Nonorientable regular maps over linear fractional groups

M. Mačaj, Comenius University, Bratislava, Slovakia
joint work with G. Jones and J. Širáň

Fields Institute, Toronto
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Maps

A *map* $\mathcal{M}$ is a 2-cell embedding of a connected graph Γ into a compact surface S.

Map $\mathcal{M}$ is of *type* (k, m) if every vertex has valency k and every face has size m. Type (k, m) is *hyperbolic* if $1/k + 1/m < 1/2$.

As surfaces are *not oriented*, basic objects are *flags* (i.e., incident vertex-edge-face triples).

An automorphism ψ of Γ which can be extended into a self-homeomorphism of S is called a *map automorphism*.

A map $\mathcal{M}$ is called *regular* if it acts regularly on the set of flags.

Existence of regular maps

Theorem. *For any hyperbolic pair* (k, m) *there exist infinitely many regular* oriented maps *of type* (k, m).

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Problem 1 Show that there exist infinitely many *nonorientable* regular maps of type (k, m).

Problem 2 Find infinitely many solutions of Problem 1.

Nonorientable regular maps

Theorem. *Regular maps of type (k, m) on nonorientable surfaces are in one-to-one correspondence with groups having presentation*

$$G = \langle r, s;\ r^k = s^m = (rs)^2 = \cdots = 1\rangle \qquad (1)$$

such that m and k are true orders of r and s, respectively, and there exists an inner automorphism ψ of G inverting both r and s.

Remark 1 Without the automorphism ψ we have regular oriented maps.

Remark 2 If we allow ψ to be an arbitrary automorphism then we obtain regular maps.

Some notations

- K – an algebraic closure of $\mathbb{Z}_p$, p coprime to $2km$,
- ξ and η – primitive $2k$th and $2m$th root of unity in K,
- $D = -(\xi^2 + \xi^{-2} + \eta^2 + \eta^{-2})$,
- $R = \pm \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $S = \pm(\xi+\xi^{-1}) \begin{pmatrix} (\eta^{-1} - \eta)\xi^{-1} & D \\ 1 & (\eta - \eta^{-1})\xi \end{pmatrix}$ – elements of $PSL(2, K)$,
- $G(\xi, \eta)$ – subgroup of $PSL(2, K)$ generated by R and S.

Previous results

Proposition. Sah 1969

1. *Orders of* R*,* S *and* RS *in* $PSL(2, K)$ *are* k*,* m *and* 2*, respectively.*

2. *Every subgroup* G *of* $PSL(2, K)$ *with presentation* (1) *is conjugate to some* $G(\xi, \eta)$*.*

Previous results

Proposition. Conder, Potočnik, Širáň 2008 *Let* $D \neq 0$*. Then*
1) There exists an integer $e = e(k,m,p)$ *such that* $G(\eta,\xi)$ *is isomorphic either to* $PSL(2,p^e)$ *or* $PGL(2,p^{e/2})$ *and which case occurs depends only on* k*,* m*, and* p*.*
2) Whether $G(\eta,\xi)$ *has an inner automorphism* ψ *inverting both* R *and* S *depends only on* k*,* m*,* p*, and* D*. In particular such* ψ *exists whenever* $G(\eta,\xi) \equiv PGL(2,p^{e/2})$*.*

Theorem. Širáň 2010 *If* $2|km$ *then there exist infinitely many nonorientable regular maps of type* (k,m) *over linear fractional groups.*

Previous results

Proposition. *Let both k and m be odd. Then*

1) D never equals 0.

2) $G(\xi, \eta)$ is always isomorphic to $PSL(2, p^e)$.

3) $G(\xi, \eta)$ has an involutory inner automorphism inverting both R and S iff D is a square in $GF(p^e)$.

Algebraic numbers

Let F be a number field of degree $[F : \mathbb{Q}] = n$, let O be the ring of algebraic integers in F, and let $\sigma_1,\ \sigma_2, \ldots,\ \sigma_n$ be all injective homomorphisms $F \to \mathbb{C}$. Recall that the *norm* of $c \in F$ is defined by $N(c) = \prod \sigma_i(c)$.

Lemma. *For any $o \in O$ we have $N(o) \in \mathbb{Z}$. Moreover $N(o) = \pm 1$ iff o is a unit in O.*

Lemma. *For any $o \in O$ and prime p there exists a maximal ideal I containing o with $|O/I| = p^d$ for some d iff $p | N(o)$.*

Computing in $\mathbb{C}$

Let (k, m) be a hyperbolic pair with km odd. Let α and β be primitive $2k$-th and $2m$-th roots of unity in $\mathbb{C}$, respectively, let $A = -(\alpha^2 + \alpha^{-2} + \beta^2 + \beta^{-2})$ and let O be the ring of algebraic integers of $\mathbb{Q}(\alpha, \beta)$.

Lemma. *If $\alpha \neq \beta$ then A is a unit in O and if $\alpha = \beta$ then $|N(A)|$ is a power of two. The number $A - n^2$ is not a unit in O for any integer $n > 2$.*

Back to finite fields

For any $n > 2$ let $I = I_n$ be a maximal ideal in O containing $A - n^2$, let $p = p_n$ be the characteristic of the field O/I and let $\xi = \alpha + I$, $\eta = \beta + I$ and $D = A + I$.

Lemma. *If n is coprime to $N(A)$ then $D = -(\xi^2 + \xi^{-2} + \eta^2 + \eta^{-2})$ is a nonzero square in $\mathbb{Z}_p$ and p is coprime to n. Moreover, if p is coprime to $2km$ then ξ and η are primitive $2m$th and $2k$th roots of unity in O/I.*

Main result

Theorem. *For any hyperbolic pair* (k, m) *there exists infinitely many nonorientable regular maps over linear fractional groups.*

Proof. It suffices to assume that both k and m are odd. Let $n_1 = 2km$ and let $n_j = 2km \prod_{i=1}^{j-1} p_i$ for $j > 1$. By the previous lemma all p_j's are distinct and there exists a nonorientable regular map over a linear fractional group in characteristic p_j for any j.

Open problems

Problem 3 For a given p determine all pairs (k, m) such that there exists a nonorientable regular map of type (k, m) over a linear fractional group in characteristic p.

Problem 4 For a given hyperbolic pair (k, m) determine all p's such that there exists a nonorientable regular map of type (k, m) over a linear fractional group in characteristic p.

Lemma. *If both k and m are powers of primes congruent to $3 \mod 4$ and p is congruent to $1 \mod 8$ then there exists a nonorientable regular map of type (k, m) over a linear fractional group in characteristic p.*

Thank You