^P_{(x,x'')}$ and $(a', a'') \in S^P_{(x',x'')}$.*

Proof: Let $C \in \mathcal{C}$ be a constraint with the scope (x, x', x''), say $C = ((x, x', x''), R)$. The projection of C to (x, x') is equal to $S^P_{(x,x')}$, therefore there exists $a'' \in A$ such that $(a, a', a'') \in R$. This element satisfies the conclusion of the lemma. ∎

C. Walking with subsets

Let $R \subseteq A_1 \times A_2$ and let $B \subseteq A_1$. We define

$$R^+[B] = \{c \in A_2 : \exists\, b \in B \;\; (b,c) \in R\}.$$

A *qoset* is a set A together with a quasi-ordering on A, i.e. a reflexive, transitive (binary) relation $\leq$ on A. The blocks of the induced equivalence $\sim$, given by $a \sim b$ iff $a \leq b \leq a$, are called *components* of the qoset. A component C is *maximal*, if $a \sim c$ for any $a \in A, c \in C$ such that $c \leq a$. A *subqoset* is a subset of A together with $\leq$ restricted to A.

For a 1-minimal instance $P = (V, A, \mathcal{C})$ we introduce a qoset $\mathrm{Qoset}(P)$ as follows. The elements are all the pairs (x, B), where $x \in V$ and B is a subset of S^P_x. We put $(x, B) \leq (x', B')$, if there exists a constraint $C \in \mathcal{C}$ whose scope contains $\{x, x'\}$ such that $C|_{(x,x')}{}^+[B] = B'$. The ordering of the qoset $\mathrm{Qoset}(P)$ is the transitive closure of $\leq$.

If the instance P is $(2,3)$-minimal, the components of $\mathrm{Qoset}(P)$ are nicely behaved:

Proposition III.3. *Let $P = (V, A, \mathcal{C})$ be a $(2,3)$-minimal instance of the CSP and let (x, B) and (x', B') be two elements of the same component of the qoset $\mathrm{Qoset}(P)$. Then $S^P_{(x,x')}{}^+[B] = B'$. In particular, if $x = x'$, then $B = B'$.*

Proof: Let $(x, B) = (x_1, B_1), (x_2, B_2), \ldots, (x_k, B_k) = (x', B')$ be a sequence of elements of $\mathrm{Qoset}(P)$ such that $S^P_{(x_i,x_{i+1})}{}^+[B_i] = B_{i+1}$ for all $i = 1, \ldots, k-1$. From Lemma III.2 it follows that

$$S^P_{(x_1,x_{i+1})}{}^+[B_1] \subseteq S^P_{(x_i,x_{i+1})}{}^+\left[S^P_{(x_1,x_i)}{}^+[B_1]\right],$$

therefore $S^P_{(x,x')}{}^+[B] \subseteq B'$. Similarly, $S^P_{(x',x)}{}^+[B'] \subseteq B$. For each $b' \in B'$ ($\subseteq S^P_{x'}$) there exists $b \in A$ such that $(b, b') \in S^P_{(x,x')}{}^+[B]$. This element b has to belong to B (since $S^P_{(x',x)}{}^+[B'] \subseteq B$), which proves the inclusion $S^P_{(x,x')}{}^+[B] \supseteq B'$. ∎

Let P be a $(2,3)$-minimal instance and $\mathcal{E} = \{E_x : x \in V\}$ be a system of subsets of A such that $E_x \subseteq S^P_x$ for each $x \in V$. A $(P, \mathcal{E})$-*strand* is a maximal subset W of V such that all the pairs (x, E_x), $x \in W$ belong to the same component of the qoset $\mathrm{Qoset}(P)$. The name of this concept is justified by the previous proposition: For example, any solution $f : V \to A$ to P with $f(x) \in E_x$ for *some* $x \in W$ satisfies $f(x) \in E_x$ for *all* $x \in W$.

D. Absorbing subuniverses

Definition III.4. *Let $\mathbf{A}$ be a finite idempotent algebra and $t \in \mathrm{Clo}(\mathbf{A})$. We say that a subalgebra $\mathbf{B}$ of $\mathbf{A}$ is an* absorbing *subalgebra of $\mathbf{A}$ with respect to t if, for any $k \leq \mathrm{ar}(t)$, any choice of $a_i \in A$ such that $a_i \in B$ for all $i \neq k$ we have $t(a_1, \ldots, a_{\mathrm{ar}(t)}) \in B$.*

We say that $\mathbf{B}$ is an absorbing *subalgebra of $\mathbf{A}$, or that $\mathbf{B}$* absorbs $\mathbf{A}$ *(and write $\mathbf{B} \lhd \mathbf{A}$), if there exists $t \in \mathrm{Clo}(\mathbf{A})$ such that $\mathbf{B}$ is an absorbing subalgebra of $\mathbf{A}$ with respect to t.*

We say that $\mathbf{A}$ is an absorption free algebra, *if it has no proper absorbing subalgebras.*

We also speak about absorbing subuniverses i.e. universes of absorbing subalgebras.

Definition III.5. *If $\mathbf{B} \lhd \mathbf{A}$ and no proper subalgebra of $\mathbf{B}$ absorbs $\mathbf{A}$, we call $\mathbf{B}$ a* minimal absorbing subalgebra *of $\mathbf{A}$ (and write $\mathbf{B} \lhd\lhd \mathbf{A}$).*

Alternatively, we can say that $\mathbf{B}$ is a minimal absorbing subalgebra of $\mathbf{A}$, if $\mathbf{B} \lhd \mathbf{A}$ and $\mathbf{B}$ is an absorption free algebra. Equivalence of these definitions follows from transitivity of $\lhd$ (see Proposition III.2 in [17]).

Algorithm 1 finds, for a given $(2,3)$-minimal instance P of the CSP, a domain restriction Q of P which is 1-minimal and satisfies $\mathbf{S}^Q_x \lhd\lhd \mathbf{S}^P_x$ for any $x \in V$.

The algorithms uses a subqoset $\mathrm{AbsQoset}(P)$ of $\mathrm{Qoset}(P)$ formed by the elements (x, B) such that B is a proper absorbing subuniverse of $\mathbf{S}^P_x$.

