Constructing self-dual chiral polytopes

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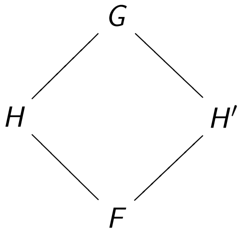
Definition of an abstract polytope

Let $\mathcal{P}$ be a ranked poset, whose elements we call faces.
Then $\mathcal{P}$ is an (abstract) n-polytope if it satisfies the following four conditions:

- Every flag (maximal chain) of $\mathcal{P}$ has $n + 2$ faces.
- There is a unique maximal face and a unique minimal face.
- If $F \leq G$ and $\mathrm{rank}(G) - \mathrm{rank}(F) > 2$, the Hasse diagram of $\{H \mid F < H < G\}$ is connected.
- If $F \leq G$ and $\mathrm{rank}(G) - \mathrm{rank}(F) = 2$, there are exactly 2 faces H such that $F < H < G$.



Schläfli symbol of a polytope

Let $\mathcal{P}$ be an n-polytope. Let F be a face of rank $(i-1)$ and G a face of rank $(i+2)$, with $F \leq G$. Then

$$G/F := \{H \mid F \leq H \leq G\}$$

is a polygon, and we define $p_i(G/F)$ to be the number of vertices of this polygon.

If each $p_i(G/F)$ depends only on i (and not on the choice of F and G), then we define $p_i := p_i(G/F)$ and we say that $\mathcal{P}$ has Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ or that $\mathcal{P}$ is of type $\{p_1, \ldots, p_{n-1}\}$.

Definition of polytope automorphisms

A function $f : \mathcal{P} \to \mathcal{Q}$ is a polytope isomorphism if:

- It is bijective, and
- $F \leq G$ in $\mathcal{P}$ if and only if $f(F) \leq f(G)$ in $\mathcal{Q}$.

(These are just the usual isomorphisms for posets.)

An isomorphism from $\mathcal{P}$ to itself is an automorphism. We denote the automorphism group of $\mathcal{P}$ by $\Gamma(\mathcal{P})$.

Definition of regular and chiral polytopes

The automorphism group of a polytope has a natural action on the flags (maximal chains). If this action is transitive, we say that the polytope is regular.

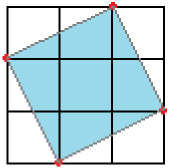
Examples: regular convex polytopes, regular tessellations, 11-cell.

The polytope $\mathcal{P}$ is chiral if

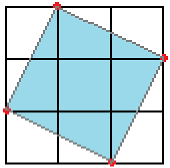
- The action of $\Gamma(\mathcal{P})$ on the flags has 2 orbits.
- Flags that differ in a single face are in different orbits.

More info on chirality

Chiral polytopes do not have mirror symmetry, but they have full rotational symmetry. The mirror-image of $\mathcal{P}$ is denoted $\overline{\mathcal{P}}$.



$\{4,4\}_{(2,1)}$



$\overline{\{4,4\}_{(2,1)}} = \{4,4\}_{(1,2)}$

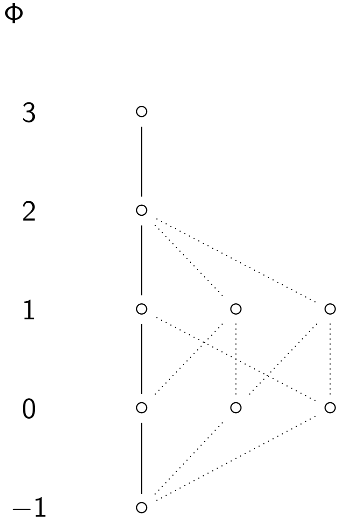
Duality

Given a polytope $\mathcal{P}$, we get its dual $\mathcal{P}^{\delta}$ by reversing the partial order. If a polytope is isomorphic to its dual, we say that it is self-dual.

For chiral polytopes, there are two types of self-duality.

