The p-adic Langlands program in the ordinary case and fundamental algebraic representations

Christophe Breuil

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AVERTISSEMENT : Il s'agit des notes des 3 exposés que j'ai donnés à l'Institut Fields de Toronto les 18, 19 et 20 avril 2012.

Contents

1 Lecture 1 (April 18) 2

2 Lecture 2 (April 19) 11

3 Lecture 3 (April 20) 19

1 Lecture 1 (April 18)

I first thank the organizers of the thematic program on Galois representations for inviting me to give these lectures. Their material is based on joint recent work with *Florian Herzig*.

In all the talks, p is a prime number.

As required by the Fields Institute, this first lecture is of "Colloquium style", and thus doesn't concern experts (except may-be the very end).

1.1 Classical local Langlands correspondence (GL_n)

Let $\mathbb{Q}_p$ be the field of p-adic numbers, $\overline{\mathbb{Q}_p}$ an algebraic closure of $\mathbb{Q}_p$, $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ the corresponding Galois group (the group of automorphisms of $\overline{\mathbb{Q}_p}$ fixing the elements of $\mathbb{Q}_p$) and $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ the Weil group of $\mathbb{Q}_p$. Recall that $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is a dense subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ defined as the inverse image of $\mathbb{Z}$ in $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$:

$$\begin{array}{ccc} \mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) & \hookrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \hookrightarrow & \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \end{array}$$

where the horizontal bottom map sends $n \in \mathbb{Z}$ to $[x \mapsto x^{p^n}] \in \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$.

Theorem 1.1.1 (Harris-Taylor, Henniart)**.** *There is a "natural" bijection between the following two sets:*

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible smooth} \\ \text{representations } \pi \text{ of} \\ \mathrm{GL}_n(\mathbb{Q}_p) \text{ over } \mathbb{C} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ n\text{-diml semi-simple} \\ \text{smooth representations} \\ \rho \text{ of } \mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \text{ over } \mathbb{C} \\ + \text{ a nilpotent operator } N \end{array} \right\}.$$

The nilpotent operator N on the underlying space of ρ (on the right hand side) is subject to a certain commutation relation with ρ that we skip. We need to explain what "smooth" means on both sides. A representation π of $\mathrm{GL}_n(\mathbb{Q}_p)$ over any vector space is smooth if every vector is fixed by a sufficiently small open subgroup H of $\mathrm{GL}_n(\mathbb{Z}_p)$. A finite dimensional representation (ρ, N) of $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over any vector space is smooth if its restriction to the inertia subgroup of $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, that is the kernel of the above map $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{Z}$, becomes trivial in restriction to an open subgroup of this inertia subgroup (equivalently the inertia acts through a finite quotient).

Remark 1.1.2. In fact, all π as in the theorem are moreover *admissible*, that is, the invariant subspaces π^H are finite dimensional for every H as above.

1.2 Rational local Langlands correspondence

The local langlands correspondence doesn't use the transcendental topology of $\mathbb{C}$: we can thus replace $\mathbb{C}$ by any algebraically closed field of characteristic 0, e.g. the algebraic closure $\overline{\mathbb{Q}_\ell}$ for ℓ a prime number distinct from p (and actually also for $\ell = p$, but for reasons which become clear in the sequel, we wish to avoid p here).

One can also normalize this correspondence so that it is *rational*, that is, it commutes with automorphisms of the coefficient field $\overline{\mathbb{Q}_\ell}$. Using this normalization, we then have for any finite extension E of $\mathbb{Q}_\ell$:

$$(\rho, N) \text{ defined over E} \quad \Longrightarrow \quad \pi = \pi(\rho, N) \text{ also defined over } E.$$

Example 1.2.1. Here is what will be for us the most important example in these lectures. Recall first that any character of $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ can be seen as a character of $\mathbb{Q}_p^\times$ via the isomorphism $\mathbb{Q}_p^\times \xrightarrow{\sim} \mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{\mathrm{ab}}$ given by local class field theory, where $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{\mathrm{ab}}$ is the maximal abelian quotient of $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ (through which any character of $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ factorizes).

We consider $(\rho, N) = (\rho, 0)$ with $\rho := \mathrm{diag}(\chi_1, \cdots, \chi_n)$ where the χ_i are E-valued smooth characters of $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ that satisfy the genericity assumption $\chi_i\chi_j^{-1} \notin \{1, |\cdot|, |\cdot|^{-1}\}$ for $i \neq j$. Here $|\cdot|$ is the ℓ-adic cyclotomic character given on $\mathbb{Q}_p^\times$ by $|x| = p^{-\mathrm{val}(x)}$ where $\mathrm{val}(p^i x) := i$ if $x \in \mathbb{Z}_p^\times$. In that case we have:

$$\pi(\rho) = \pi(\rho, 0) = \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \big(\chi_1|\cdot|^{1-n} \otimes \chi_2|\cdot|^{2-n} \otimes \cdots \otimes \chi_n\big).$$

I explain this representation of $\mathrm{GL}_n(\mathbb{Q}_p)$: $B^-(\mathbb{Q}_p) \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is the subgroup of lower triangular matrices, $\chi_1|\cdot|^{1-n} \otimes \chi_2|\cdot|^{2-n} \otimes \cdots \otimes \chi_n$ is seen as an E valued character of $B^-(\mathbb{Q}_p)$ via $B^-(\mathbb{Q}_p) \twoheadrightarrow T(\mathbb{Q}_p)$ ($T(\mathbb{Q}_p)$ = diagonal matrices) and:

$$(\chi_1|\cdot|^{-(n-1)} \otimes \cdots \otimes \chi_n)(\mathrm{diag}(x_i)) := \chi_1(x_1)|x_1|^{1-n}\chi_2(x_2)|x_1|^{2-n} \cdots \chi_n(x_n),$$

the underlying space of $\pi(\rho)$ is the E-vector space of all locally constant functions $f : \mathrm{GL}_n(\mathbb{Q}_p) \to E$ such that for all $b \in B^-(\mathbb{Q}_p)$ and all $g \in \mathrm{GL}_n(\mathbb{Q}_p)$:

$$f(bg) = (\chi_1|\cdot|^{1-n} \otimes \cdots \otimes \chi_n)(b)f(g),$$

and finally the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ is given for $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ by $(g \cdot f)(g') := f(g'g)$.

Such a representation is called a *principal series*. In the rest of this lecture, we denote by θ the character which sends $\mathrm{diag}(x_i)$ to $x_1^{n-1}x_2^{n-2} \cdots x_n$ (it is an algebraic character) and we can write the above representation as:

$$\pi(\rho) = \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \chi_1 \otimes \cdots \otimes \chi_n \cdot (|\cdot|^{-1} \circ \theta).$$

Now comes the crucial fact: *the order of the χ_i doesn't matter in the definition of $\pi(\rho)$, that is, all the above principal series for all permutations of the χ_i are isomorphic!*

As we will see, this crucial fact will completely break down in the (continuous) p-adic world and this will be an essential point in these lectures.

1.3 ℓ-adic local Langlands correspondence, $\ell \neq p$

First, recall that a smooth representation (ρ, N) defined over E is the same thing as an *ℓ-adic* representation ρ of $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, that is, a continuous E-linear representation of the topological group $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on a finite dimensional E-vector space (endowed with the p-adic topology coming from any isomorphism with E^n). This is a result of Deligne, based on a famous theorem of Grothendieck called the *ℓ-adic monodromy theorem*.

Let $\mathcal{O}_E$ be the ring of integers in E, by an *$\mathcal{O}_E$-lattice* in an E-vector space (of enumerable dimension), we mean a $\mathcal{O}_E$-module that generates the E-vector space and doesn't contain any E-line. We say that a (smooth) representation (ρ, N) (resp. π) is *integral* if its underlying E-vector space contains an $\mathcal{O}_E$-lattice which is invariant under $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and N (resp. under $\mathrm{GL}_n(\mathbb{Q}_p)$). A representation (ρ, N) is integral if and only if the corresponding ℓ-adic representation ρ (uniquely) extends from $\mathrm{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. A representation π is integral if and only if has an invariant (ℓ-adic) norm $\|\cdot\|$ (i.e. $\|g \cdot v\| = \|v\|$ for all $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ and all v in the underlying E-vector space).

Theorem 1.3.1 (Vignéras)**.** *(i) The rational local Langlands correspondence respects integrality.*

(ii) Up to equivalence there is only one invariant norm on an integral π.

One can then define an *ℓ-adic correspondence* as follows starting from any n-dimensional ℓ-adic representation ρ of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E:

$$\rho \rightsquigarrow (\rho, N) \rightsquigarrow (\rho^{\mathrm{ss}}, N) \rightsquigarrow \pi(\rho^{\mathrm{ss}}, N) \rightsquigarrow \Pi(\rho)$$

where ss means semi-simplified and where $\Pi(\rho)$ is the completion of $\pi(\rho, N)$ with respect to its unique equivalence class of invariant norms. In particular $\Pi(\rho)$ is a Banach representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E and we can choose a norm on it so that the corresponding unit ball is stable under $\mathrm{GL}_n(\mathbb{Q}_p)$: we say $\Pi(\rho)$ is *unitary.* Moreover, $\Pi(\rho)$ is an absolutely topologically irreducible representation and contains $\pi(\rho^{\mathrm{ss}}, N)$ as a smooth (dense) subrepresentation.