Fig. 1. Algorithm 1: Minimal absorbing subuniverses

Input: $(2,3)$-minimal instance $P = (V, A, \mathcal{C})$ of $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$
Output: $\mathcal{E} = \{E_x : x \in V\}$ such that $E_x \lhd\lhd \mathbf{S}^P_x$, $x \in V$, and $P|_{\mathcal{E}}$ is 1-minimal
1: **while** some S^P_x has a proper absorbing subuniverse **do**
2:     find a maximal component $\mathcal{F} = \{(x, E_x) : x \in W\}$ of the qoset $\mathrm{AbsQoset}(P)$
3:     $P := P|_{\mathcal{F}}$
4: **return** $\{S^P_x : x \in V\}$

Theorem III.6. *Algorithm 1 is correct and, for a fixed idempotent algebra $\mathbf{A}$, works in polynomial time.*

Proof: The qoset $\mathrm{AbsQoset}(P)$ contains at most $2^{|A|}|V|$ elements, therefore its maximal component can be found in a polynomial time. In each while loop at least one of the sets S^P_x becomes smaller, thus the while loop is repeated at most $|V||A|$ times, and the algorithm is therefore polynomial.

The correctness follows from a slightly generalized results from [9] (the generalized version will be in [10]): In the beginning of the while loop, P is so called *Prague strategy*. An analogue of Proposition III.3 remains valid for Prague strategies (Lemma IV.10 in [9], Lemma V.5 part (iii) in Section V), in particular, for each variable $x \in V$, there is at most

one element (x, E_x) in the maximal component, therefore the definition of $\mathcal{F}$ in step 2 makes sense. Finally, the restriction of P to $\mathcal{F}$ is again a Prague strategy (Theorem IV.15 in [9], Lemma V.6 in Section V). The details are in Section V. ∎

E. Rectangularity

The core result for proving correctness of our algorithm for conservative CSPs is the "Rectangularity Theorem". We state the theorem here, its proof spans Section VI.

We need one more notion. Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be sets such that $B_i \subseteq A_i$ and let $R \subseteq_S A_1 \times \cdots \times A_n$. We define a quasi-ordering $\preceq$ on the set $\{1, 2, \ldots, n\}$ by

$$i \preceq j \quad \text{if} \quad R|_{(i,j)}{}^+[B_i] \subseteq B_j.$$

Components of this qoset are called (R, B)-*strands*.

Theorem III.7. *Let* $\mathbf{A}$ *be a finite Taylor algebra, let* $\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathrm{V_{fin}}(\mathbf{A})$ *be conservative algebras such that* $\mathbf{B}_i \vartriangleleft\!\vartriangleleft \mathbf{A}_i$ *for all* i, *let* $R \leq_S \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ *and assume that* $R \cap (B_1 \times \ldots, B_n) \neq \emptyset$. *Then a tuple* $\mathbf{a} \in B_1 \times \cdots \times B_n$ *belongs to* R *whenever* $\mathbf{a}|_K \in R|_K$ *for each* (R, B)-*strand* K.

F. Maltsev instances

Our final ingredient is the polynomial time algorithm by Bulatov and Dalmau [6] for the CSPs over constraint languages with a Maltsev polymorphism. Their algorithm can be used without any change in the following setting:

Theorem III.8. *[6] Let* $\mathbf{A}$ *be a finite algebra with a ternary term operation* m. *Then there is a polynomial time algorithm which correctly decides every* 1-*minimal instance* P *of* $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$ *such that, for every variable* x, m *is a Maltsev operation of* $\mathbf{S}_x^P$.

IV. ALGORITHM

The algorithm for conservative CSPs is in Figure 2. It uses a subqoset $\mathrm{NafaQoset}(P)$ of the qoset $\mathrm{Qoset}(P)$ formed by the elements (x, B) such that $B \subseteq S_x^P$ and $\mathbf{B}$ has a proper absorbing subalgebra (where $\mathbf{B}$ stands for the subalgebra of $\mathbf{A}$ with universe B).

Theorem IV.1. *If* $\mathbf{A}$ *is a conservative finite algebra, then Algorithm 2 is correct and works in polynomial time.*

Proof: By induction on k we show that the algorithm works in polynomial time for all instances such that $|S_x^P| \leq k$. The base case of the induction is obvious: if every S_x^P is at most one-element, then the algorithm proceeds directly to Step 15 (where the algorithm answers YES iff every S_x^P is one-element).

Step 1 can be done in polynomial time as discussed in Subsection III-B. In Step 3 the qoset has size at most $2^{|A|}|V|$, therefore its maximal component can be found in polynomial time. Step 5 is polynomial according to Theorem III.6. There are at most $|V|$ repetitions of the **for** cycle in Step 6. Step 7 is polynomial by the induction hypothesis, since every E_x is a minimal absorbing subuniverse of $\mathbf{D}_x$ and $\mathbf{D}_x$ has, as a

Input: Instance $P = (V, A, \mathcal{C})$ of $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$
Output: "YES" if P has a solution, "NO" otherwise
1: Transform P to a $(2, 3)$-minimal instance with the same solution set
2: **if** some subalgebra of S_x^P has a proper absorbing subalgebra **then**
3:     Find a maximal component $\mathcal{D} = \{D_x : x \in W\}$ of $\mathrm{NafaQoset}(P)$
4:     $Q := (P|_W)|_{\mathcal{D}}$
5:     $\mathcal{E} :=$ the result of Algorithm 1 for the instance Q
6:     **for** each $(Q, \mathcal{E})$-strand U **do**
7:         Use this algorithm for the instance $(Q|_U)|_{\{E_x : x \in U\}}$
8:         **if** no solution exists **then**
9:             $\mathcal{F} := \{S_x^P - E_x : x \in U\}$
10:            $P := P|_{\mathcal{F}}$
11:            **goto** step 1
12:     $\mathcal{F} := \{S_x^P - (D_x - E_x) : x \in W\}$
13:     $P := P|_{\mathcal{F}}$
14:     **goto** step 1
15: Use the algorithm for Maltsev instances (Theorem III.8)

member of $\mathrm{NafaQoset}(P)$, a proper absorbing subuniverse. Before we return to Step 1 (either in Step 11 or in Step 14) at least one of the sets S_x^P becomes strictly smaller. It follows that there are at most $|A||V|$ returns to the first step. Finally, the last step is polynomial by Theorem III.8.