1. improperly self-dual – the dual is equal to the mirror image
2. properly self-dual – the dual is equal to the polytope itself

Automorphism group of a chiral polytope



Let $\mathcal{P}$ be a chiral n-polytope, and fix a base flag Φ. Then

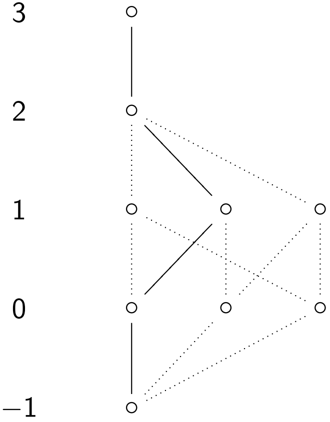
- $\Gamma(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$
- σ_i sends Φ to $\Phi^{i,i-1}$
- For $i < j$, the automorphism $\sigma_i \sigma_{i+1} \cdots \sigma_j$ has order 2.
- $\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$, where

$$\Gamma_I = \langle \sigma_i \cdots \sigma_j \mid i \leq j \text{ and } i{-}1, j \in I \rangle$$

$$(I \subseteq \{0, 1, \ldots, n-1\})$$

Automorphism group of a chiral polytope

Φ^1



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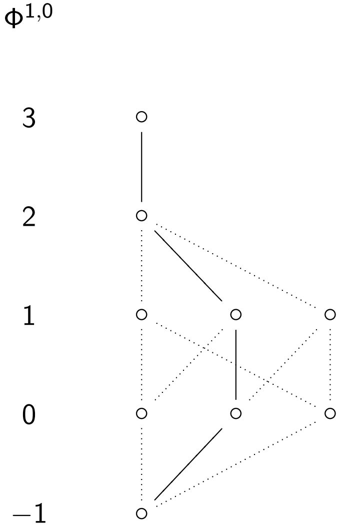

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Building a chiral polytope from a group

Given a group $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$, we can build a poset $\mathcal{P}(\Gamma)$ in a natural way. We set

- $\Gamma_0 = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle$
- $\Gamma_i = \langle \sigma_1, \ldots, \sigma_{i-1}, \sigma_i \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{n-1} \rangle$ $(1 \leq i \leq n-2)$
- $\Gamma_{n-1} = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle$

The faces of rank k are the cosets of Γ_k, and two cosets are incident if they intersect.

If Γ is "nice enough", then $\mathcal{P}(\Gamma)$ is a chiral polytope and $\Gamma(\mathcal{P}(\Gamma)) = \Gamma$.

Mixing groups

Given groups

$$\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$$

$$\Gamma' = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle,$$

we define $\alpha_i = (\sigma_i, \sigma'_i) \in \Gamma \times \Gamma'$ $(1 \leq i \leq n-1)$.

The mix of Γ and Γ' is defined to be

$$\Gamma \diamond \Gamma' = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle.$$

(The diagonal subgroup of the direct product)

If $\Gamma = G/N$ and $\Gamma' = G/N'$, then $\Gamma \diamond \Gamma' = G/(N \cap N')$.

Mixing polytopes

Let $\mathcal{P}$ and $\mathcal{Q}$ be chiral polytopes. We define the mix of $\mathcal{P}$ and $\mathcal{Q}$ (denoted $\mathcal{P} \diamond \mathcal{Q}$) to be the poset built from $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$.

Unfortunately,

1. $\mathcal{P} \diamond \mathcal{Q}$ may be regular instead of chiral.
2. $\mathcal{P} \diamond \mathcal{Q}$ may not even be a polytope!

Mixing and duality

Proposition

$(\mathcal{P} \diamond \mathcal{Q})^\delta = \mathcal{P}^\delta \diamond \mathcal{Q}^\delta$.

Corollary

$\mathcal{P} \diamond \mathcal{P}^\delta$ *is properly self-dual.*

Under what conditions is $\mathcal{P} \diamond \mathcal{P}^\delta$ a chiral polytope?

When is $\mathcal{P} \diamond \mathcal{P}^\delta$ chiral?