A further theorem of Vignéras states that the subspace of *smooth* vectors in $\Pi(\rho)$, i.e. the subspace of vectors on which a sufficiently small open subgroup $H \subset \mathrm{GL}_n(\mathbb{Z}_p)$ acts trivially, is in fact exactly the smooth subrepresentation $\pi(\rho^{\mathrm{ss}}, N)$ (no new smooth vectors appear when completing).

All this shows that this ℓ-adic correspondence doesn't contain anything more than the usual classical (rational) local langlands correspondence: we have just managed to add some ℓ-adic topology on both sides of the classical correspondence. Although usually one would rather do the converse (!), these considerations are important for the analogy with the p-adic world.

Example 1.3.2. Let us consider again the above example: $(\rho, N) = (\rho, 0)$ with $\rho := \mathrm{diag}(\chi_1, \cdots, \chi_n)$ and $\chi_i\chi_j^{-1} \notin \{1, |\cdot|, |\cdot|^{-1}\}$ for $i \neq j$. One finds:

$$\Pi(\rho) = \Big(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \chi_1 \otimes \cdots \otimes \chi_n \cdot (|\cdot|^{-1} \circ \theta)\Big)^{\mathcal{C}^0}$$

where the representation on the right hand side is defined exactly as in the previous example except that one takes continuous functions $f : \mathrm{GL}_n(\mathbb{Q}_p) \to E$ instead of locally constant functions. An invariant norm on (the underlying vector space of) $\Pi(\rho)$ is simply given by:

$$\|f\| = \mathrm{Max}_{g\in \mathrm{GL}_n(\mathbb{Q}_p)}|f(g)|_\ell$$

where $|\cdot|_\ell$ is the usual ℓ-adic absolute value on E (defined by $|x|_\ell := \ell^{-\mathrm{val}_\ell(x)}$ where val_ℓ is normalized by $\mathrm{val}_\ell(\ell) = 1$).

Remark 1.3.3. Emerton has a refinement of this ℓ-adic correspondence producing a Banach representation $\Pi(\rho)$ which is not always topologically irreducible (although it is most of the time and then coincides with the above $\Pi(\rho)$, for instance in the above example). The advantage of Emerton's $\Pi(\rho)$ is that it is really this representation which occurs in suitable ℓ-adic completions of cohomology spaces.

1.4 p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

For $\mathrm{GL}_2(\mathbb{Q}_p)$, there is an analogue of the above ℓ-adic correspondence $\rho \rightsquigarrow \Pi(\rho)$ where now E is a finite extension of $\mathbb{Q}_p$, ρ is a continuous E-linear representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on a 2-dimensional E-vector space (i.e. a (2-dimensional) p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$) and $\Pi(\rho)$ is a unitary Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ which is topologically of finite length. This correspondence was established over the past ten years by the work of many people (including Colmez, Emerton, Kisin, Paskunas, myself, ...). We give below an explicit example, which is all we need to know in these lectures.

This p-adic Langlands correspondence has several serious complications with respect to the ℓ-adic correspondence. Let me just mention two of them:

(i) Even when $\Pi(\rho)$ is topologically irreducible, it is *not true* that the smooth vectors are always dense in $\Pi(\rho)$ (as in the ℓ-adic local Langlands correspondence). In fact they are most of the time 0! One could refine this by looking for *locally algebraic vectors* in $\Pi(\rho)$, that is, vectors on which a sufficiently small H acts through the restriction of a finite dimensional algebraic representation of $\mathrm{GL}_n(\mathbb{Z}_p)$, but in fact here again, this subspace can be zero. To have a nonzero dense subspace, by a theorem of Schneider and Teitelbaum one has to consider *locally analytic* vectors, which is much more complicated to define (and won't be used in these lectures).

(ii) One can't only restrict to (absolutely) topologically irreducible $\Pi(\rho)$ (as in the ℓ-adic local Langlands correspondence) for the following obvious reason. Let us consider our favourite example: $\rho = \mathrm{diag}(\chi_1, \chi_2)$ where the $\chi_i : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to E^\times$ are p-adic characters such that $\chi_1\chi_2^{-1} \notin \{1, \varepsilon, \varepsilon^{-1}\}$ with ε the p-adic cyclotomic character. Then one has *two* natural topologically irreducible Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to ρ, namely:

$$\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0} \text{ and } \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0}$$

(where again $\mathcal{C}^0$ means continuous functions $\mathrm{GL}_2(\mathbb{Q}_p) \to E$). These two representations are not at all isomorphic. So what can be done? The simple idea is: take both of them! Indeed, in that case, one has:

$$\Pi(\rho) = \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0} \oplus \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0}.$$

But now, in this p-adic world, we can also have a ρ which is a non-split extension of, say, χ_2 by χ_1 (such a non-split extension doesn't occur in the ℓ-adic world due to our assumption on the χ_i):

$$\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

and moreover this non-split extension is unique under our hypothesis on the χ_i. Correspondingly, in that case one has:

Theorem 1.4.1. *There is a unique non-split extension of unitary Banach representations:*

$$0 \to \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0} \to \Pi(\rho) \to \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0} \to 0.$$

For ρ absolutely irreducible, $\Pi(\rho)$ is also absolutely topologically irreducible. In these talks, I will only consider reducible ρ's.

1.5 The functor of Colmez

I now describe a crucial ingredient which is used in the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Although, strictly speaking, I won't use it, it is quite important for these lectures to have it in mind.

Now, in the above p-adic correspondence for (generic) reducible ρ's, we see that a length 2 split (resp. non-split) ρ goes to a length 2 split (resp. non-split) $\Pi(\rho)$. So something functorial seems to be going on. Indeed, Colmez could define a covariant exact functor:

$$\left\{\begin{array}{c}\text{finite length admissible}\\ \text{unitary Banach}\\ \text{representations of}\\ \mathrm{GL}_n(\mathbb{Q}_p)\text{ over } E\end{array}\right\} \xrightarrow{F} \left\{\begin{array}{c}\text{finite dimensional}\\ p\text{-adic representations}\\ \rho \text{ of } \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)\text{ over } E\end{array}\right\}.$$

I need to explain the word "admissible" for Banach representations. It was defined by Schneider and Teitelbaum (from the work of Lazard). In our context, the fastest definition is the following: a unitary Banach representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E is admissible if $\Pi^0 \otimes_{\mathcal{O}_E} \mathcal{O}_E/\varpi_E$ is a (smooth) admissible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over the finite field $\mathcal{O}_E/\varpi_E$. Here ϖ_E is any uniformizer of $\mathcal{O}_E$ and Π^0 is any invariant unit ball in the Banach Π. Note that the $\mathrm{GL}_n(\mathbb{Q}_p)$-representation $\Pi^0 \otimes_{\mathcal{O}_E} \mathcal{O}_E/\varpi_E$ is trivially checked to be smooth (see section 1.1) and admissibility is thus meant in the sense of Remark 1.1.2.

Colmez proved that $F(\Pi(\rho)) = \rho$. In our reducible example, we have more precisely:

$$\begin{aligned} F\left(\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0}\right) &= \chi_1 \\ F\left(\left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1 \cdot (\varepsilon^{-1} \circ \theta)\right)^{\mathcal{C}^0}\right) &= \chi_2. \end{aligned}$$

In fact, the functor F doesn't directly produce a p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ from a unitary Banach representation Π of $\mathrm{GL}_n(\mathbb{Q}_p)$. Instead it rather produces what is called an *étale* (φ, Γ)*-module* (a structure defined by Fontaine) which is known by a theorem of Fontaine to be the same thing as a p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

1.6 Serre weights

The previous p-adic correspondence $\rho \mapsto \Pi(\rho)$ for $\mathrm{GL}_2(\mathbb{Q}_p)$ also works in characteristic p and gives a correspondence $\overline{\rho} \mapsto \Pi(\overline{\rho})$ where $\overline{\rho}$ is a 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over $k_E := \mathcal{O}_E/\varpi_E$ and $\Pi(\overline{\rho})$ is a finite length smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k_E.

Example 1.6.1. If $\overline{\rho} = \overline{\chi}_1 \oplus \overline{\chi}_2$ with $\overline{\chi}_1\overline{\chi}_2^{-1} \notin \{1, \omega, \omega^{-1}\}$ (where ω is the reduction mod p of ε), then:

$$\Pi(\overline{\rho}) = \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \overline{\chi}_1 \otimes \overline{\chi}_2 \cdot (\omega^{-1} \circ \theta) \oplus \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \overline{\chi}_2 \otimes \overline{\chi}_1 \cdot (\omega^{-1} \circ \theta)$$

where the principal series are smooth, that is, defined with locally constant functions $f : \mathrm{GL}_2(\mathbb{Q}_p) \to k_E$.

