Lecture 6: Some recent progress on regular and chiral polytopes

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Marston Conder

University of Auckland

m.conder@auckland.ac.nz

This final lecture will have three parts:

- A summary of some recent research on regular polytopes
- A summary of some recent research on chiral polytopes
- Finding the smallest finite regular polytopes of all ranks

Constructions for regular polytopes

- C-group permutation representation graphs (CPR graphs)

These used (as a tool) by Daniel Pellicer to construct regular polyhedra with alternating groups A_n as the automorphism groups (2008), and regular polytopes with given facets and prescribed (even) last entry of the Schläfli symbol (2010).

- The mix of two polytopes

Egon Schulte and Peter McMullen (2002) introduced a new group-theoretic method for constructing a new regular polytope from two given regular polytopes $\mathcal{P}$ and $\mathcal{Q}$, called the 'mix' of $\mathcal{P}$ and $\mathcal{Q}$.

- Polytopes of given type

For example, Dimitri Leemans and Michael Hartley (2009) constructed various regular 4-polytopes with type $[5, 3, 5]$.

Similarly, many families of examples (of type $[3, 5, 3]$ etc.) arise from quotients of groups associated with hyperbolic 3-manifolds of small volume (by Lorimer, Jones, Conder, Torstensson et al, 1990s–).

- Amalgamation of polytopes

Michael Hartley constructed regular polytopes with given facets and given vertex-figures, in some special cases (2010).

Collecting small examples of regular polytopes

• Michael Hartley has created a web-based atlas of regular polytopes with automorphism group of order at most 2000, except those with autom group of order 512, 1024 or 1536 — see `http://www.abstract-polytopes.com/atlas` for this.

• Dimitri Leemans and Laurence Vauthier have found all regular polytopes whose automorphism group G is an almost simple group with $S \leq G \leq \text{Aut}(S)$ for some simple group S of order less than 900,000 — for the complete list, see `http://cso.ulb.ac.be/ dleemans/polytopes`.

Both of these two atlases were first published in 2006.

Regular polytopes with given group

• Dimitri Leemans and Laurence Vauthier proved (in 2006) that the group $\mathrm{PSL}(2,q)$ cannot be the automorphism group of a regular n-polytope for any $n \geq 5$.

• Dimitri Leemans and Egon Schulte determined all regular 4-polytopes with automorphism group $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ (in 2007 and 2009).

• Daniel Pellicer (2008) used CPR graphs to construct regular polyhedra with automorphism group A_n (and other groups related to A_n and S_n), and Dimitri Leemans, Maria Elisa Fernandes and Mark Mixer have extended this (2011).

• Barry Monson and Egon Schulte (2009) used modular reduction techniques to construct new regular 4-polytopes of hyperbolic types $\{3, 5, 3\}$ and $\{5, 3, 5\}$ with a finite orthogonal group as automorphism group.

• Peter Brooksbank and Deborah Vicinsky (2010) showed that regular polytopes that have a 3-dimensional classical group as automorphism group come from orthogonal groups.

• Ann Kiefer and Dimitri Leemans (2010) determined the regular polyhedra whose automorphism group is a Suzuki simple group Sz(q).

• Dimitri Leemans and Maria Elisa Fernandes (2011) proved that for every $n > 3$, the symmetric group S_n is the automorphism group of some regular r-polytope, for each r such that $3 \leq r \leq n-1$, and hence for any given $r \geq 3$, all but finitely many S_n are the automorphism group of a regular r-polytope.

Geometric and other considerations

• Barry Monson and Egon Schulte wrote a series of five papers (2004–2009) on reflection groups and polytopes over finite fields, producing (for example) a catalogue of modular polytopes of small rank that are spherical or Euclidean.

• Peter McMullen (2004) classified all regular n-polytopes (and apeirotopes) that are faithfully realisable in a Euclidean space of minimum dimension n (resp. $n-1$).

• Peter McMullen used similar techniques in order to classify 4-dimensional finite regular polyhedra (2007), and regular apeirotopes of dimension 4 (2009).

- Michael Hartley and Gordon Williams (2010) used methods for finding quotients of regular polytopes to obtain representations of the 14 sporadic Archimedean polyhedra.