Theorem

Let

$$\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{P}^\delta) = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$$

$$\Gamma(\mathcal{P}) \diamond \Gamma(\overline{\mathcal{P}}) = \langle \beta_1, \ldots, \beta_{n-1} \rangle$$

If there is no epimorphism

$$\varphi : \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{P}^\delta) \to \Gamma(\mathcal{P}) \diamond \Gamma(\overline{\mathcal{P}})$$

that sends α_i to β_i, then $\mathcal{P} \diamond \mathcal{P}^\delta$ is chiral.

When is $\mathcal{P} \diamond \mathcal{P}^\delta$ chiral?

Let $\mathcal{P}$ be a finite chiral polyhedron of type $\{3, 7\}$ such that $\Gamma(\mathcal{P})$ is simple. Then

- $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{P}^\delta) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{P}^\delta)$
- $\Gamma(\mathcal{P}) \diamond \Gamma(\overline{\mathcal{P}}) = \Gamma(\mathcal{P}) \times \Gamma(\overline{\mathcal{P}})$

So $|\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{P}^\delta)| = |\Gamma(\mathcal{P}) \diamond \Gamma(\overline{\mathcal{P}})|$. Thus, the function sending α_i to β_i is only well-defined if it has a trivial kernel. But α_1 has order 21, while β_1 has order 3. So the function is not well-defined, and thus $\mathcal{P} \diamond \mathcal{P}^\delta$ is chiral.

When is $\mathcal{P} \diamond \mathcal{P}^\delta$ chiral?

Theorem

Let $\mathcal{P}$ be a finite chiral polytope of type $\{p_1, \ldots, p_{n-1}\}$. If $\Gamma(\mathcal{P})$ is simple and there is some i such that $p_i \neq p_{n-i}$, then $\mathcal{P} \diamond \mathcal{P}^\delta$ is chiral.

$$\begin{array}{cccccc} \{ & p_1, & p_2, & \ldots, & p_{n-2}, & p_{n-1} & \} \\ \{ & p_{n-1}, & p_{n-2}, & \ldots, & p_2, & p_1 & \} \end{array}$$

When is $\mathcal{P} \diamond \mathcal{P}^\delta$ a polytope?

Proposition

The mix of chiral polyhedra is a polyhedron.

Theorem

Let $\mathcal{P}$ be a finite chiral polyhedron of type $\{p, q\}$. Let $g = \gcd(p, q)$, and suppose that

$$|\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{P}^\delta)| < \frac{pq}{g^2} |\Gamma(\mathcal{P}) \diamond \Gamma(\overline{\mathcal{P}})|.$$

Then $\mathcal{P} \diamond \mathcal{P}^\delta$ is a properly self-dual chiral polyhedron.

When is $\mathcal{P} \diamond \mathcal{P}^\delta$ a polytope?

Theorem

If $\mathcal{P}$ is of type $\{p_1, p_2, \ldots, p_{n-1}\}$ and each p_i is relatively prime to p_{n-i}, then $\mathcal{P} \diamond \mathcal{P}^\delta$ is a polytope.

Corollary

If $\mathcal{P}$ is of type $\{p_1, p_2, \ldots, p_{n-1}\}$, each p_i is relatively prime to p_{n-i}, and $\Gamma(\mathcal{P})$ is simple, then $\mathcal{P} \diamond \mathcal{P}^\delta$ is a self-dual chiral polytope.

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(Of course, this is only possible when n is odd.)

When is $\mathcal{P} \diamond \mathcal{P}^\delta$ not a polytope?

Theorem

If $\mathcal{P}$ is of type $\{p, q, r\}$, with q odd and p coprime to r, then $\mathcal{P} \diamond \mathcal{P}^\delta$ is not a polytope.
(The group $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{P}^\delta)$ fails to have the "intersection property".)

Some generalizations

- $\mathcal{P} \diamond \overline{\mathcal{P}}^{\delta}$ is improperly self-dual
- The mix of a regular polytope with its dual is self-dual
- The mix of a regular polyhedron with its Petrie dual is self-Petrie

Some generalizations

- $\mathcal{P} \diamond \overline{\mathcal{P}}^{\delta}$ is improperly self-dual
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Thank you!