But it turns out that in this char. p setting, one can attach to $\overline{\rho}$ a piece of information which is much simpler than $\Pi(\overline{\rho})$ and still significant: a finite set of *Serre weights*.

Definition 1.6.2. A Serre weight for $\mathrm{GL}_n(\mathbb{F}_p)$ is an irreducible representation of $\mathrm{GL}_n(\mathbb{F}_p)$ over k_E.

Any Serre weight for $\mathrm{GL}_n(\mathbb{F}_p)$ is absolutely irreducible and defined over $\mathbb{F}_p$. Serre weights for $\mathrm{GL}_2(\mathbb{F}_p)$ are given by:

$$\left(\mathrm{Sym}^{a_1-a_2}\, k_E^2\right) \otimes_{k_E} \det{}^{a_2}$$

where a_i are integers such that $0 \leq a_1 - a_2 \leq p-1$ and where $\mathrm{GL}_2(\mathbb{F}_p)$ acts in the obvious way on the canonical basis of k_E^2. Note that, since $\mathrm{Ker}(\mathrm{GL}_n(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_p))$ is a pro-p-group, the Serre weights for $\mathrm{GL}_n(\mathbb{F}_p)$ are also the irreducible representations of $\mathrm{GL}_n(\mathbb{Z}_p)$ over k_E.

Definition 1.6.3. The Serre weights of $\overline{\rho}$ (2-diml over k_E) is the set of Serre weights (up to isomorphism) that appear in the socle of $\Pi(\overline{\rho})|_{\mathrm{GL}_2(\mathbb{Z}_p)}$.

Recall that the socle means the maximal semisimple subrepresentation. it follows from the admissibility of $\Pi(\overline{\rho})$ that the set of Serre weights of $\overline{\rho}$ is always finite (in fact generically it has cardinality 1 or 2).

Example 1.6.4. If $\overline{\rho} = \overline{\chi}_1 \oplus \overline{\chi}_2$ with $\overline{\chi}_1\overline{\chi}_2^{-1} \notin \{1, \omega, \omega^{-1}\}$ we find 2 Serre weights as the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of each principal series in $\Pi(\overline{\rho})$ is irreducible. If $\overline{\rho}$ is a non-split extension of $\overline{\chi}_2$ by $\overline{\chi}_1$, we find one Serre weight.

What is the point of looking at Serre weights when we have $\Pi(\overline{\rho})$? This is the following: although we don't know $\Pi(\overline{\rho})$ when we deal with $\mathrm{GL}_n(\mathbb{Q}_p)$ and $n > 2$, we *do know*, at least conjecturally and for many $\overline{\rho}$, what the Serre weights of $\overline{\rho}$ should be (this follows from work of Buzzard-Diamond-Jarvis, Herzig, Gee, Schein and others). And it turns out that knowing the set of Serre weights of $\overline{\rho}$ already gives a *strong* input on what $\Pi(\overline{\rho})$ should look like.

1.7 $\mathrm{GL}_n(\mathbb{Q}_p)$ and fundamental algebraic representations

Now, what we would like to do is extend the correspondences $\rho \mapsto \Pi(\rho)$ and $\overline{\rho} \mapsto \Pi(\overline{\rho})$ from $\mathrm{GL}_2(\mathbb{Q}_p)$ to $\mathrm{GL}_n(\mathbb{Q}_p)$ (and from there to more general reductive groups). In particular, in view of our previous examples, we would like to understand what $\Pi(\rho)$ and $\Pi(\overline{\rho})$ look like in the case where ρ and $\overline{\rho}$ are upper triangular:

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to B(E) \subset \mathrm{GL}_n(E), \quad \overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to B(k_E) \subset \mathrm{GL}_n(k_E)$$

where B is the Borel subgroup of upper triangular matrices.

Let us assume for one moment that we have such representations $\Pi(\rho)$ and $\Pi(\overline{\rho})$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ and also a covariant exact functor F analogous to the one of section 1.5. In particular, we thus have $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-representations $F(\Pi(\rho))$ and $F(\Pi(\overline{\rho}))$ associated to ρ and $\overline{\rho}$. The basic question we address now is:

Can we guess what $F(\Pi(\rho))$ and $F(\Pi(\overline{\rho}))$ should be?

Hint 1: When $n = 2$, we know that $F(\Pi(\rho)) = \rho$ and $F(\Pi(\overline{\rho})) = \overline{\rho}$.

Hint 2: When $n \geq 2$ is arbitrary and $\rho = \mathrm{diag}(\chi_1, \cdots, \chi_n)$ with $\chi_i\chi_j^{-1} \notin \{1, \varepsilon, \varepsilon^{-1}\}$ for $i \neq j$, it is highly probable that, as in the $n = 2$ case, $\Pi(\rho)$ will contain as a direct summand the direct sum of $n!$ principal series, namely all the principal series:

$$I(\rho)_w := \Big(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} w^{-1}(\chi) \cdot (\varepsilon^{-1} \circ \theta)\Big)^{\mathcal{C}^0}$$

where $\chi := \chi_1 \otimes \cdots \otimes \chi_n$, w is an element of the Weyl group of GL_n, that is, a permutation on $\{1, \cdots n\}$, and $w^{-1}(\chi) := \chi_{w(1)} \otimes \cdots \otimes \chi_{w(n)}$. All these principal series can be proved to be non-isomorphic. Thus $F(\Pi(\rho))$ should contain as a direct summand at least $\oplus_w F(I(\rho)_w)$.

Hint 3: Schneider and Vignéras have defined a candidate for the functor F in a quite general setting. Unfortunately, almost no explicit example is known of the value of their functor applied to a representation of another group than $\mathrm{GL}_2(\mathbb{Q}_p)$. Nevertheless, from their work, it seems natural to expect that we should have for $F(I(\rho)_w)$ a 1-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, and even more precisely $F(I(\rho)_w) = \chi_{w(1)}^{n-1}\chi_{w(2)}^{n-2} \cdots \chi_{w(n-1)}$. So, from Hint 2, we should have $\oplus_w \chi_{w(1)}^{n-1}\chi_{w(2)}^{n-2} \cdots \chi_{w(n-1)}$ appearing as a direct summand of $F(\Pi(\rho))$. In particular we see that $F(\Pi(\rho))$ should be different from ρ when $n > 2$.

Hint 4: The same considerations hold of course in characteristic p, and this is indeed compatible with the Serre weights of $\overline{\rho}$: for a generic diagonal $\overline{\rho}$, the $n!$ Serre weights corresponding to the $\mathrm{GL}_n(\mathbb{Z}_p)$-socles of the $n!$ principal series

$I(\overline{\rho})_w$ are all Serre weights of $\overline{\rho}$. *But*, as soon as $n > 2$, it is expected that *other* Serre weights should also be there. Going back to characteristic 0, this suggests that $\oplus_w \chi_{w(1)}^{n-1}\chi_{w(2)}^{n-2}\cdots\chi_{w(n-1)}$ should be a *strict* direct summand of $F(\Pi(\rho))$ if and only if $n > 2$.

Hint 5: Although we don't consider this case in this lecture, if L is a finite unramified extension of $\mathbb{Q}_p$ and if $\overline{\rho}$ is a continuous sufficiently generic 2-dimensional semi-simple representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ over k_E, considerations of Serre weights again quite strongly suggest that we should have for $F(\Pi(\overline{\rho}))$ the *tensor induction* from $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ of $\overline{\rho}$, that is, the tensor product of all the conjugates of $\overline{\rho}$ under $\mathrm{Gal}(L/\mathbb{Q}_p)$.

So the idea is to find for $F(\Pi(\rho))$ a representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ that is functorial in ρ and interpolates all the above hints. And there is indeed (at least) one, which is:

$$F(\Pi(\rho)) \overset{?}{\simeq} \wedge_E^1\rho \otimes_E \wedge_E^2\rho \otimes_E \cdots \otimes_E \wedge_E\rho^{n-1}.$$

It also has the nice advantage that it can be generalized to more general reductive groups than GL_n. Indeed, the algebraic representations $\wedge^i$ are the so-called *fundamental algebraic representations* of the algebraic group GL_n, and such fundamental representations exist (at least) for any split connected reductive algebraic group such that its dual has a connected center.

In the next lecture, I will start by studying properties of the tensor product of these fundamental algebraic representations. Then, I will show that the "ordinary part" of this tensor product suggests the definition of a Banach representation $\Pi(\rho)^{\mathrm{ord}}$ (resp. a smooth representation $\Pi(\overline{\rho})^{\mathrm{ord}}$) of $\mathrm{GL}_n(\mathbb{Q}_p)$ which hopefully should be the maximal subrepresentation of the unknown $\Pi(\rho)$ (resp. of the unknown $\Pi(\overline{\rho})^{\mathrm{ord}}$) such that its irreducible constituents are all constituents of principal series.