- Isabel Hubard (2010) investigated 'two-orbit' polytopes, determining when the automorphism group is transitive on the faces of each rank, and used this to completely characterise the groups of two-orbit polyhedra (3-polytopes).

- Mark Mixer (PhD) investigated the layer graphs (showing incidence between two layers) of regular polytopes, esp. the medial layer graph of regular n-polytopes for even n.

Properties of chiral polytopes

- Asia Weiss and Isabel Hubard (2005) proved that every self-dual chiral polytope of odd rank admits a polarity, but that this is not true for even ranks.

- Asia Weiss, Egon Schulte and Isabel Hubard (2006) then showed how to construct chiral polyhedra from improperly self-dual chiral polytopes of rank 4, and regular polyhedra from properly self-dual ones.

Construction of chiral polytopes

• Isabel Hubard, Marston Conder and Tomo Pisanski (2008) used computational group-theoretic methods to find subgroups of small index in Coxeter groups that are normal in the orientation-preserving subgroup but not in the group itself. This produced the smallest examples of finite chiral 3- and 4-polytopes, and also the first known finite chiral 5-polytopes, in both the self-dual and non-self-dual cases.

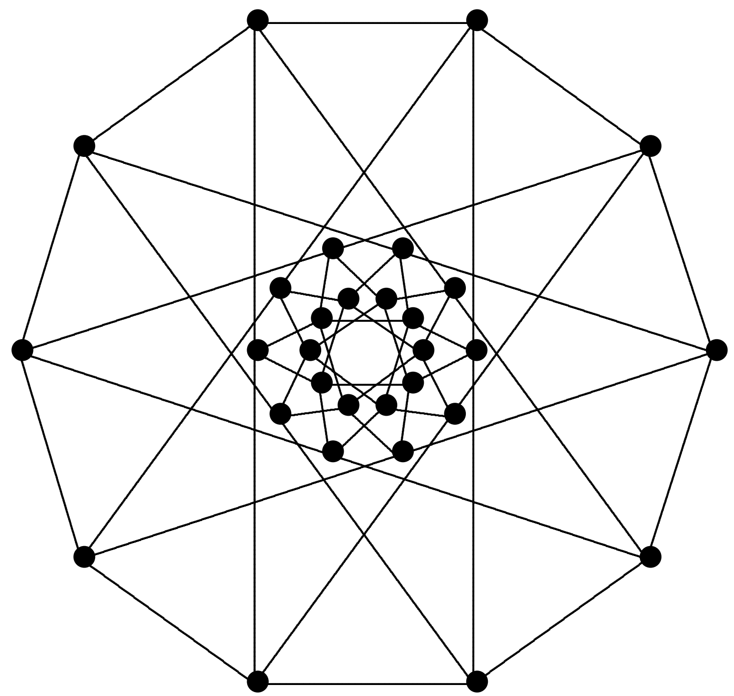
• Alice Devillers and Marston Conder (2009) found the first known finite chiral 6-, 7- and 8-polytopes, by group-theoretic construction for types $[3, 3, \ldots, 3, k]$.

• Daniel Pellicer (2010) devised a construction for chiral polytopes with prescribed regular facets, and used this to prove the existence of chiral d-polytopes, for all $d \geq 3$.

Smallest (known) chiral polytopes

Rank n	Properly self-dual	Improperly self-dual	Non-self-dual
3	Type $\{7,7\}$ $\|\text{Aut}(\mathcal{P})\| = 56$	Type $\{4,4\}$ $\|\text{Aut}(\mathcal{P})\| = 20$	Type $\{3,6\}$ $\|\text{Aut}(\mathcal{P})\| = 42$
4	Type $\{4,4,4\}$ $\|\text{Aut}(\mathcal{P})\| = 120$	Type $\{4,4,4\}$ $\|\text{Aut}(\mathcal{P})\| = 400$	Type $\{3,4,4\}$ $\|\text{Aut}(\mathcal{P})\| = 120$
5	Type $\{3,8,8,3\}$ $\|\text{Aut}(\mathcal{P})\| = 20!/2$	Type $\{3,4,4,3\}$ $\|\text{Aut}(\mathcal{P})\| = 720$	Type $\{3,4,4,6\}$ $\|\text{Aut}(\mathcal{P})\| = 1440$

The *medial layer graph* (showing incidences between 1- and 2-faces) of the smallest PSD chiral 4-polytope is interesting, and can be defined in terms of a 1-factorisation of K_6:



The smallest regular polytopes in all ranks

Daniel Pellicer asked this question at SIGMAP in Oaxaca, in June 2010:

For each $n \geq 3$, what are the regular n-polytopes with the smallest numbers of flags? Call the smallest number M_n.