Now we show the correctness of the algorithm.

First, we observe that no solution is lost in Step 10. As the pairs (x, E_x), $x \in U$ are in one component of the qoset $\mathrm{Qoset}(Q)$ and the instance Q is the restriction of P to elements of the same component of $\mathrm{Qoset}(P)$, it follows that all the pairs (x, E_x), $x \in U$ lie in the same component of $\mathrm{Qoset}(P)$. Therefore, if $f : V \to A$ is a solution to P such that $f(x) \in E_x$ for some $x \in U$, then $f(x) \in E_x$ for all $x \in U$ (see Proposition III.3 and the discussion bellow). But the restriction of such a function f to the set U would be a solution to the instance $(Q|_U)|_{\{E_x : x \in U\}}$, thus we would not get to this step. We have shown that in Step 10 every solution to P misses all the sets E_x, $x \in U$, and hence we do not lose any solution when we restrict P to $\mathcal{F}$.

Next, we show that if P has a solution before Step 13, then the restricted instance $P|_{\mathcal{F}}$ has a solution as well. If $f : V \to A$ is a solution to P such that $f(x) \notin D_x$ for some $x \in W$, then $f(x) \in D_x$ for all $x \in W$, because (x, D_x), $x \in W$ are in the same component of $\mathrm{Qoset}(P)$ and we can use Proposition III.3 as above. In this case f is a solution to the restricted instance. Now we assume that f is a solution to P such that $f(x) \in D_x$ for all $x \in W$. For each $(Q, \mathcal{E})$-strand U let $g_U : U \to A$ be a solution to the instance $(Q|_U)|_{\{E_x : x \in U\}}$. Let $h : V \to A$ be the mapping satisfying $h|_{V-W} = f|_{V-W}$ and $h|_U = f|_U$ for each $(Q, \mathcal{E})$-strand U. We claim that this mapping is a solution to the instance $P|_{\mathcal{F}}$.

Clearly, $h(x) \in S_x^P - (D_x - E_x)$ for every $x \in W$.

We define D_x for $x \in V - W$ by $D_x = S^P_{(y,x)}{}^+[D_y]$, where y is an arbitrarily chosen element of W. The definition of D_x does not depend on the choice of y: Let $y, y' \in W$ and take an arbitrary $a \in S^P_{(y,x)}{}^+[D_y]$. From the choice of a it follows that there is $b \in D_y$ such that $(b,a) \in S^P_{(y,x)}$. Lemma III.2 provides us with an element $b' \in A$ such that $(b', a) \in S^P_{(y',x)}$ and $(b, b') \in S^P_{(y,y')}$. The latter fact together with Proposition III.3 implies $b' \in D_{y'}$, therefore $a \in S^P_{(y',x)}{}^+[D_{y'}]$. We have proved the inclusion $S^P_{(y,x)}{}^+[D_y] \subseteq S^P_{(y',x)}{}^+[D_{y'}]$, the opposite inclusion is proved similarly.

We put $E_x = D_x$ for $x \in V - W$. Let $\mathbf{D}_x$ (resp. $\mathbf{E}_x$) denote the subalgebra of $\mathbf{A}$ with universe D_x (resp. E_x), $x \in V$. For any $x \in V - W$ and $y \in W$, the pair (x, D_x) is greater than or equal to (y, D_y) in the qoset $\mathrm{Qoset}(P)$. Since $\mathcal{D}$ is a maximal component and $x \notin W$, it follows that D_x is outside the qoset $\mathrm{NafaQoset}(P)$ and thus $\mathbf{D}_x$ has no proper absorbing subuniverse. Therefore $E_x \lhd\lhd \mathbf{D}_x$ for all $x \in V$ (for $x \in W$ it follows from the fact that $\mathcal{E}$ is the result of Algorithm 1).

Now we are ready to show that h is a solution to P, i.e. h satisfies all the constraints in $\mathcal{C}$. So, let $C = (\mathbf{x}, R) \in \mathcal{C}$, $\mathbf{x} = (x_1, \ldots, x_n)$ be an arbitrary constraint. For each $i \in V$ let $\mathbf{A}_i = \mathbf{D}_{x_i}$ and $\mathbf{B}_i = \mathbf{E}_{x_i}$, let $\mathbf{a} = (h(x_1), \ldots, h(x_n))$, and let $L = \{l_1, \ldots, l_k\} := \{i : x_i \in W\}$. By the choice of D_xs, the relation R is subdirect in $A_1 \times \cdots \times A_n$. Since $Q|_\mathcal{E}$ is 1-minimal (it is the result of Algorithm 1), the projection of C to $(x_{l_1}, \ldots, x_{l_k})$ has a nonempty intersection with $B_{l_1} \times \cdots \times B_{l_k}$. By the choice of E_x, $x \in V - W$ it follows that the relation R has a nonempty intersection with $B_1 \times \cdots \times B_n$. For any $i \in L$ and $j \in \{1, \ldots, n\} - L$ we have $R|_{(j,i)}{}^+[B_j] = A_i \not\subseteq B_i$, therefore no element of L is in the same (R, B)-strand as an element outside L. Moreover, $i, j \subseteq L$ are in the same (R, B)-strand if and only if x_i, x_j are in the same $(Q, \mathcal{E})$-strand, since $R|_{(i,j)} = S^P_{(x_i,x_j)}$. It follows that $\mathbf{a}|_K \in R|_K$ for each (R, B)-strand $K \subseteq L$, and the same is of course true for each (R, B)-strand $K \subseteq \{1, 2, \ldots, n\} - L$ as $f|_{V-W} = h|_{V-W}$. We have checked all the assumptions of Theorem III.7, which gives us $\mathbf{a} \in R$. In other words, h satisfies the constraint C.

From the fact that $\mathbf{A}$ is conservative it easily follows that after both Step 10 and Step 13 the restricted instance is still an instance of $\mathrm{CSP}(\mathrm{Inv}\,\mathbf{A})$.