2 Lecture 2 (April 19)

(This lecture is no more of "colloquium style".)

In the two remaining lectures, I denote by $G/\mathbb{Q}_p$ a split connected reductive algebraic group such that both G and its dual $\widehat{G}$ have a connected center (e.g. $G = \mathrm{GL}_n$ or $G = \mathrm{GSp}_{2n}$). I fix in G a maximal split torus T and a Borel subgroup B containing T. The triple $T \subset B \subset G$ gives rise to a based root datum $(X(T), S, X^\vee(T), S^\vee)$ where S is the simple positive roots associated to B in $X(T) := \mathrm{Hom}_{\mathrm{groups}}(T, \mathbb{G}_{\mathrm{m}})$, $S^\vee$ the simple positive coroots, etc. The dual based root datum $(X^\vee(T), S^\vee, X(T), S)$ then corresponds to a dual triple $\widehat{T} \subset \widehat{B} \subset \widehat{G}$. I let B^- be the Borel in G corresponding to $-S$ (thus opposite to B) and W the Weyl group of G or $\widehat{G}$.

In this talk I will define a finite length admissible Banach representation $\Pi(\rho)^{\mathrm{ord}}$ of $G(\mathbb{Q}_p)$ over E ($[E : \mathbb{Q}_p] < +\infty$) associated to a generic continuous $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \widehat{B}(E) \subset \widehat{G}(E)$ and state an important theorem concerning the analogue $\Pi(\overline{\rho})^{\mathrm{ord}}$ of $\Pi(\rho)^{\mathrm{ord}}$ in characterictic p. This last theorem will be used in the next lecture to prove that the representations $\Pi(\overline{\rho})^{\mathrm{ord}}$ essentially occur (up to multiplicities issues) in some cohomology spaces.

2.1 The algebraic representation $L^{\otimes}$ (Galois side)

Let $(\lambda_{\alpha^\vee})_{\alpha\in S}$ be fundamental weights for $\widehat{G}$, that is elements of $X(\widehat{T})$ such that for all $\beta \in S$:

$$\langle \beta, \lambda_{\alpha^\vee} \rangle = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta. \end{cases}$$

The $\lambda_{\alpha^\vee}$ are actually defined up to an element of $X(\widehat{G}) = \mathrm{Hom}_{\mathrm{groups}}(\widehat{G}, \mathbb{G}_{\mathrm{m}})$ but we will ignore this (as this plays no role in the sequel). They are obviously dominant so that we have algebraic representations $L(\lambda_{\alpha^\vee})$ of $\widehat{G}$ of highest weight $\lambda_{\alpha^\vee}$ (we see them as defined over E). We set:

$$L^{\otimes} := \otimes_{\alpha\in S} L(\lambda_{\alpha^\vee})$$

It is a reducible algebraic representation of $\widehat{G}$ over E with highest weight $\lambda := \sum_{\alpha\in S} \lambda_{\alpha^\vee}$. The weights $w(\lambda)$ for $w \in W$ all appear in $L^{\otimes}|_{\widehat{T}}$.

Definition 2.1.1. An ordinary weight of $L^{\otimes}|_{\widehat{T}}$ is a weight $w(\lambda)$ for $w \in W$.

There are plenty of weights in $L^{\otimes}|_{\widehat{T}}$ which are not ordinary. One can prove:

Theorem 2.1.2. *The only weights that occur with multiplicity* 1 *in* $L^{\otimes}|_{\widehat{T}}$ *are the ordinary weights.*

Let $R^{+\vee} \subset X(\widehat{T})$ be the positive coroots and $C \subseteq R^{+\vee}$ a closed subset (recall that a subset $C \subseteq R^{+\vee}$ is closed if $\alpha^\vee, \beta^\vee \in C$ and $\alpha^\vee + \beta^\vee \in R^{+\vee}$ imply $\alpha^\vee + \beta^\vee \in C$). We let $\widehat{B}_C$ be the Zariski closed subgroup of $\widehat{B}$ such that the roots of $\widehat{B}_C$ are exactly C. We denote by:

$$(L^\otimes|_{\widehat{B}_C})^{\mathrm{ord}} \subseteq L^\otimes|_{\widehat{B}_C}$$

the maximal $\widehat{B}_C$-subrepresentation of $L^\otimes|_{\widehat{B}_C}$ such that all its weights are ordinary.

Example 2.1.3. For $C = \emptyset$, one has $\widehat{B}_C = \widehat{T}$ and $(L^\otimes|_{\widehat{T}})^{\mathrm{ord}} = \oplus_{w \in W} w(\lambda)$.

One can completely work out the structure of $L^\otimes|_{\widehat{T}}$. Let:

$$W_C := \{w \in W, w^{-1}(C) \subseteq R^{+\vee}\} = \{w \in W, \dot{w}^{-1}\widehat{B}_C\dot{w} \subseteq \widehat{B}\}$$

(where $\dot{w}$ is any representative in $\mathrm{Normal}(T)$ of $w \in W = \mathrm{Normal}(T)/T$). Fix $w \in W_C$ and let $I \subseteq w(S^\vee) \cap C$ be a subset of *pairwise orthogonal* coroots (that is, if $\alpha^\vee, \beta^\vee \in I$ then $\langle \beta, \alpha^\vee \rangle = 0$ or equivalently $\langle \alpha, \beta^\vee \rangle = 0$). We write $I\perp$ for such a set of coroots. We denote by $\widehat{G}_I \subset \widehat{G}$ the Levi subgroup containing $\widehat{T}$ with roots exactly $\pm I$ (such a Levi subgroup exists). With our assumptions on G and $\widehat{G}$, one can prove there is a decomposition:

$$\widehat{G}_I \simeq \widehat{T}'_I \times \prod_{\alpha^\vee \in I} \mathrm{GL}_2$$

for some subtorus $\widehat{T}'_I \subset \widehat{T}$. Moreover $\widehat{B} \cap \widehat{G}_I$ (resp. $\widehat{T} \cap \widehat{G}_I = \widehat{T}$) also decomposes as $\widehat{T}'_I$ times the product of the induced Borel $\widehat{B}_{\alpha^\vee}$ in each GL_2 (resp. times the product of the induced split torus $\widehat{T}_{\alpha^\vee}$ in each GL_2).

With these data in mind we set:

$$L_I := w(\lambda)|_{\widehat{T}'_I} \otimes_E \big(\otimes_{\alpha^\vee \in I} L_{\alpha^\vee} \big)$$

where $L_{\alpha^\vee}$ is the $\widehat{B}_{\alpha^\vee}$-representation defined as the unique non-split extension of $w(\lambda)|_{\widehat{T}_{\alpha^\vee}}$ by $(s_\alpha w)(\lambda)|_{\widehat{T}_{\alpha^\vee}}$, or equivalently the restriction to $\widehat{B}_{\alpha^\vee}$ of the simple GL_2-module of highest weight $w(\lambda)|_{\widehat{T}_{\alpha^\vee}}$. Here $s_\alpha \in W$ is the reflection associated to $\alpha^\vee$ (or α) and L_I is seen as a $\widehat{B}_C$-representation via the canonical surjection $\widehat{B}_C \twoheadrightarrow \widehat{B} \cap \widehat{G}_I$ (recall $I \subseteq C$). If $I' \subseteq I$, one has $L_{I'} \subseteq L_I$ and we set:

$$L_w^{\mathrm{ord}} := \varinjlim_I L_I$$

where the limit is over all $I \subseteq w(S^\vee) \cap C$, $I\perp$. The $\widehat{B}_C$-socle of L_w^{ord} is the weight $w(\lambda)$.

Theorem 2.1.4. *We have* $(L^{\otimes}|_{\widehat{B}_C})^{\mathrm{ord}} \cong \oplus_{w \in W_C} L_w^{\mathrm{ord}}$.

All this becomes much more clear with examples:

Example 2.1.5. For $G = \widehat{G} = \mathrm{GL}_n$ and $B = \widehat{B} =$ the upper triangular matrices, write $X(\widehat{T}) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ and recall that $S^\vee = \{e_i - e_{i+1}, 1 \leq i \leq n-1\}$.