By regularity, this number M_n is the order of the smallest good quotient of an n-generator Coxeter group $[k_1, \ldots, k_{n-1}]$ — with 'good' meaning that the orders of the generators ρ_i and their pairwise products $\rho_i \rho_j$ are preserved, and the intersection condition holds.

Also we may assume that $k_i > 2$ for all i (for otherwise the question is not very interesting).

A lower bound for the number of flags of a regular n-polytope

Suppose $\mathcal{P}$ is a regular n-polytope, of type $\{k_1, \ldots, k_{n-1}\}$, with automorphism group $G = \langle \rho_0, \rho_1, \ldots, \rho_{n-1} \rangle$.

Then $H = \langle \rho_0, \rho_1, \ldots, \rho_{n-2} \rangle$ is the automorphism group of a regular $(n-1)$-polytope (a facet of $\mathcal{P}$), and $D = \langle \rho_{n-2}, \rho_{n-1} \rangle$ is dihedral of order $2k_{n-1}$, with $H \cap D = \langle \rho_{n-1} \rangle$ of order 2.

By the intersection property,

$$|G| \geq |HD| = |H||D|/|H \cap D| = |H|(2k_{n-1})/2 = |H|k_{n-1},$$

and by induction, $|\mathrm{Aut}(\mathcal{P})| \geq 2k_1k_2 \ldots k_{n-1}$.

If this lower bound is attained, we will say $\mathcal{P}$ is tight.

Small ranks

These results achievable by computation (using MAGMA):

Rank n	$(n+1)!$	Min # flags M_n	Types of polytopes achieving minimum
2	6	6	$\{3\}$
3	24	24	$\{3,3\}$, $\{3,4\}$, $\{4,3\}$
4	120	96	$\{4,3,4\}$
5	720	432	$\{3,6,3,4\}$, $\{4,3,6,3\}$
6	5040	1728	$\{4,3,6,3,4\}$

Note that all but one of these examples are 'tight', but surprisingly(?), the minimum type is not always $\{3,3,\ldots,3\}$.

Is there a pattern evident here? Are extensions possible?

Two new families

MAGMA computations give also defining presentations for the automorphism groups of small examples. Patterns in these give rise to constructions for two infinite families:

- A regular n-polytope of type $\{4,3,6,3,6,3,6,...,3,6,3\}$, with $8 \cdot 3^{(n-1)/2} \cdot 6^{(n-3)/2}$ flags, for every **odd** $n > 2$
- A regular n-polytope of type $\{4,3,6,3,6,3,6,...,3,6,3,4\}$, with $32 \cdot 3^{(n-2)/2} \cdot 6^{(n-4)/2}$ flags, for every **even** $n > 2$.

Thus $M_n \leq \begin{cases} 24 \cdot 18^{(n-3)/2} & \text{for } n \text{ odd} \\ 96 \cdot 18^{(n-4)/2} & \text{for } n \text{ even.} \end{cases}$

Are these the best?

All the polytopes constructed in the families above (of types $\{4,3,6,3,6,...,3,6,3\}$ and $\{4,3,6,3,6,...,3,6,3,4\}$) are tight.

Can we prove these give the smallest numbers of flags for all n? or are there too many '6's in the type?

In the course of trying to prove the above were the best, another family emerged ...

Tight regular polytopes of type {4,4,...,4}

There exist regular polytopes of types {4,4}, {4,4,4} and {4,4,4,4}, with 32, 128 and 512 flags. Closer inspection of these (and their automorphism groups) gives a new family:

For every $n > 2$, take the Coxeter group $[4, \overset{n-1}{\ldots}, 4]$, with n involutory generators $\rho_0, \rho_1, ..., \rho_{n-1}$, and add relations of the form $[(\rho_{i-1}\rho_i)^2, \rho_j] = 1$ to make the squares $(\rho_{i-1}\rho_i)^2$ all central. This gives a group G whose centre $Z(G)$ is generated by the $n-1$ involutions $(\rho_{i-1}\rho_i)^2$.