Finally, we prove that P satisfies the assumptions of Theorem III.8 when we get to Step 15. Note that at this point we know that no subalgebra of $\mathbf{S}^P_x$ has a proper absorbing subalgebra. Let t be a cyclic term operation of the algebra $\mathbf{A}$ (guaranteed by Theorem II.2). If $t(a, a, \ldots, a, b) = a$ for some $x \in V$, $a, b \in S^P_x$, then $t(a, a, \ldots, a, b) = t(a, a, \ldots, a, b, a) = \cdots = t(b, a, a, \ldots, a)$, and hence $\{a\}$ is an absorbing subuniverse of $\{a, b\}$ with respect to t, a contradiction. Therefore, as $\mathbf{A}$ is conservative, $t(a, a, \ldots, a, b) = b = t(b, a, a, \ldots, a)$ for any $x \in V, a, b \in S^P_x$. Now the term operation $m(x, y, z) = t(x, y, y, \ldots, y, z)$ satisfies the assumptions of Theorem III.8 and the proof is concluded. ∎

V. Prague strategies

This section fills the gaps in the proof of Theorem III.6.

Definition V.1. *Let* $P = (V, A, \mathcal{C})$ *be a* 1-*minimal instance of the CSP. A* pattern *in* P *is a tuple* $(x_1, C_1, x_2, C_2, \ldots, C_{n-1}, x_n)$, *where* $x_1, \ldots, x_n \in V$ *and, for every* $i = 1, \ldots, n - 1$, C_i *is a constraint whose scope contains* $\{x_i, x_{i+1}\}$. *The pattern* w *is* closed *with base* x, *if* $x_1 = x_n = x$. *We define* $[\![w]\!] = \{x_1, \ldots, x_n\}$.

A sequence $a_1, \ldots, a_n \in A$ *is a* realization *of* w *in* P, *if* $(a_i, a_{i+1}) \in C_i|_{(x_i, x_{i+1})}$ *for any* $i \in \{1, \ldots, n-1\}$. *We say that two elements* $a, a' \in A$ *are* connected *via* w *(in* P*), if there exists a realization* $a = a_1, a_2, \ldots, a_{n-1}, a_n = a'$ *of the pattern* w.

For two patterns $w = (x_1, C_1, \ldots, x_n)$, $w' = (x'_1, C'_1, \ldots, x'_m)$ *with* $x_n = x'_1$ *we define their concatenation by* $wv = (x_1, C_1, \ldots, x_n, C'_1, \ldots, x'_m)$. *We write* w^k *for a* k-*fold concatenation of a closed pattern* w *with itself.*

Definition V.2. *A* 1-*minimal instance* $P = (V, A, \mathcal{C})$ *is a* Prague strategy, *if for every* $x \in V$, *every pair of closed patterns* v, w *in* P *with base* x *such that* $[\![v]\!] \subseteq [\![w]\!]$, *and every* $a, a' \in S^P_x$ *connected via the pattern* v *in* P, *there exists a natural number* k *such that* a *is connected to* a' *via the pattern* w^k.

First we show that every $(2, 3)$-minimal instance is a Prague strategy. We need an auxiliary lemma.

Lemma V.3. *Let* $P = (V, A, \mathcal{C})$ *be a* $(2, 3)$-*minimal instance, let* $x, x' \in V$ *and let* $w = (x = x_1, C_1, x_2, \ldots, x_n = x')$ *be a pattern. Then* a *is connected to* a' *via* w *in* P *for any* $a, a' \in A$ *such that* $(a, a') \in S^P_{(x, x')}$.

Proof: Using Lemma III.2 we obtain $a_2 \in A$ such that $(a, a_2) \in S^P_{(x_1, x_2)}$ and $(a_2, a') \in S^P_{(x_2, x_n)}$. The element a_2 is the second (after a) element of a realization of the pattern w. Similarly, there exists an element $a_3 \in A$ such that $(a_2, a_3) \in S^P_{(x_2, x_3)}$ and $(a_3, a') \in S^P_{(x_3, x_n)}$. Repeated applications of this reasoning produce a realization of the pattern w connecting a to a'. ∎

Lemma V.4. *Every* $(2, 3)$-*minimal instance is a Prague strategy.*

Proof: Let $x \in V$, let v, w be closed patterns in P with base x such that $[\![v]\!] \subseteq [\![w]\!]$, and let $a, a' \in S^P_x$ be elements connected via $v = (x_1, \ldots, x_n)$. Let $a = a_1, \ldots a_n = a'$ be a realization of v. Since x_2 appears in w there exists an initial part of w, say w', starting with x and ending with x_2. Since $(a, a_2) \in S^P_{(x, x_2)}$ we use Lemma V.3 to connect a to a_2 via w'. Since x_3 appears in w there exists w'' such that $w'w''$ is an initial part of w^2 and such that w'' ends in x_3. Since $(a_2, a_3) \in S^P_{(x_2, x_3)}$ we use Lemma V.3 again to connect a_2 to a_3 via the pattern w''. Now a_1 and a_3 are connected via the pattern $w'w''$. By continuing this reasoning we obtain the pattern w^k (for some k) connecting a to a'. ∎

Part (iii) of the following lemma generalizes Proposition III.3.

Lemma V.5. *Let* $P = (V, A, \mathcal{C})$ *be a* 1-*minimal instance. The following are equivalent.*

(i) P *is a Prague strategy.*

(ii) *For every $x \in V$, every pair of closed patterns v, w in P with base x such that $\llbracket v \rrbracket \subseteq \llbracket w \rrbracket$, and every $a, a' \in S_x^P$ connected via the pattern v in P, there exists a natural number m such that, for all $k \geq m$, the elemenets a, a' are connected via the pattern w^k;*

(iii) *For every two elements $(x, B), (x', B')$ in the same component of the qoset $\mathrm{Qoset}(P)$ and every constraint $C \in \mathcal{C}$ whose scope contains $\{x, x'\}$, we have $C|_{(x,x')}{}^+[B] = B'$.*

Proof: For (i) $\implies$ (ii) it is clearly enough to prove the claim for $a = a'$. To do so, we obtain (using (i)) a natural number p such that a is connected to a via w^p. Let b be an element of A such that a is connected to b via w and b is connected to a via w^{p-1}. We use the property (i) for a, b and the pattern w^p to find a natural number q such that a is connected to b via w^{pq}. From the facts that a is connected to a via w^p and also via w^{pq+p-1} (as a is connected to b via w^{pq} and b to a via w^{p-1}) we get that a is connected to a via $w^{ip+j(pq+p-1)}$ for arbitrary i, j. Since p and $pq + p - 1$ are coprime, the claim follows.