(i) Assume $n = 3$ and:

$$\widehat{B}_C := \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \subset \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

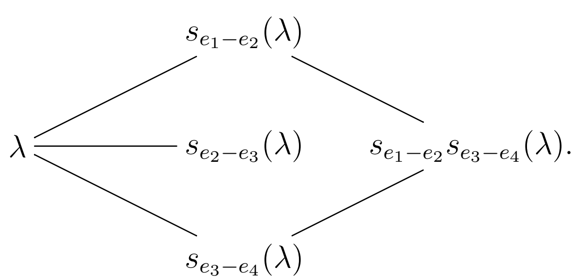
(that is, $C = \{e_1 - e_2\}$). We find $W_C = \{1, s_{e_2-e_3}, s_{e_2-e_3}s_{e_1-e_2}\}$ and the corresponding conjugates of $\widehat{B}_C$ are respectively:

$$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

We have for $(L^{\otimes}|_{\widehat{B}_C})^{\mathrm{ord}}$ (we write a stroke between two weights if and only if the unique non-split extension as $\widehat{B}_C$-representations between these weights occurs as subquotient in $(L^{\otimes}|_{\widehat{B}_C})^{\mathrm{ord}}$):

$$\lambda \text{ --- } s_{e_1-e_2}(\lambda) \;\oplus\; s_{e_2-e_3}(\lambda) \;\oplus\; s_{e_2-e_3}s_{e_1-e_2}(\lambda) \text{ --- } s_{e_1-e_2}s_{e_2-e_3}s_{e_1-e_2}(\lambda)\,.$$

(ii) Assume $n = 4$ and $\widehat{B}_C = \widehat{B}$ (so $C = R^{+\vee}$). We have for $(L^{\otimes}|_{\widehat{B}_C})^{\mathrm{ord}}$:



In general, one can describe the socle filtration $\mathrm{Fil}_j L_w^{\mathrm{ord}}$ of the $\widehat{B}_C$-representation L_w^{ord} as:

$$\mathrm{Fil}_j L_w^{\mathrm{ord}} / \mathrm{Fil}_{j-1} L_w^{\mathrm{ord}} \cong \bigoplus_{\substack{I \subseteq w(S^\vee) \cap C \\ I\perp \;\; |I|=j}} E\Big(\prod_{\alpha^\vee \in I} s_\alpha\Big) w(\lambda).$$

2.2 The $G(\mathbb{Q}_p)$-representation $\Pi(\rho)^{\mathrm{ord}}$

We let $\theta := \sum_{\alpha\in S}\lambda_\alpha$ where λ_α are fundamental weights for G (not $\widehat{G}$ here). We fix a continuous homomorphism:

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \widehat{B}(E) \subset \widehat{G}(E)$$

and we let $\widehat{\chi}(\rho) : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \widehat{B}(E) \twoheadrightarrow \widehat{T}(E)$. We define a continuous character $\chi(\rho) : T(\mathbb{Q}_p) \to E^\times$ as in the classical Langlands correspondence for tori:

$$T(\mathbb{Q}_p) \cong X(\widehat{T}) \otimes_{\mathbb{Z}} \mathbb{Q}_p^\times \hookrightarrow X(\widehat{T}) \otimes_{\mathbb{Z}} \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{\mathrm{ab}} \to X(\widehat{T}) \otimes_{\mathbb{Z}} \widehat{T}(E) \to E^\times.$$

From now on we make the following assumption on ρ:

Genericity assumption: $\alpha^\vee \circ \widehat{\chi}(\rho) \notin \{1, \varepsilon, \varepsilon^{-1}\}$ for all $\alpha \in R^+$.

Let $C(\rho) \subseteq R^{+\vee}$ be the minimal closed subset such that ρ factors through $\widehat{B}_{C(\rho)}(E) \subseteq \widehat{B}(E)$. Replacing ρ by a conjugate in $\widehat{B}(E)$, one can assume (at least under the genericity assumption) that $C(\rho)$ is minimal under conjugation by $\widehat{B}(E)$.

Remark 2.2.1. In fact, such a minimal $C(\rho)$ is *not* an invariant of the conjugacy class of ρ in $\widehat{G}(E)$. However, our definition of $\Pi(\rho)^{\mathrm{ord}}$ below won't depend on which equivalence class of ρ we start from (taking values in $\widehat{B}(E)$ as above), so we can work with this $C(\rho)$ and ignore this problem in the sequel.

From the previous lecture, recall the principal series associated to ρ:

$$I(\rho)_w := \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w^{-1}(\chi(\rho)) \cdot (\varepsilon^{-1}\circ\theta)\right)^{\mathcal{C}^0}$$

where $w \in W$. This is a finite length admissible unitary Banach representation of $G(\mathbb{Q}_p)$ over E which is conjecturally topologically irreducible. Following the "philosophy" at the end of the previous lecture, the idea to construct $\Pi(\rho)^{\mathrm{ord}}$ is the following:

Basic idea: $\Pi(\rho)^{\mathrm{ord}}$ is a successive extension of some of the $I(\rho)_w$ in such a way that, if $w(\lambda)$ appears in $(L^\otimes|_{\widehat{B}_{C(\rho)}})^{\mathrm{ord}}$, then $I(\rho)_w$ appears in $\Pi(\rho)^{\mathrm{ord}}$ "at the same place".

We now define $\Pi(\rho)^{\mathrm{ord}}$ via parabolic induction. As previously, fix $w \in W_{C(\rho)}$ and $I \subseteq w(S^\vee) \cap C(\rho)$, $I\perp$. We set $J := w^{-1}(I)^\vee \subseteq S$ and denote by $G_J \subset G$ the Levi subgroup containing T with roots $\pm J$. As for $\widehat{G}_I$, we have a decomposition $G_J \simeq T'_J \times \prod_{\beta\in J}\mathrm{GL}_2$ and analogous decompositions for $B^- \cap G_J$ and T. We mirror the definition of L_I above and define the following Banach representation of $G_J(\mathbb{Q}_p)$ over E:

$$\widetilde{\Pi}(\rho)_I := \big(w^{-1}(\chi(\rho)) \cdot (\varepsilon^{-1} \circ \theta)\big)|_{T'_J} \otimes_E \big(\widehat{\otimes}_{\beta \in J} E_\beta\big)$$

where E_β is the unique admissible unitary Banach representation over E of the copy of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to β which is a non-split extension of $\Big(\mathrm{Ind}_{B_\beta^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(w^{-1}(\chi(\rho)) \cdot (\varepsilon^{-1} \circ \theta))|_{T_\beta(\mathbb{Q}_p)}\Big)^{\mathcal{C}^0}$ by $\Big(\mathrm{Ind}_{B_\beta^-(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} s_{w(\beta)}(w^{-1}(\chi(\rho)) \cdot (\varepsilon^{-1} \circ \theta))|_{T_\beta(\mathbb{Q}_p)}\Big)^{\mathcal{C}^0}$ (see Theorem 1.4.1, here $B_\beta^- := B^- \cap \mathrm{GL}_2$ and $T_\beta := T \cap \mathrm{GL}_2$) and $\widehat{\otimes}$ is the completed tensor product.

We then set $\Pi(\rho)_I := \big(\mathrm{Ind}_{B^-(\mathbb{Q}_p)G_J(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \widetilde{\Pi}(\rho)_I\big)^{\mathcal{C}^0}$ (a continuous parabolic induction). If $I' \subseteq I$, then $\Pi(\rho)_{I'} \subseteq \Pi(\rho)_I$ and we define:

$$\Pi(\rho)_w^{\mathrm{ord}} := \varinjlim_I \Pi(\rho)_I$$

where the limit is over all $I \subseteq w(S^\vee) \cap C(\rho)$, $I\perp$. Finally, we set:

$$\Pi(\rho)^{\mathrm{ord}} := \oplus_{w \in W_{C(\rho)}} \Pi(\rho)_w^{\mathrm{ord}}.$$

By construction, $\Pi(\rho)^{\mathrm{ord}}$ satisfies the above basic idea. Note that, when $n = 2$, we have $\Pi(\rho)^{\mathrm{ord}} = \Pi(\rho)$.

Example 2.2.2. We go back to the two examples of $(L^\otimes|_{\widehat{B}_C})^{\mathrm{ord}}$ given above.

(i) If $n = 3$ and $\rho = \begin{pmatrix} \chi_1 & * & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}$ (and ρ doesn't take values in any smaller subgroup up to conjugation by the upper Borel), then $\chi(\rho) = \chi_1 \otimes \chi_2 \otimes \chi_3$ and $\Pi(\rho)^{\mathrm{ord}}$ is:

$$I(\rho)_1 \,\text{---}\, I(\rho)_{s_{e_1-e_2}} \quad \oplus \quad I(\rho)_{s_{e_2-e_3}}$$
$$\oplus \quad I(\rho)_{s_{e_2-e_3}s_{e_1-e_2}} \,\text{---}\, I(\rho)_{s_{e_1-e_2}s_{e_2-e_3}s_{e_1-e_2}}$$

where again a stroke between two representations means that a non-split extension between these representations occurs as subquotient in $\Pi(\rho)^{\mathrm{ord}}$.