In particular, $Z(G)$ and $G/Z(G)$ are elementary abelian, of orders 2^{n-1} and 2^n, so G has order $2^{2n-1} = 2 \cdot 4^{n-1}$. Also the intersection property holds, so G is the automorphism group of a tight regular n-polytope of type $\{4, 4, ..., 4\}$.

Improved upper bounds on M_n

Tight polytopes of type $\{4, ..., 4\}$ give $M_n \leq 2 \cdot 4^{n-1}$ for all n.

This is better than our earlier upper bound of $24 \cdot 18^{(n-3)/2}$ for n odd, and $96 \cdot 18^{(n-4)/2}$ for n even, whenever $n > 8$.

Question: Is the bound $M_n \leq 2 \cdot 4^{n-1}$ sharp for all $n > 8$?

Question: We know M_3 to M_6. What are M_7 and M_8?

Key observation

Suppose $\mathcal{P}$ is a regular n-polytope, of type $\{k_1, \ldots, k_{n-1}\}$.

Then each of the sections of $\mathcal{P}$ is also a regular polytope. In fact, if A and B are i- and j- faces of $\mathcal{P}$ with $A \leq B$, then the section $[A, B] = \{F \in \mathcal{P} : A \leq F \leq B\}$ is a regular $(j-i-1)$-polytope with automorphism group $\langle \rho_{i+1}, \ldots, \rho_{j-1} \rangle$.

Next, for any i, let $L_i = \langle \rho_0, \ldots, \rho_i \rangle$ and $R_i = \langle \rho_{i-1}, \ldots, \rho_{n-1} \rangle$. By the intersection property, $L_i \cap R_i = \langle \rho_{i-1}, \rho_i \rangle \cong D_{k_i}$ and so $|\mathrm{Aut}(\mathcal{P})| \geq |L_i R_i| = |L_i||R_i|/|L_i \cap R_i| = |L_i||R_i|/|D_{k_i}|$.

It follows that $M_n \geq \frac{M_{i+1} M_{n-i+1}}{2k_i}$ for $1 \leq i \leq n-1$.

As $L_i \cap R_{i+1} = \langle \rho_i \rangle$, also $M_n \geq \frac{M_{i+1} M_{n-i}}{2}$ for $1 \leq i \leq n-2$.

Application

Suppose $M_n = 2 \cdot 4^{n-1}$ for all n in the range $i < n \le 2i$.

Then

$$M_{2i+1} \ge \frac{M_{i+1} M_{i+1}}{2} = \frac{(2 \cdot 4^i)^2}{2} = 2 \cdot 4^{2i}$$

and similarly

$$M_{2i+2} \ge \frac{M_{i+1} M_{i+2}}{2} = \frac{(2 \cdot 4^i)(2 \cdot 4^{i+1})}{2} = 2 \cdot 4^{2i+1}$$

and so $M_n = 2 \cdot 4^{n-1}$ for all n in the range $i < n \le 2i + 2$.

This gives a possible basis for induction. We just have to find a starting value of i ...

Finding M_n for small $n \geq 7$

With the help of the `LowIndexNormalSubgroups` algorithm in MAGMA (applied to Coxeter groups), we can find:

- all regular 3-polytopes with at most 100 flags
- all regular 4-polytopes with at most 300 flags
- all regular 5-polytopes with at most 900 flags
- all regular 6-polytopes with at most 2700 flags.

Then multiple applications of the intersection property show:

- the only regular 7-polytopes with fewer than $2 \cdot 4^6$ flags have type $\{4,3,6,3,6,3\}$ or $\{3,6,3,6,3,4\}$ (and 7776 flags)
- the only regular 8-polytope with fewer than $2 \cdot 4^7$ flags has type $\{4,3,6,3,6,3,4\}$ (and 31104 flags), and
- for $9 \leq n \leq 16$, the smallest regular n-polytope is a tight one of type $\{4,4,...,4\}$ (with $2 \cdot 4^{n-1}$ flags).