For (i) $\implies$ (iii) let $(x, B) = (x_1, B_1), (x_2, B_2), \ldots, (x_n, B_n) = (x', B') = (x_{n+1}, B_1'), (x_{n+2}, B_2'), \ldots, (x_m, B_m) = (x, B)$ be a sequence of elements of $\mathrm{Qoset}(P)$ and $C_1, \ldots, C_{m-1} \in \mathcal{C}$ be constraints such that $C|_{(x_i, x_{i+1})}{}^+[B_i] = B_{i+1}$ for every $i = 1, \ldots, m - 1$.

Assume that there exists $a, a' \in A$ such that $(a, a') \in C|_{(x,x')}$ and $a' \in B'$ while $a \notin B$. We can find an element $b \in B$ such that b is connected to a' via the pattern $(x_1, C_1, \ldots, x_n)$. The elements b, a are connected via the pattern $(x_1, C_1, \ldots, C_{n-1}, x_n, C, x_1)$, therefore, by (i), they must be connected via a power of the pattern $(x_1, C_1, \ldots, C_{m-1}, x_m)$, which contradicts the last property from the last paragraph. This contradiction shows that $C|_{(x',x)}{}^+[B'] \subseteq B$. Similarly $C|_{(x,x')}{}^+[B] \subseteq B$ and the proof can be finished as in Proposition III.3.

We do not need the implication (iii) $\implies$ (i) in this paper, therefore we omit the proof. ∎

The following lemma covers the last gap.

Lemma V.6. *Let $P = (V, A, \mathcal{C})$ be an instance of $\mathrm{CSP}(\mathrm{Inv}\, \mathbf{A})$ which is a Prague strategy and let $\mathcal{F} = \{(x, E_x) : x \in W\}$ be a maximal component of the qoset $\mathrm{AbsQoset}(P)$. Then the restriction $Q = P|_{\mathcal{F}}$ is a Prague strategy.*

Proof: It is easy to see that, for any $x, x' \in V$, any $B \lhd \mathbf{S}_x^P$ and any constraint C whose scope contains $\{x, x'\}$, the set $C|_{(x,x')}{}^+[B]$ is an absorbing subuniverse of $\mathbf{S}_{x'}^P$ (with respect to the same term operation of $\mathbf{A}$). Therefore $C|_{(x,x')}{}^+[E_x] = S_{x'}^P$ whenever $x \in W$ and $x' \in V - W$. From this fact and Lemma V.5 part (iii) it follows that Q is 1-minimal.

To prove that Q is a Prague strategy let v and $w = (x = x_1, C_1, x_2, \ldots, C_{n-1}, x_n = x)$ be closed patterns with base x such that $\llbracket v \rrbracket \subseteq \llbracket w \rrbracket$ and let $a, a' \in S_x^Q$ be elements connected via v in Q. Let t be a k-ary term operation providing the absorptions $E_x \lhd \mathbf{S}_x^P$. By Lemma V.5 part (ii) we can find a natural number m such that any two elements b, b', which are

connected in P via some closed pattern v' with base x such that $\llbracket v' \rrbracket \subseteq \llbracket w \rrbracket$, are connected via w^m.

We form a matrix with k rows and $(km(n-1)+1)$ columns. The i-th row is formed as follows. We find a realization (1) of the pattern $w^{(i-1)m}$ connecting a to an element b in Q. This is possible since Q is 1-minimal. (For $i = 1$ we consider the empty sequence.) Then we find a realization (3) of the pattern $w^{(k-i)m}$ connecting some element b' to a' in Q. Finally we find a realization (2) of the pattern w^m connecting b to b' in the strategy P (which is possible by the last sentence in the previous paragraph). Finally we join the realizations (1),(2),(3). When we apply the operation t to the columns of this matrix, we get a realization of the pattern w^{km} connecting $a = t(a, \ldots, a)$ to $a' = t(a', \ldots, a')$ in Q, which finishes the proof. ∎

VI. Proof of Theorem III.7

For the entire section we fix a finite idempotent Taylor algebra $\mathbf{A}$.

Two absorptions can be provided by different term operations. A simple trick can unify them:

Lemma VI.1. *Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2 \in \mathrm{V_{fin}}(\mathbf{A})$ and $\mathbf{B}_1 \lhd \mathbf{A}_1, \mathbf{B}_2 \lhd \mathbf{A}_2$. Then there exists a term operation t of $\mathbf{A}$ such that both absorptions are with respect to the operation t.*

Proof: If $\mathbf{B}_i$ is an absorbing subalgebra of $\mathbf{A}_i$ with respect to an n_i-ary operation t_i, $i = 1, 2$, then the $n_1 n_2$-ary operation defined by $t(a_1, \ldots, a_{n_1 n_2}) = t_1(t_2(a_1, \ldots, a_{n_2}), t_2(a_{n_2+1}, \ldots), \ldots)$ satisfies the conclusion. ∎

The main tool for proving Theorem III.7 is the Absorption Theorem (Theorem III.6. in [17]). We require a definition of a linked subdirect product:

Definition VI.2. *Let $R \subseteq_S A_1 \times A_2$. We say that two elements $a, a' \in A_1$ are R-linked via $c_0, c_1, \ldots, c_{2n}$, if $a = c_0, c_{2n} = a'$ and $(c_{2i}, c_{2i+1}) \in R$ and $(c_{2i+2}, c_{2i+1}) \in R$ for all $i = 0, 1, \ldots, n - 1$.*

We say that R is linked, *if any two elements $a, a' \in A_1$ are R-linked.*

Theorem VI.3. *[17], [18] Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{V_{fin}}(\mathbf{T})$ be absorption free algebras and let $R \leq_S \mathbf{A}_1 \times \mathbf{A}_2$ be linked. Then $R = A_1 \times A_2$.*

We will need the following consequence.