(ii) If $n = 4$ and $\rho = \begin{pmatrix} \chi_1 & * & * & * \\ 0 & \chi_2 & * & * \\ 0 & 0 & \chi_3 & * \\ 0 & 0 & 0 & \chi_4 \end{pmatrix}$ (and ρ doesn't take values in any smaller subgroup up to conjugation by the upper Borel), then $\chi(\rho) = \chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \chi_4$

and $\Pi(\rho)^{\mathrm{ord}}$ is:

$$\begin{array}{ccccc} & & I(\rho)_{s_{e_1-e_2}} & & \\ & \diagup & & \diagdown & \\ I(\rho)_1 & \text{------} & I(\rho)_{s_{e_2-e_3}} & & I(\rho)_{s_{e_1-e_2}s_{e_3-e_4}}. \\ & \diagdown & & \diagup & \\ & & I(\rho)_{s_{e_3-e_4}} & & \end{array}$$

2.3 The $G(\mathbb{Q}_p)$-representation $\Pi(\overline{\rho})^{\mathrm{ord}}$ and Serre weights

We now slightly modify the setting so as to deal with characteristic p. We take $T \subset B \subset G$ all over $\mathbb{Z}_p$ and $\widehat{T} \subset \widehat{B} \subset \widehat{G}$ all over $\mathcal{O}_E$ (with G and $\widehat{G}$ having a connected center). We fix a continuous homomorphism:

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \widehat{B}_{C(\overline{\rho})}(k_E) \subseteq \widehat{B}(k_E) \subset \widehat{G}(k_E)$$

and define $\widehat{\chi}(\overline{\rho})$ and $\chi(\overline{\rho})$ as in characteristic 0. We make the following assumptions:

(i) $\alpha^\vee \circ \widehat{\chi}(\overline{\rho}) \notin \{1, \omega, \omega^{-1}\}$ for all $\alpha \in R^+$

(ii) $C(\overline{\rho})$ is minimal under conjugation by $\widehat{B}(k_E)$.

The previous construction of $\Pi(\rho)^{\mathrm{ord}}$ then extends essentially verbatim and gives a finite length admissible smooth representation:

$$\Pi(\overline{\rho})^{\mathrm{ord}} = \oplus_{w \in W_{C(\overline{\rho})}} \Pi(\overline{\rho})^{\mathrm{ord}}_w$$

of $G(\mathbb{Q}_p)$ over k_E which is a successive extension of absolutely irreducible smooth principal series:

$$I(\overline{\rho})_w := \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w^{-1}(\chi(\overline{\rho})) \cdot (\omega^{-1} \circ \theta)$$

for some $w \in W$ (their irreducibility follows from work of Ollivier and Abe). More precisely $\Pi(\overline{\rho})^{\mathrm{ord}}_w$ for $w \in W_{C(\overline{\rho})}$ has a socle filtration (for $G(\mathbb{Q}_p)$) $\mathrm{Fil}_j \Pi(\overline{\rho})^{\mathrm{ord}}_w$ such that:

$$\mathrm{Fil}_j \Pi(\overline{\rho})^{\mathrm{ord}}_w / \mathrm{Fil}_{j-1} \Pi(\overline{\rho})^{\mathrm{ord}}_w \cong \bigoplus_{\substack{I \subseteq w(S^\vee) \cap C(\overline{\rho}) \\ I \perp \ |I| = j}} I(\overline{\rho})_{(\prod_{\alpha^\vee \in I} s_\alpha) w}.$$

In particular the $G(\mathbb{Q}_p)$-socle of $\Pi(\overline{\rho})^{\mathrm{ord}}_w$ is $I(\overline{\rho})_w$.

I now want to state a very important theorem involving the representations $\Pi(\overline{\rho})^{\mathrm{ord}}_w$ and which will be key to the local-global compatibility result of the next lecture (i.e. the link with the global theory of automorphic forms in characteristic p).

I will assume a stronger hypothesis on $\overline{\rho}$ (which could in fact probably be relaxed with some more work but which makes life much simpler):

Inertial genericity assumption: $\alpha^\vee \circ \widehat{\chi}(\overline{\rho})|_{\text{inertia}} \notin \{1, \omega, \omega^{-1}\}$ for all $\alpha \in R^+$.

This assumption implies that p is large enough. For instance, if $G = \mathrm{GL}_n$, it implies $p > 2n$.

Under this assumption, each $I(\overline{\rho})_w$ for $w \in W$ has an irreducible $G(\mathbb{Z}_p)$-socle (a Serre weight) that we denote $\sigma(\overline{\rho})_w$ and each $\Pi(\overline{\rho})_w^{\text{ord}}$ for $w \in W_{C(\overline{\rho})}$ has also $\sigma(\overline{\rho})_w$ as irreducible $G(\mathbb{Z}_p)$-socle.

If σ is any Serre weight for $G(\mathbb{Z}_p)$ (or $G(\mathbb{F}_p)$), recall that:

$$\mathcal{H}_G(\sigma) := \mathrm{End}_{G(\mathbb{Q}_p)}\left(\text{c-Ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)} \sigma\right)$$

is a commutative k_E-algebra of finite type (where $\text{c-Ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)} \sigma$ is the usual smooth induction with compact support). This follows from work of Herzig. If π is a smooth representation of $G(\mathbb{Q}_p)$ over k_E, then the k_E-vector space:

$$\mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma, \pi|_{G(\mathbb{Z}_p)}) \cong \mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\text{c-Ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)} \sigma, \pi\right)$$

(Frobenius reciprocity) is naturally an $\mathcal{H}_G(\sigma)$-module. If $\eta : \mathcal{H}_G(\sigma) \to k_E$ is a character, we denote by $\mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma, \pi|_{G(\mathbb{Z}_p)})[\eta] \subseteq \mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma, \pi|_{G(\mathbb{Z}_p)})$ the subspace where $\mathcal{H}_G(\sigma)$ acts by η. Finally, if U^- is the unipotent radical of B^-, Herzig has defined a localization map (between commutative k_E-algebras of finite type):

$$\mathcal{H}_G(\sigma) \hookrightarrow \mathcal{H}_T(\sigma_{U^-(\mathbb{Z}_p)})$$

where $\sigma_{U^-(\mathbb{Z}_p)}$ is the quotient of σ of coinvariants under $U^-(\mathbb{Z}_p)$.

Definition 2.3.1. We say that $\eta : \mathcal{H}_G(\sigma) \to k_E$ is ordinary if it factors (necessarily uniquely) through $\mathcal{H}_T(\sigma_{U^-(\mathbb{Z}_p)})$.

The following theorem enables us to write the above principal series $I(\overline{\rho})_w$ as quotients of compact inductions:

Theorem 2.3.2 (Herzig)**.** *For all $w \in W$ there is a unique ordinary character $\eta(\overline{\rho})_w : \mathcal{H}_G(\sigma(\overline{\rho})_w) \to k_E$ such that:*

$$\left(\text{c-Ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)} \sigma(\overline{\rho})_w\right) \otimes_{\mathcal{H}_G(\sigma(\overline{\rho})_w), \eta(\overline{\rho})_w} k_E \cong I(\overline{\rho})_w.$$

We finally let $\mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma, \pi|_{G(\mathbb{Z}_p)})^{\mathrm{ord}} \subseteq \mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma, \pi|_{G(\mathbb{Z}_p)})$ be the maximal subspace on which the action of $\mathcal{H}_G(\sigma)$ extends to $\mathcal{H}_T(\sigma_{U^-(\mathbb{Z}_p)})$.

The following theorem will be crucial:

Theorem 2.3.3. *Let Π be an admissible smooth representation of $G(\mathbb{Q}_p)$ over k_E such that $\Pi|_{G(\mathbb{Z}_p)}$ is an injective object in the category of smooth representations of $G(\mathbb{Z}_p)$ over k_E. Fix $w \in W_{C(\overline{\rho})}$ and assume that:*

$$\mathrm{Hom}_{G(\mathbb{Z}_p)}\left(\sigma(\overline{\rho})_{(\prod_{\alpha^\vee \in I} s_\alpha)w}, \Pi|_{G(\mathbb{Z}_p)}\right)^{\mathrm{ord}} = 0$$

for all non-empty $I \subseteq w(S^\vee) \cap C(\overline{\rho})$, $I\perp$. Then restriction to the $G(\mathbb{Z}_p)$-socle induces an isomorphism:

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(\Pi(\overline{\rho})_w^{\mathrm{ord}}, \Pi) \xrightarrow{\sim} \mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma(\overline{\rho})_w, \Pi|_{G(\mathbb{Z}_p)})[\eta(\overline{\rho})_w].$$

Note that the statement implies that any map in $\mathrm{Hom}_{G(\mathbb{Q}_p)}(\Pi(\overline{\rho})_w^{\mathrm{ord}}, \Pi)$ is either injective or zero. We will sketch the proof of this theorem in the next lecture as well as give a global application.