Example: $n = 9$ (to show what happens)

Suppose there is a regular 9-polytope of type $\{k_1, k_2, ..., k_8\}$ with fewer than $2 \cdot 4^8 = 131072$ flags.

By taking the dual if necessary, we can assume that some 5-face F has fewer than $2 \cdot 4^4 = 512$ flags. Then F must have exactly 432 flags and have type $\{3, 6, 3, 4\}$ or $\{4, 3, 6, 3\}$, and its co-5-face must have at most 606 flags, with its type $\{k_5, k_6, k_7, k_8\}$ coming from a known list.

Then the given 9-polytope has type $\{3, 6, 3, 4, k_5, k_6, k_7, k_8\}$ or $\{4, 3, 6, 3, k_5, k_6, k_7, k_8\}$, but from our lists of small regular 6-polytopes we find no 6-section of type $\{3, 4, k_5, k_6, k_7\}$ or $\{6, 3, k_5, k_6, k_7\}$ small enough to give fewer than $2 \cdot 4^8$ flags.

Theorem

For $n \geq 9$, the smallest regular n-polytopes are the tight polytopes of type $\{4, \overset{n-1}{\ldots}, 4\}$, with $2 \cdot 4^{n-1}$ flags.

For $n \leq 8$, the smallest have the following parameters:

n	M_n	Type(s)
2	6	$\{3\}$
3	24	$\{3,3\}$, $\{3,4\}$ (and dual $\{4,3\}$)
4	96	$\{4,3,4\}$
5	432	$\{3,6,3,4\}$ (and dual $\{4,3,6,3\}$)
6	1728	$\{4,3,6,3,4\}$
7	7776	$\{3,6,3,6,3,4\}$ (and dual $\{4,3,6,3,6,3\}$)
8	31104	$\{4,3,6,3,6,3,4\}$.

Regular polytopes with the fewest elements

The same kind of approach can be taken to find for all n the regular n-polytopes with the smallest numbers of elements.

Let E_n be the smallest such number, for given $n \geq 1$, and suppose that this is attained by the regular n-polytope $\mathcal{P}$. Also suppose that $\mathcal{P}$ has f_j distinct j-faces, for $0 \leq j < n$.

Then $1 + f_0 + f_1 + \cdots + f_{n-3} + 1$ is at least the number of elements of an $(n-2)$-face of $\mathcal{P}$, which is at least E_{n-2}, so

$$E_n = 1+f_0+f_1+\cdots+f_{n-2}+f_{n-1}+1 \geq E_{n-2} + f_{n-2} + f_{n-1}.$$

Since $f_{n-1} \geq k_{n-1}$ and $f_{n-2} \geq k_{n-2}$, again this gives a basis for induction ...

Theorem: For all $n \geq 9$, the smallest number of elements in a regular n-polytope is $8n-6$, and this is achieved by tight polytopes of type $\{4, \overset{n-1}{\ldots}, 4\}$.

For $n \leq 8$, the fewest elements are achieved as follows:

n	E_n	Type(s)
2	8	$\{3\}$
3	15	$\{3,4\}$ (and dual $\{4,3\}$)
4	22	$\{4,3,4\}$
5	33	$\{3,6,3,4\}$ (and dual $\{4,3,6,3\}$)
6	40	$\{4,3,6,3,4\}$
7	50	$\{4,4,4,4,4,4\}$
8	58	$\{4,3,6,3,6,3,4\}$ and $\{4,4,4,4,4,4,4\}$.

Similarly ...

Theorem: For all $n \geq 7$, the smallest number of direct incidences in a regular n-polytope is $32n - 56$, and this is achieved by tight polytopes of type $\{4, \overset{n-1}{\ldots}, 4\}$.

For $n \leq 6$, the fewest direct incidences are as follows:

n	L_n	Type(s)
2	12	$\{3\}$
3	31	$\{3, 4\}$ (and dual $\{4, 3\}$)
4	56	$\{4, 3, 4\}$
5	100	$\{3, 6, 3, 4\}$ (and dual $\{4, 3, 6, 3\}$
6	131	$\{4, 3, 6, 3, 4\}$.