Lemma VI.4. *Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{V_{fin}}(\mathbf{A})$ be absorption free algebras, let $R \leq_S \mathbf{A}_1 \times \mathbf{A}_2$ and let α_1 be a maximal congruence of $\mathbf{A}_1$. Then either $\{R^+[C] : C$ is an α_1-block$\}$ is the set of blocks of a maximal congruence α_2 of $\mathbf{A}_2$, or $R^+[C] = A_2$ for every α_1-block C.*

Proof: If the sets $R^+[C]$ are disjoint, then they are blocks of an equivalence on A_2, and it is straightforward to check that this equivalence is indeed a maximal congruence of $\mathbf{A}_2$.

In the other case we consider the factor algebra $\mathbf{A}_1' = \mathbf{A}_1/\alpha_1$ and the subdirect subalgebra $R' = \{([a_1]_{\alpha_1}, a_2) :$

$(a_1, a_2) \in R\}$ of $\mathbf{A}_1 \times \mathbf{A}_2$. Since α_1 is maximal, the algebra $\mathbf{A}_1'$ has only trivial congruences. Also, $\mathbf{A}_1'$ is an absorption free algebra, because the preimage of any absorbing subalgebra $\mathbf{C} \leq \mathbf{A}'$ is an absorbing subalgebra of $\mathbf{A}$.

We define a congruence $\sim$ on $\mathbf{A}_1'$ by $[a_1] \sim [a_2]$, if $[a_1], [a_2]$ are R'-linked. As not all of the sets $R^+[C]$ are disjoint, $\sim$ is not the diagonal congruence, therefore $\sim$ must be the full congruence, and it follows that R' is linked. By Theorem VI.3 $R' = A_1' \times A_2$. In other words, $R^+[C] = A_2$ for every α_1-block C. ∎

Links are absorbed to absorbing subuniverses:

Lemma VI.5. *Let* $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{V}_{\mathrm{fin}}(\mathbf{A})$, *let* $R \leq_S \mathbf{A}_1 \times \mathbf{A}_2$, *let* $\mathbf{B}_1 \triangleleft \mathbf{A}_1$, $\mathbf{B}_2 \triangleleft \mathbf{A}_2$ *and let* $S = R \cap (B_1 \times B_2)$ *be subdirect in* $B_1 \times B_2$. *Then every pair* $b_1, b_1' \in B_1$ *of* R-*linked elements is also* S-*linked.*

Proof: By Lemma VI.1 there exists a term operation t such that both absorptions are with respect to t. Let $b_1, b_1' \in B_1$ be arbirary. Since S is subdirect, there exist $b_2, b_2' \in B_2$ such that $(b_1, b_2), (b_1', b_2') \in S$. Let b_1, b_1' be R-linked via $c_0, c_1, \ldots, c_{2n}$. Now the following sequence S-links b_1 to b_1':

$$b_1 = t(b_1 = c_0, b_1, \ldots, b_1), t(c_1, b_2, \ldots, b_2), t(c_2, b_1, \ldots, b_1),$$
$$\ldots, t(b_1' = c_{2n}, b_1, \ldots, b_1), t(b_2', c_1, b_2, \ldots, b_2), \ldots,$$
$$t(b_1', b_1', b_1, \ldots, b_1), t(b_2', b_2', c_1, b_2, \ldots, b_2), \ldots, \ldots,$$
$$t(b_1', \ldots, b_1') = b_1'.$$

∎

A subalgebra of a conservative absorption free algebra which hits all block of a proper congruence is absorption free:

Lemma VI.6. *Let* $\mathbf{A}_1 \in \mathrm{V}_{\mathrm{fin}}(\mathbf{A})$ *be a conservative absorption free algebra and let* α *be a proper congruence of* $\mathbf{A}_1$. *Then any subalgebra* $\mathbf{B}$ *of* $\mathbf{A}_1$ *which has a nonempty intersection with every* α-*block is an absorption free algebra.*

Proof: For a contradiction, consider a proper absorbing subuniverse C of $\mathbf{B}$. Let $D_1, \ldots, D_k$ be all the α-blocks whose intersections with B and C are equal and let $E_1, \ldots, E_l$ be the remaining α-blocks which intersect C nonempty.

We claim that, for every $m \leq l$, the set $F = D_1 \cup \cdots \cup D_k \cup E_1 \cup \cdots \cup E_m$ is an absorbing subuniverse of $\mathbf{A}$: Let t be a term operation providing the absorption $C \triangleleft \mathbf{B}$ and let $\mathbf{a}$ be a tuple of elements in A with all the coordinates in F with the exception of, say, a_i. We take any tuple $\mathbf{b}$ such that $b_j \, \alpha \, a_j$ for all coordinates j, $b_i \in B - C$ and $b_j \in C$ for all $j \neq i$. As $B \triangleleft \mathbf{C}$, $t(\mathbf{b})$ is an element of C and, due to conservativity, $t(\mathbf{b}) \in F$. Therefore $t(\mathbf{a}) \in F$ as this element is α-congruent to $t(\mathbf{b})$.

For an appropriate choice of $m \leq l$, F is a proper nonempty subset of $\mathbf{A}$ and $F \triangleleft \mathbf{A}$, a contradiction. ∎

A subdirect product of conservative absorption free algebras is absorption free:

Lemma VI.7. *Let* $\mathbf{R} \leq_S \mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n$, *where every* $\mathbf{A}_i \in \mathrm{V}_{\mathrm{fin}}(\mathbf{A})$ *is a conservative absorption free algebra. Then* $\mathbf{R}$ *is an absorption free algebra.*

Proof: We take a minimal counterexample to the lemma in the following sense: We assume that the lemma holds true for every smaller n, and also for every $\mathbf{R}' \leq_S \mathbf{A}_1' \times \cdots \times \mathbf{A}_n'$ such that $|A_i'| \leq |A_i|$, $i = 1, \ldots, n$, where at least one inequality is strict. We can assume that no A_i is one-element, otherwise we can employ the minimality assumption and use the lemma for the projection to the remaining coordinates.

Let S be a proper absorbing subuniverse of $\mathbf{R}$. It is easily seen that the projection of S to any coordinate i is an absorbing subuniverse of $\mathbf{A}_i$, thus S is subdirect. Let α_1 be a maximal congruence of $\mathbf{A}_1$.