3 Lecture 3 (April 20)

I quickly recall some of the notation of the previous lecture: $(T \subset B \subset G)/\mathbb{Z}_p$, $(\widehat{T} \subset \widehat{B} \subset \widehat{G})/\mathcal{O}_E$, G and $\widehat{G}$ with a connected center, $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \widehat{B}_{C(\overline{\rho})}(k_E) \subseteq \widehat{B}(k_E) \subset \widehat{G}(k_E)$ inertially generic (with $C(\overline{\rho})$ minimal), $\chi(\overline{\rho}) : T(\mathbb{Q}_p) \to k_E^\times$. For $w \in W$ we have the irreducible smooth principal series:

$$I(\overline{\rho})_w := \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w^{-1}(\chi(\overline{\rho})) \cdot (\omega^{-1} \circ \theta)$$

with $G(\mathbb{Z}_p)$-socle a Serre weight $\sigma(\overline{\rho})_w$ and associated (ordinary) character of $\mathcal{H}_G(\sigma(\overline{\rho})_w)$ denoted by $\eta(\overline{\rho})_w$. For $w \in W_{C(\overline{\rho})}$ we have associated to $\overline{\rho}$ an indecomposable $G(\mathbb{Q}_p)$-representation $\Pi(\overline{\rho})_w^{\mathrm{ord}}$ with $G(\mathbb{Q}_p)$-socle $I(\overline{\rho})_w$ and constituents some $I(\overline{\rho})_{w'}$ for $w' \neq w$.

3.1 A local theorem

We now sketch the proof of the following theorem:

Theorem 3.1.1. *Let $w \in W_{C(\overline{\rho})}$ and Π an admissible smooth representation of $G(\mathbb{Q}_p)$ over k_E such that:*

(i) $\Pi|_{G(\mathbb{Z}_p)}$ is a smooth injective representation of $G(\mathbb{Z}_p)$ over k_E

(ii) $\mathrm{Hom}_{G(\mathbb{Z}_p)}\big(\sigma(\overline{\rho})_{w'}, \Pi|_{G(\mathbb{Z}_p)}\big)^{\mathrm{ord}} = 0$ for $w' \neq w$ as above.

Then restriction to $\sigma(\overline{\rho})_w$ induces an isomorphism:

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(\Pi(\overline{\rho})_w^{\mathrm{ord}}, \Pi) \xrightarrow{\sim} \mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma(\overline{\rho})_w, \Pi|_{G(\mathbb{Z}_p)})[\eta(\overline{\rho})_w].$$

Note that one always has $\mathrm{Hom}_{G(\mathbb{Q}_p)}(I(\overline{\rho})_w, \Pi) \xrightarrow{\sim} \mathrm{Hom}_{G(\mathbb{Z}_p)}(\sigma(\overline{\rho})_w, \Pi|_{G(\mathbb{Z}_p)})[\eta(\overline{\rho})_w]$ by Frobenius reciprocity combined with Herzig's theorem writing $I(\overline{\rho})_w$ as a quotient of the compact induction $\text{c-Ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)} \sigma(\overline{\rho})_w$ (this doesn't require any assumption on Π). But $I(\overline{\rho})_w$ is only the $G(\mathbb{Q}_p)$-socle of $\Pi(\overline{\rho})_w^{\mathrm{ord}}$, so the whole point is to show that any map $I(\overline{\rho})_w \to \Pi$ automatically extends to $\Pi(\overline{\rho})_w^{\mathrm{ord}} \to \Pi$ under assumptions (i) and (ii) on Π.

1) The injectivity in the isomorphism of the theorem is easy: if $f \mapsto 0$, then f is not injective and thus $f|_{I(\overline{\rho})_w} = 0$ which implies $\mathrm{Hom}_{G(\mathbb{Q}_p)}(I(\overline{\rho})_{w'}, \Pi) \neq 0$ for some $w' \neq w$. But this is impossible since:

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(I(\overline{\rho})_{w'}, \Pi) = \mathrm{Hom}_{G(\mathbb{Z}_p)}\big(\sigma(\overline{\rho})_{w'}, \Pi\big)[\eta(\overline{\rho})_{w'}] \subseteq \mathrm{Hom}_{G(\mathbb{Z}_p)}\big(\sigma(\overline{\rho})_{w'}, \Pi\big)^{\mathrm{ord}}$$

which is zero by assumption (ii). We are left to prove surjectivity, and, going back to the definition of $\Pi(\overline{\rho})_w^{\mathrm{ord}}$, it is enough to prove it replacing $\Pi(\overline{\rho})_w^{\mathrm{ord}}$ by $\Pi(\overline{\rho})_I$ for any $I \subseteq w(S^\vee) \cap C(\overline{\rho})$, $I\perp$ (see the previous lecture for $\Pi(\overline{\rho})_I$).

2) Set $J := w^{-1}(I)^\vee \subseteq S$, $G_J \subset G$ the Levi subgroup containing T with roots $\pm J$ (recall $G_J \simeq T'_J \times \prod_{\beta\in J} \mathrm{GL}_2$), $P_J := BG_J$ and $P_J^- := B^-G_J$ (parabolic subgroups with Levi G_J) and $U_J^- \subset P_J^-$ the unipotent radical of P_J^-. The character $\eta(\overline{\rho})_w$ being ordinary factors through the localizations maps:

$$\mathcal{H}_G(\sigma(\overline{\rho})_w) \hookrightarrow \mathcal{H}_{G_J}\big((\sigma(\overline{\rho})_w)_{U_J^-(\mathbb{Z}_p)}\big) \hookrightarrow \mathcal{H}_T\big((\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}\big) \overset{\eta(\overline{\rho})_w}{\longrightarrow} k_E.$$

We now consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{G(\mathbb{Q}_p)}\big(\Pi(\overline{\rho})_I, \Pi\big) & \overset{\sim}{\longrightarrow} & \mathrm{Hom}_{G_J(\mathbb{Q}_p)}\big(\widetilde{\Pi}(\overline{\rho})_I, \mathrm{Ord}_{P_J}(\Pi)\big) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{G(\mathbb{Z}_p)}\big(\sigma(\overline{\rho})_w, \Pi\big)[\eta(\overline{\rho})_w] & \overset{\sim}{\longrightarrow} & \mathrm{Hom}_{G_J(\mathbb{Z}_p)}\big((\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}, \mathrm{Ord}_{P_J}(\Pi)\big)[\eta(\overline{\rho})_w]. \end{array}$$

I explain this diagram.

First Ord_{P_J} is Emerton's functor of ordinary parts, which associates a smooth representation of $P_J(\mathbb{Q}_p)$ over k_E to any admissible smooth representation of $G(\mathbb{Q})$ over k_E. Then the top isomorphism is Emerton's adjunction formula (recall that $\Pi(\overline{\rho})_I = \mathrm{Ind}_{P_J^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \widetilde{\Pi}(\overline{\rho})_I$). The two vertical isomorphisms are restrictions to respectively the $G(\mathbb{Q}_p)$- and $G_J(\mathbb{Q}_p)$-socles (which are principal series) followed by Herzig's theorem (writing such a principal series as a quotient of a compact induction) and Frobenius reciprocity. The bottom isomorphism is again Emerton's adjunction formula (together with Frobenius reciprocity). Isomorphisms like the one at the bottom but replacing $\sigma(\overline{\rho})_w$ by $\sigma(\overline{\rho})_{w'}$ also show that $\mathrm{Ord}_{P_J}(\Pi)$ satisfies assumption (ii) replacing $\sigma(\overline{\rho})_{w'}$ by its $U_J^-(\mathbb{Z}_p)$-coinvariants. Moreover, one can prove that $\Pi|_{G(\mathbb{Z}_p)}$ injective implies $\mathrm{Ord}_{P_J}(\Pi)|_{G_J(\mathbb{Z}_p)}$ injective.

All this shows that one can replace $G(\mathbb{Q}_p)$ by $G_J(\mathbb{Q}_p)$, $\Pi(\overline{\rho})_I$ by $\widetilde{\Pi}(\overline{\rho})_I$ and Π by $\mathrm{Ord}_{P_J}(\Pi)$ in the original statement. The advantage is that, now, we essentially have to deal with a product of $\mathrm{GL}_2(\mathbb{Q}_p)$ (up to a harmless torus part) which is much easier.

