Day 11

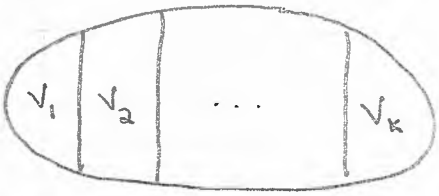
6/26/11

Colorings and Homomorphisms for Graphs and Finite structures — Jaroslav Nesetril

Combinatorics Surrounding CSP
Paradoxes of coloring

$G = (V, E)$ $\quad \chi(G) =$ min # of colors which suffice to color G s.t. adj. vertices receive diff. colors.

G



V_i no edges "easy"
min # of partitions into "easy" parts.

edges are "between"

$$\chi(G) \le \overset{\Delta(G)}{\overset{\shortparallel}{\max d_G(x)}} + 1$$
$$\le \max \underset{\text{degeneracy}}{\underset{\shortparallel}{\text{average degree}}} + 1$$

$\chi(G) \ge \omega(G) = \max\{k : K_k$ contained in $G\}$.

ohnson
↳ If G contains no triangle △, then $\chi(G) \le \dfrac{|\Delta(G)|}{\log_2 |\Delta(G)|}$

$\chi(G) \le \dfrac{3\Delta(G)}{4} \le c \cdot \Delta(G)$

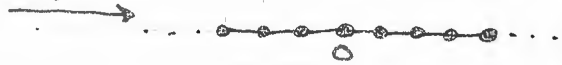
Homomorphism

$G = (V, E)$ $\quad G' = (V', E')$ (undirected)

$f : G \longrightarrow G'$

$f : V \longrightarrow V'$ $\quad xy \in E \Rightarrow f(x)f(y) \in E'$

$G = Q_d$ (d-dimensional cube) $\xrightarrow{f}$



J. Kahn : Expected range $\le c_0 = 5$

D. Galvin

G, H $\quad \mathrm{Hom}(G,H) = \{ f \mid f : G \to H \}$

$\mathrm{hom}(G,H) = |\mathrm{Hom}(G,H)|$

Thm| (L^{2}) $H_1 \simeq H_2 \iff \forall G \ \ \mathrm{hom}(G, H_1) = \mathrm{hom}(G, H_2)$

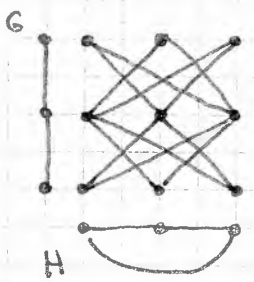
$t(G,H) = \dfrac{\mathrm{hom}(G,H)}{|H|^{|G|}} \qquad (|H| = |V(H)|)$

← all maps

so this is probability that a map is a homomorphism

direct

$G \times H$ = categorical product of G and H



Thm| $SP\{K_k \mid k = 2, 3, \ldots\}$ = all graphs

$G \underset{ind}{\leq} \bigtimes_{i \in I} K_{u(i)}$

$\underset{ind}{\leq}$ = induced subgraph.

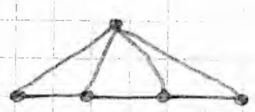
Defn| $\dim(G) = \min |I| = \min \{ d \mid G \underset{ind}{\leq} K_{|G|}^{d} \}$

Thm| $\forall k \ \exists A_k \quad \chi(G) \leq k \iff \underset{\exists t}{\wedge} G \underset{ind}{\leq} A_k^{t}$

minimal examples

A_2

A_3

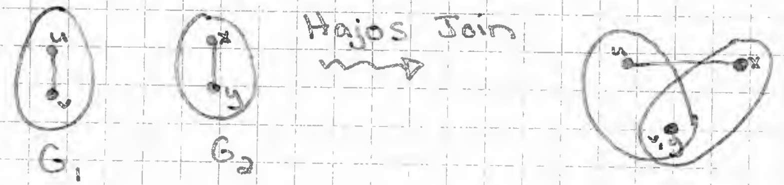


⋮

Remark) Any $G \not\to K_k$ can be generated by adding a vertex or edge or identifying 2 vertices

Starting from K_{k+1}

$$\left.\begin{array}{l} G \longrightarrow G+v \\ G \longrightarrow G+e \\ G \longrightarrow G \,|\, x=y \\ G_1, G_2 \text{ Hajos Join} \end{array}\right\} \text{no homomorphism to } K_k$$



We don't know what $\chi(G)$ large means

Conjectures

① Product conjecture (Hedetniemi) $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$

know $\leq$

$f(k) = \min\{\chi(G \times H) \mid \chi(G), \chi(H) \geq k\}$
conjecture says $f(k) = k$. (true for $k = 2, 3, 4$)

? $f(k) \to \infty$ Poljak, Rodl

X. Zhu $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$

[χ_f = fractional chromatic number
$\chi_f(G) = \min\left\{ \frac{n}{k} \;\middle|\; G \to K\binom{n}{k} \right\}$

$K\binom{n}{k} = (V, E)$ V = all k-elt. subsets of n-elt. set $\{1, \dots, n\}$
Kneser Graphs E = pairs of disjoint subsets

$\chi\left(K\binom{n}{k}\right) = n - 2k + 2$ (L^2 (Lovas?))]

Thm) (Erdös, 1958) $\forall k \; \forall g \; \exists G$ s.t. $\chi(G) \geq k$ and $\underbrace{G \not\supseteq \Delta, \Box, \dots, C_g}_{\text{girth } G > g}$

How large G?
Construction of G?

means χ is not "local"

All $|G|$ not primitive recursive in k, g

② If $\chi(G) > \overset{f(k,g)}{\text{large}}$ then $\exists\, H \overset{\text{weak}}{\subseteq} G$ s.t. $\chi($[illegible] and

girth$(H) \geq g$. (Erdös, Hajnal)

$g=3$ true Rödl 1977