To hold a convex body by a circle

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1 To hold a ball by a net/box/cage

Theorem (A. S. Besicovitch 1957). The total length of a net that holds a unit ball is greater than 3π, and 3π is the lower bound.

Theorem (A. S. Besicovitch and H. G. Eggleston 1957). The total length of the edges of a convex polyhedron that contains a unit ball is at least 24 and 24 is attained only by a cube. (This settled a conjecture of L. Fejes Tóth.)

Coxeter posed a problem that asks the total length of the edges of a cage (1-skeleton of a convex polyhedron) that can holds a unit ball, and his conjecture was $9\sqrt{3} \approx 15.5884$ (the case of a regular triangular prism with all edges of length $\sqrt{3}$).

Theorem (A. S. Besicovitch 1963, O. Aberth 1963). The total length of a cage that can hold a unit ball is greater than $\gamma = \frac{8}{3}\pi + 2\sqrt{3} \approx 11.8417$, and γ is the lower bound.

Besicovitch constructed a cage of total length $\gamma + \varepsilon$ that can hold a unit ball (hence Coxeter's conjecture is false), and Aberth proved that γ is the lower bound.

2 To hold a convex body by a circle

- A circle Γ is said to hold a convex body K if
  (1) $\Gamma \cap int(K) = \emptyset$, $conv(\Gamma) \cap int(K) \neq \emptyset$, and
  (2) it is impossible to move Γ (or K) with keeping $\Gamma \cap int(K) = \emptyset$ until K goes far away from Γ.
- A convex body is called **circle-free** if no circle can hold the convex body. For example, every ball is circle-free.

Theorem (T. Zamfirescu 1995). The set of circle-free convex bodies in $\mathbb{R}^3$ is a nowhere dense subset of the set of all convex bodies in $\mathbb{R}^3$ with Haussdorff metric. Thus, most convex bodies can be held by circles.

Example. 1. Every circular cone is circle-free.
2. Every circular cylinder is circle-free.

Theorem (Maehara 2010). For every planar compact convex set X and a ball B in $\mathbb{R}^3$, the Minkowski sum $X + B$ is circle-free.

Problem. Is it true that for every circle-free convex body K, the set $K + B$ is also circle-free?

• The **slope** ρ of a regular pyramid is defined by $\rho = \frac{\text{height}}{\text{circum-radius of the base}}$.

Theorem (Maehara 2010). Every regular pyramid with slope $\rho \geq 1$ can be held by a circle. Moreover, for every $0 < \varepsilon < 1$, there is a circle-free regular pyramid with slope $\rho = 1 - \varepsilon$.

Theorem (Tanoue 2009, Maehara 2010). A regular pyramid with equilateral triangular base can be held by a circle if and only if $\rho > \sqrt{\frac{3\sqrt{17}-5}{32}} \approx 0.4799$. (Yuichi Tanoue proved the if part.)

Theorem (Maehara 2010). A regular pyramid with square base can be held by a circle if and only if $\rho > \sqrt{\frac{\sqrt{33}-3}{4}} \approx 0.828$.

The great pyramid of Giza has base-edge $230m$ and height $140m$. Since $140\sqrt{2}/230 \approx 0.860 > 0.828$, it can be held by a circle.

3 Sizes of circles for Platonic solids

Theorem (Itoh, Tanoue, Zamfirescu 2006). A circle of diameter d can hold a regular tetrahedron of unit edge if and only if $\frac{1}{\sqrt{2}} \leq d < \phi_t \approx 0.8956$, where ϕ_t is the minimum value of $\frac{2(x^2-x+1)}{\sqrt{3x^2-4x+4}}$.

Theorem (Maehara 2010). A circle of diameter d can hold a unit cube if and only if $\sqrt{2} \leq d < \phi_c \approx 1.53477$, where ϕ_c is the minimum value of $\frac{\sqrt{2}(x^2+2)}{\sqrt{x^2+2x+3}}$.

Theorem (Maehara 2010). A circle of diameter d can hold a regular octahedron of unit edge if and only if $1 \leq d < \phi_o \approx 1.1066$, where ϕ_o is the minimum value of $\frac{2(x^2+1)}{\sqrt{3x^2+2x+3}}$.

Y. Tanoue obtained the same result independently.

Problem. Find similar results for the regular dodecahedron and the regular icosahedron.

4 String/round-hole/holding-circle

• A loop of string can change its shape freely, but its length never changes.

Theorem (A. Fruchard 2009). A loop of string winding on a convex body can slip out of the convex body.

By this theorem, it seems impossible to hold a convex body by *flex-cuff*. (A flex-cuff is a flexible handcuff made of nylon or plastic.)

Theorem (Itoh, Tanoue, Zamfirescu 2006). The minimum diameter of a round-hole in a plane through which a tetrahedron of unit edge can pass is $\phi_t \approx 0.8956$.

Remark. The minimum diameter of a round-hole in a plane through which a unit cube can pass is $\sqrt{2}$.

Example (T. Zamfirescu). For every $d > \varepsilon > 0$, there is a convex body that can pass through a round-hole of diameter ε, and yet can be held by a circle of diameter d.

References

[1] O. Aberth, An isoperimetric inequality for polyhedra and its application to an extremal problem, *Proc. London Math. Soc.* (3) 13 (1963) 322–336.

[2] A. S. Besicovitch, A net to hold a sphere, *Math. Gaz.* 41 (1957) 106–107.

[3] A. S. Besicovitch, A cage to hold a unit-sphere, *Proc. Sympos. Pure Math. Vol. VII Amer. Math. Soc.* (1963) 19–20.

[4] A. S. Besicovitch, H. G. Eggleston, The total length of the edges of a polyhedron, *Quart. J. Math. Oxford Ser. (2)* 8 (1957) 172–190.

[5] A. Fruchard, Some remarks about strings and rigid curves holding convex bodies, preprint 2009.

[6] J. Itoh, Y. Tanoue, T. Zamfirescu, Tetrahedra passing through a circular or square hole, *Rend. Circ. Mat. Palermo (2) Suppl. No. 77* (2006) 349–354.

[7] J. Itoh, T. Zamfirescu, Simplices passing through a hole, *J. Geom.* 83 (2005) 65–70.

[8] H. Maehara, On the diameter of a circle to hold a cube, to appear.

[9] H. Maehara, To hold a regular pyramid by a circle, preprint.

[10] Y. Tanoue, Regular triangular pyramids held by a circle, *Journal of Geometry* 94 (2009) 151–157.

[11] T. Zamfirescu, How to hold a convex body, *Geometriae Dedicata* 54 (1995) 313–316.