For every $i \in \{1, 2, \ldots, n\}$ we have two possibilities (see Lemma VI.4):

(i) $\{R|_{(1,i)}^+[C] : C$ is an α_1-block$\}$ are blocks of a maximal congruence α_i of $\mathbf{A}_i$

(ii) $R|_{(1,i)}^+[C] = A_i$ for every α_1-block C

Let G (resp. W) denote the set of is for which the first (resp. the second) possibility takes place. By using Lemma VI.4 again, we get that $R|_{(i,j)}^+[C] = A_j$ for any $i \in G$, $j \in W$ and any α_i-block C.

We take an arbitrary tuple $(a_1, \ldots, a_n) \in R$ and we aim to show that this tuple belongs to S as well. The proof splits into two cases.

Assume first that for every $i \in G$, α_i is the diagonal congruence. Let $A_j' = R|_{(1,j)}^+[\{a\}]$, $j = 1, 2, \ldots, n$, let $R' = R \cap (A_1' \times \cdots \times A_n')$ and let $S' = S \cap (A_1' \times \cdots \times A_n')$. Note that A_i' is one element (for $i \in G$) or equal to A_i (for $i \in W$), and S' absorbs R'. Therefore $R' = S'$ (by the minimality assumption) and hence $(a_1, \ldots, a_n) \in S$.

Now assume that some α_i, $i \in G$ is not the diagonal congruence. Take a proper subset B of A_i which contains a_i and which intersects all α_i-blocks nonempty. Let $A_j' = R|_{(i,j)}^+[B]$, $j = 1, \ldots, n$, and let R', S' be as in the previous paragraph. By Lemma VI.7 every $\mathbf{A}_j'$, $j \in G$ is an absorption free algebra, and $\mathbf{A}_j' = A_j$ for $j \in W$ is absorption free as well. Now, by the minimality assumption, $S' = R'$, hence $(a_1, \ldots, a_n) \in S$. ∎

The following lemma proves a special case of Theorem III.7. Note that we do not require $\mathbf{A}_2$ to be conservative.

Lemma VI.8. *Let* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2 \in \mathrm{V}_{\mathrm{fin}}(\mathbf{A})$ *be algebras such that* $\mathbf{A}_1$ *is conservative,* $\mathbf{B}_1 \triangleleft\triangleleft \mathbf{A}_1$ *and* $\mathbf{B}_2 \triangleleft\triangleleft \mathbf{A}_2$. *Let* $R \leq_S \mathbf{A}_1 \times \mathbf{A}_2$. *If* $R \cap (B_1 \times B_2) \neq \emptyset$ *and there exists a pair* $(a_1, b_2) \in R$ *such that* $a_1 \in A_1 - B_1$ *and* $b_2 \in B_2$, *then* $B_1 \times B_2 \subseteq R$.

Proof: Let $S = R \cap (B_1 \times B_2)$. As before, the projection of S to the first (resp. second) coordinate is an absorbing subuniverse of $\mathbf{B}_1$ (resp. $\mathbf{B}_2$), and, by the assumption, S is nonempty, therefore $S \leq_S B_1 \times B_2$. Let $b_1 \in B_1$ be such that $(b_1, b_2) \in R$. We define a congruence on $\mathbf{A}_1$ by putting $c \sim d$, if c and d are R-linked.

Let C denote the set of all the elements of B_1 which are not R-linked to b_1. If C is empty, then, by Lemma VI.5, S is linked and therefore $S = B_1 \times B_2$ by Theorem VI.3.

Otherwise, C is a proper subuniverse of $\mathbf{B}_1$ and we claim that $C \triangleleft \mathbf{B}_1$: Let t be a term operation providing the absorption

$B_1 \lhd \mathbf{A}_1$ and let $\mathbf{c}$ be a tuple of elements of B_1 with all the coordinates but one, say c_i, in C. Let $\mathbf{d}$ be the tuple defined by $d_i = a_1$, and $d_j = c_j$ for $j \neq i$. As $d_i \sim c_i$ for all i we have $t(\mathbf{c}) \sim t(\mathbf{d})$. But $t(\mathbf{d})$ lies inside C (as B_1 absorbs $\mathbf{A}_1$ and $\mathbf{A}_1$ is conservative), hence also $t(\mathbf{c}) \in C$.

We have found a proper absorbing subuniverse C of $\mathbf{B}_1$, a contradiction. $\blacksquare$

The next lemma generalizes the previous one. Recall the definition of the quasi-ordering $\preceq$ introduced in Subsection III-E.

Lemma VI.9. *Let $\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}_1, \ldots, \mathbf{B}_n \in V_{\mathrm{fin}}(\mathbf{A})$ be algebras such that $\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ are conservative and $\mathbf{B}_i \lhd\!\!\lhd \mathbf{A}_i$ for all $i = 1, \ldots, n$. If $\{1, 2, \ldots, n-1\}$ is an (R, B)-strand and there exists a tuple $(a_1, a_2, \ldots, a_{n-1}, b_n) \in R$ such that $b_n \in B_n$ and $a_i \in A_i - B_i$ for some (equivalently every) $i \in \{1, 2, \ldots, n-1\}$, then every tuple $\mathbf{c} \in B_1 \times \cdots \times B_n$ such that $\mathbf{c}|_{\{1,2,\ldots,n-1\}} \in R|_{\{1,2,\ldots,n-1\}}$ belongs to R.*

Proof: We take a minimal counterexample in the same sense as in Lemma VI.7, i.e., we assume that the lemma holds if n is smaller and also if some A_i is smaller.

We may assume that all B_is are at least two-element and let us also assume that if some of the algebras $\mathbf{B}_1, \ldots, \mathbf{B}_n$ has a nontrivial congruence, then $\mathbf{B}_1$ has a nontrivial congruence (otherwise we just change the indices).

Let α_1 be a maximal congruence of $\mathbf{B}_1$. By applying Lemma VI.4 as in the proof of Lemma VI.7 we get that, for each $i \in \{1, 2, \ldots, n-1\}$, either $R|_{(1,i)}{}^+[C] = B_i$ for every α_1-block C, or $\{R|_{(1,i)}^+[C] : C$ is an α-block $\}$ are blocks of a maximal congruence α_i on $\mathbf{B}_i$.

Let $\mathbf{c} \in B_1 \times \cdots \times B_n$ be an arbitrary tuple such that $\mathbf{c}|_{\{1,2,\ldots,n-1\}} \in R|_{\{1,2,\ldots,n-1\}}$.

