[Day 3] 6/29/11

Colorings and Homomorphisms for Graphs and Finite Structures

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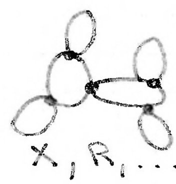
Density for undirected graphs

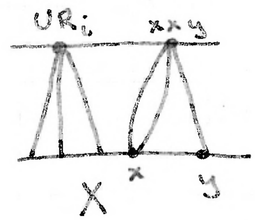
Thm (N., Tardif 2000) For a connected relational structure A TFAE

(1) There exists a predecessor of A in $\mathcal{C}$ (hom.-order) i.e. $P_A < A$ and there is no B s.t. $P_A < B < A$.

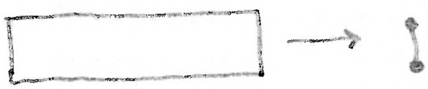
(2) A is a relational tree



(xxy)



Pf

$G_2 \not\to$ 

Thm (N., Zhu 2003-4) $\forall k\ \forall H$ connected $\exists G$:

(1) $c: G \longrightarrow H$

(2) G is sparse (girth $\geq \ell$)

(3) $H \longrightarrow H' \iff G \longrightarrow H'$ $|H'| \leq k$

(4)
$$\begin{tikzcd} H \arrow[r, "h"] & H' \\ G \arrow[u, "c"] \arrow[ur, "g"'] & \end{tikzcd}$$

$\forall G\ \exists!\ h$ and $h \circ c = g$
$|H'| \leq k$ H' to H is <u>pointed</u>

<u>pointed</u>: $h_1 : H \to H'$, $h_2 : H \to$
$h_1(x) = h_2(x)$ $x \neq x_0 \Rightarrow h_1(x_0) = h_2(x_0)$

Cor
$$\begin{tikzcd} H \arrow[r] & H \\ G \arrow[u] \arrow[ur] & \end{tikzcd}$$
There are sparse uniquelly H-colorable graphs.

Pf of Thm $G = H \times H_0$
$\chi(H_0) \gg$

$\{x, y\} \in H_0$, $f_x = f_y \Rightarrow f_x$ homomorphism (last time)

notes by Megan Heenehan

Suppose f_x is a homomorphism, $(x,y) \in E(H_0)$
$\Rightarrow f_y$ is a homomorphism $f_x = f_y$

It suffices to show $f_x = f_y$.

Suppose $f_x(x_0) \neq f_y(x_0)$. Define $\varphi(z) = f_x(z)$ $z \neq x_0$
$\varphi(x_0) = f_y(x_0)$

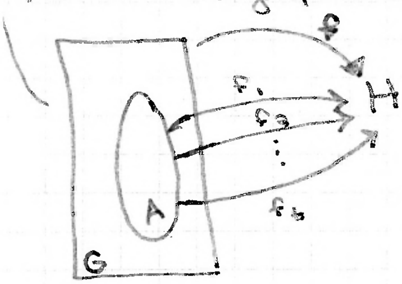
[redacted] φ is a homomorphism.
$\Rightarrow$ [redacted] $\varphi(x_0) = f_x(x_0)$ [redacted] by pointed $\Rightarrow\!\Leftarrow$
$\square$

Müller Extension Theorem (MET)| Suppose H is projective,
$|H| \geq 2$, ℓ, A, $f_1, \dots, f_t : A \to V(H)$.
$\exists G$: ① $A \subseteq V(G)$
② odd girth $> \ell$
③ $\forall i \; \exists! \; g : G \to H$ $g|_A = f_i$ (unique ext. to hom.)
→ ④ $f : G \to H$ $\exists i \; \exists \alpha$: $f = \alpha \circ g_i$

H is a projective graph if $H^t \xrightarrow{f} H \Rightarrow f$ is projection.

Thm| K_k for $k \geq 3$ is projective.

Almost all graphs are projective. (N. Lovžak)



Pf Sketch of MET| H^t $A \longrightarrow V(H)^t$
$a \longmapsto (f_1(a), f_2(a), \dots, f_t(a))$
1-1 (injection) so $A \subseteq V(H)^t$

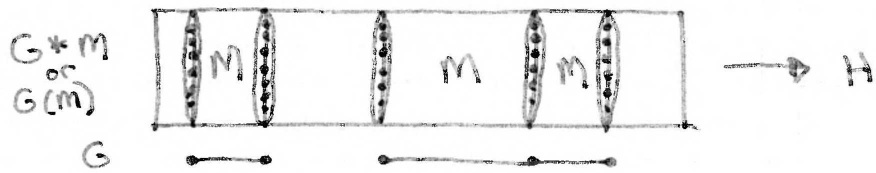
(by prev. thm) $\exists$ $c : G \longrightarrow H^t$

$c^{-1}(A)$

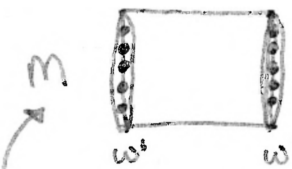
$$\begin{array}{ccc} H^t \times K_M & \xrightarrow{\text{proj.}} & H \\ \uparrow c & \nearrow g & \\ A \subseteq G & & \end{array}$$

$\square$

Thm| MET $\Leftrightarrow$ H projective. $\Leftrightarrow$ any 2 elt. subset is constructable



relational structure



Fiber construction

gadget
K_3-partition

$\underbrace{G \to K_3}$ $\Leftrightarrow$ $G * M \to H$

- NP-complete if $\Delta(G) \leq 4$
- NP-complete for large girth

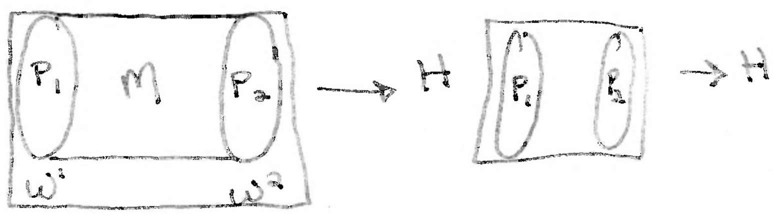
M K_3-partition for H. $\quad M \to H$

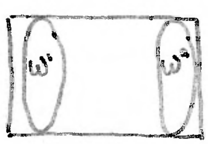
(1) $V(M) \supseteq w^1, w^2$ disjoint $\quad w^1 \sim w^*$

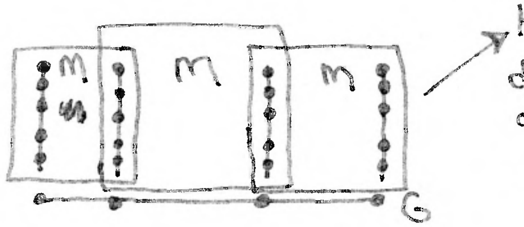
(2) 3 disjoint sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ of H-patterns
$\mathcal{P}_i \subseteq w^* \to V(H)$

(3) $\phi : M \to H \Rightarrow \phi|_{w^i} \in \mathcal{P}_{\tau(i)} \quad \tau(1) \neq \tau(2)$

(4) $\exists\ P_i \in \mathcal{P}_i$







define coloring of G $f|_{w^*} \in \mathcal{P}_i$

$\Leftrightarrow$ color of x is i. is good by (3).

$\to K_3$

$G * M \to H$

If H is projective, then $\mathcal{M}$ exists.

If H is block projective, then $\mathcal{M}$ exists.