If α_1 is the diagonal congruence, then we put $D = \{c_1\} \cup (A_1 - B_1)$. Otherwise, we take an arbitrary D such that $(A_1 - B_1) \cup \{c_1\} \subseteq D \subsetneq A_1$ and D intersects every α_1-block nonempty. Let $A_i' = R|_{(1,i)}{}^+[D]$, $B_i' = B_i \cap A_i'$, $i = 1 \ldots, n$ and $R' = R \cap (A_1' \times \cdots \times A_n')$.

For every $i \in \{1, \ldots, n-1\}$, B_i' is an absorption free algebra, either because B_i' is a singleton, or B_i' intersects every α_i block nonempty and we can apply Lemma VI.6. From Lemma VI.8 it follows that $B_1 \times B_n \subseteq R|_{(1,n)}$, therefore $B_n' = B_n$, in particular, $(a_1, a_2, \ldots, a_{n-1}, b_n) \in R'$. Obviously B_i' is an absorbing sublalgebra of $\mathbf{A}_i'$ for every $i = 1, \ldots, n$. Now $\mathbf{c} \in R' (\subseteq R)$ follows from the minimality of our counterexample. $\blacksquare$

We are ready to prove Theorem III.7.

Theorem VI.10. *Let $\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}_1, \ldots, \mathbf{B}_n \in V_{\mathrm{fin}}(\mathbf{A})$ be conservative algebras such that $\mathbf{B}_i \lhd\!\!\lhd \mathbf{A}_i$ for all $i = 1, \ldots, n$, let $R \leq_S \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ and assume that $R \cap (B_1 \times \ldots, B_n) \neq \emptyset$. Then a tuple $\mathbf{a} \in B_1 \times \cdots \times B_n$ belongs to R whenever $\mathbf{a}|_K \in R|_K$ for each (R, B)-strand K.*

Proof: We again use the minimality assumption, i.e., we assume that the theorem holds if n is smaller, or if some A_i is smaller. We can assume that there are at least two (R, B)-strands and that $|B_i| > 1$ for all $i = 1, \ldots, n$. Let

$\mathbf{a} \in B_1 \times \cdots \times B_n$ be a tuple such that $\mathbf{a}|_K \in R|_K$ for each (R, B)-strand K, but $\mathbf{a} \notin R$. Note that $\mathbf{a}|_L \in R|_L$ for every proper subset L of $\{1, 2, \ldots, n\}$, because of the minimality assumption – we can apply the theorem to $R|_L$.

Let D be a minimal (R, B)-strand and let $l \notin D$. Since $l \not\preceq D$, there exists a tuple $\mathbf{c} \in R$ such that $c_l \in B_l$ and $c_i \notin B_i$ for all $i \in D$. Let $E = \{i \in \{1, \ldots, n\} : c_i \notin B_i\} - D$ and $F = \{i \in \{1, \ldots, n\} : c_i \in B_i\}$. Clearly, E and F are unions of (R, B)-strands.

Our aim now is to find a tuple $\mathbf{c}' \in R$ such that $c_i' \in A_i - B_i$ for all $i \in D$, and $c_i' \in B_i$ for all $i \notin D$. If $E = \emptyset$, we can take $\mathbf{c}' = \mathbf{c}$, so suppose otherwise. We consider the following subset of R:

$$R' = \{\mathbf{b} \in R : b_i \in \{a_i\} \cup (A_i - B_i) \text{ for all } i \in D\}.$$

For all $i \in \{1, \ldots, n\}$, let $A_i' = R'|_{\{i\}}$. Let $B_i' = \{a_i\}$ for all $i \in D$, and $B_i' = B_i$ for $i \notin D$.

We have $B_i' \subseteq A_i$ for every $i \notin D$: for any $b \in B_i$ we apply the theorem for $R|_{D \cup \{i\}}$ to obtain a tuple $\mathbf{e} \in R$ such that $e_i = b$ and $e_j = d_j$ for every $j \in D$.

We know that $\mathbf{a}|_{D \cup E} \in R|_{D \cup E}$, therefore $a|_E \in R'|_E$. Similarly, $a|_F \in R'|_F$.

Observe that any $i \in E$, $j \in F$ are in different (R', B')-strands, since $\mathbf{c} \in R'$. Therefore, the theorem, used for R' and the minimal absorbing subuniverses B_i' of A_i', proves $\mathbf{a}|_{E \cup F} \in R'|_{E \cup F}$. Let $\mathbf{c}'$ be a tuple from R' with $\mathbf{c}'|_{E \cup F} = \mathbf{a}|_{E \cup F}$. The tuple $\mathbf{c}'$ cannot be equal to $\mathbf{a}$ as $\mathbf{a} \notin R$, therefore $c_i' \in A_i - B_i$ for all $i \in D$, $c_i' \in B_i$ for $i \notin D$.

Now, when we have the sought after tuple $\mathbf{c}'$, we can finish the proof by applying Lemma VI.9 for the following choice: $n' = |D| + 1$; $\mathbf{A}_i' = \mathbf{A}_{d_i}$ and $\mathbf{B}_i' = \mathbf{B}_{d_i}$ for $i = 1, \ldots, l$, where $D = \{d_1, \ldots, d_l\}$; $\mathbf{A}_{n'}' = R|_{E \cup F}$; $\mathbf{B}_{n'}' = S|_{E \cup F}$, where $S = \{\mathbf{b} \in R : b_i \in B_i \text{ for all } i \in E \cup F\}$; and $(a_1', a_2', \ldots, a_{n'-1}', b_n') = (c_{d_1}, c_{d_2}, \ldots, c_{d_l}, \mathbf{c}'|_{E \cup F})$. All the assumptions are satisfied, the only nontrivial fact is that $\mathbf{B}_{n'}'$ is absorption free and this follows from Lemma VI.7. $\blacksquare$

VII. CONCLUSION

We have presented a new, simple algorithm for solving tractable CSPs over conservative languages. We believe that this simplification can help in the final attack on the dichotomy conjecture.

No effort has been made to optimize the algorithm, we has not computed its time complexity and we has not compared the complexity with the algorithm of Bulatov. This can be a topic of further research.

We note that some reductions can be done using a trick from [26], it would be interesting to see whether this trick can improve the running time.

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