3) The exists a $G_J(\mathbb{Z}_p)$-representation $\widetilde{\sigma}(\overline{\rho})_I \subset \widetilde{\Pi}(\overline{\rho})_I$ such that its constituents are exactly the $G_J(\mathbb{Z}_p)$-socles of all the $G_J(\mathbb{Q}_p)$-constituents of $\widetilde{\Pi}(\overline{\rho})_I$ (in particular its socle is $(\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}$). This follows from the fact that $\widetilde{\Pi}(\overline{\rho})_I$ is an exterior tensor product of non-split extensions for groups $\mathrm{GL}_2(\mathbb{Q}_p)$ (see previous lecture) and known properties of these non-split $\mathrm{GL}_2(\mathbb{Q}_p)$-extensions. We then set:

$$\widetilde{X}(\overline{\rho})_I := \big(\,\text{c-Ind}_{G_J(\mathbb{Z}_p)}^{G_J(\mathbb{Q}_p)} \widetilde{\sigma}(\overline{\rho})_I\big) \oplus_{\text{c-Ind}_{G_J(\mathbb{Z}_p)}^{G_J(\mathbb{Q}_p)}(\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}} \widetilde{I}(\overline{\rho})_w$$

where $\widetilde{I}(\overline{\rho})_w$ is the $G_J(\mathbb{Q}_p)$-socle of $\widetilde{\Pi}(\overline{\rho})_I$ (an irreducible principal series with $G_J(\mathbb{Z}_p)$-socle $(\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}$). We then have the following:

(i) The map $\text{c-Ind}_{G_J(\mathbb{Z}_p)}^{G_J(\mathbb{Q}_p)} \widetilde{\sigma}(\overline{\rho})_I \to \widetilde{\Pi}(\overline{\rho})_I$ (Frobenius reciprocity) factors through $\widetilde{X}(\overline{\rho})_I$ and gives rise to a short exact sequence of $G_J(\mathbb{Q}_p)$-representations:

$$0 \to \text{Ker} \to \widetilde{X}(\overline{\rho})_I \to \widetilde{\Pi}(\overline{\rho})_I \to 0.$$

(ii) The injectivity of $\text{Ord}_{P_J}(\Pi)|_{G_J(\mathbb{Z}_p)}$ and Frobenius reciprocity imply that restriction to $\text{c-Ind}_{G_J(\mathbb{Z}_p)}^{G_J(\mathbb{Q}_p)}(\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}$ is surjective:

$$\text{Hom}_{G_J(\mathbb{Q}_p)}\big(\text{c-Ind}_{G_J(\mathbb{Z}_p)}^{G_J(\mathbb{Q}_p)} \widetilde{\sigma}(\overline{\rho})_I, \text{Ord}_{P_J}(\Pi)\big) \twoheadrightarrow \text{Hom}_{G_J(\mathbb{Z}_p)}\big((\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}, \text{Ord}_{P_J}(\Pi)\big).$$

Taking the pull-back under the injection:

$$\begin{array}{c} \text{Hom}_{G_J(\mathbb{Z}_p)}\big((\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}, \text{Ord}_{P_J}(\Pi)\big) \\ \uparrow \\ \text{Hom}_{G_J(\mathbb{Z}_p)}\big((\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}, \text{Ord}_{P_J}(\Pi)\big)[\eta(\overline{\rho})_w] \end{array}$$

thus yields a surjection:

$$\text{Hom}_{G_J(\mathbb{Q}_p)}\big(\widetilde{X}(\overline{\rho})_I, \text{Ord}_{P_J}(\Pi)\big) \twoheadrightarrow \text{Hom}_{G_J(\mathbb{Z}_p)}\big((\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}, \text{Ord}_{P_J}(\Pi)\big)[\eta(\overline{\rho})_w].$$

Now let $\overline{f} \in \text{Hom}_{G_J(\mathbb{Z}_p)}\big((\sigma(\overline{\rho})_w)_{U^-(\mathbb{Z}_p)}, \text{Ord}_{P_J}(\Pi)\big)[\eta(\overline{\rho})_w]$ and f lifting $\overline{f}$. Consider the pushout:

$$(*) \qquad 0 \to f(\text{Ker}) \to \widetilde{X}(\overline{\rho})_I \oplus_{\text{Ker}} f(\text{Ker}) \to \Pi(\overline{\rho})_I \to 0$$

and assume that the following hold:

$$\text{(H)} \qquad (*) \text{ splits and } \text{Hom}_{G_J(\mathbb{Q}_p)}\big(\widetilde{\Pi}(\overline{\rho})_I, f(\text{Ker})\big) = 0$$

then the map:

$$\widetilde{\Pi}(\overline{\rho})_I \overset{\text{section}}{\hookrightarrow} \widetilde{X}(\overline{\rho})_I \oplus_{\text{Ker}} f(\text{Ker}) \overset{f}{\to} \text{Ord}_{P_J}(\Pi)$$

lifts $\overline{f}$ and we are done.

4) It thus remains to prove (H) above. First, it follows from a result of Paskunas for $\text{GL}_2(\mathbb{Q}_p)$ that, if $\eta : \mathcal{H}_{G_J}\big((\sigma(\overline{\rho})_{w'})_{U_J^-(\mathbb{Z}_p)}\big) \to k_E$ is *not* ordinary, then any of the $G_J(\mathbb{Q}_p)$-representations:

$$(**) \qquad \big(\text{c-Ind}_{G_J(\mathbb{Z}_p)}^{GJ(\mathbb{Q}_p)}(\sigma(\overline{\rho})_{w'})_{U^-(\mathbb{Z}_p)}\big) \otimes_{\mathcal{H}_{G_J}((\sigma(\overline{\rho})_{w'})_{U_J^-(\mathbb{Z}_p)})} \eta$$

(for $w' \in W$) has only split extensions with analogous representations but with ordinary $\eta's$ (the result of Paskunas is that any supersingular representation of $\text{GL}_2(\mathbb{Q}_p)$ has only split extensions with principal series of $\text{GL}_2(\mathbb{Q}_p)$). Then one shows that $f(\mathrm{Ker}) \subseteq \mathrm{Ord}_{P_J}(\Pi)$ can be filtered by $G_J(\mathbb{Q}_p)$-representations with graded pieces as in $(**)$ for $w' \neq w$ as in the statement of the theorem and various η. Hypothesis (ii) for $\mathrm{Ord}_{P_J}(\Pi)$, the above result and a staightforward induction then imply that all the characters η appearing there have to be *non-ordinary*. But then, since only ordinary η's appear in $\widetilde{\Pi}(\overline{\rho})_I$ by construction, the same argument (together with an obvious dévissage) gives that any extension $(*)$ has to split. Likewise, there can't be any nonzero morphism $\widetilde{\Pi}(\overline{\rho})_I \to f(\mathrm{Ker})$.

3.2 Global application

I start with the global setting: F^+ is a totally real number field and F/F^+ is a quadratic totally imaginary extension where p splits completely. For technical reasons (due to the temporary status of base change in the classical Langlands program), we have to assume that F/F^+ is moreover everywhere unramified (which rules out $F^+ = \mathbb{Q}$ for instance). However, it should just be a matter of time for this hypothesis to disappear. We set $\mathcal{O}_{F^+,p} := \mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{v|p} \mathbb{Z}_p$.

We fix $G_{/\mathcal{O}_{F^+}[1/N]}$ (where N is an integer prime to p) a connected reductive algebraic group such that $G \times_{\mathcal{O}_{F^+}[1/N]} \mathcal{O}_F[1/N] \cong \mathrm{GL}_n$ and $G \times_{\mathcal{O}_{F^+}[1/N]} F^+$ is an outer form of GL_n. We moreover assume that G is quasi-split at all finite places of F^+ and isomorphic to $U_n(\mathbb{R})$ at all infinite places.

For M a k_E-vector space endowed with a linear action of the compact group $G(\mathcal{O}_{F^+,p}) \simeq \prod_{v|p} \mathrm{GL}_n(\mathbb{Z}_p)$ and for $U \subset G(\mathbb{A}_{F^+}^{\infty,p}) \times G(\mathcal{O}_{F^+,p})$ we define the usual space of algebraic mod p automorphic forms of level U and weight M:

$$S(U,M) := \{f : G(F)\backslash G(\mathbb{A}_{F^+}^{\infty}) \to M,\ f(gu) = u_p^{-1}(f(g)), u \in U,$$
$$u_p := \mathrm{Image}(u) \text{ in } G(\mathcal{O}_{F^+,p})\}.$$

If U is sufficiently small, then $S(U,M) \cong M^{\oplus d(U)}$ for some integer $d(U)$ which doesn't depend on M.

For $U^p \subset G(\mathbb{A}_{F^+}^{\infty,p})$ a compact open subgroup we set:

$$S(U^p, k_E) := \lim_{\overrightarrow{U_p}} S(U^pU_p, k_E)$$

where U_p runs among compact open subgroups of $G(\mathcal{O}_{F^+,p})$. Then $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ naturally acts on $S(U^p, k_E)$ and the resulting representation is smooth and admissible. If U^p is sufficiently small, it follows from $S(U^pG(\mathcal{O}_{F^+,p}), M) \cong M^{\oplus d(U^pG(\mathcal{O}_{F^+,p}))}$ that $S(U^p, k_E)|_{G(\mathcal{O}_{F^+,p})}$ is an *injective representation.*

If Σ is a finite set of finite places of F^+ containing the places that split in F and either divide pN or at which U^p is not maximal, then a natural Hecke algebra $\mathbb{T}^{\Sigma}$

(a formal polynomial algebra) acts on $S(U^p, k_E)$ by usual double cosets at finite places $v \notin \Sigma$. This action commutes with that of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$.

Now we let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(k_E)$ continuous absolutely irreducible. If Σ moreover contains the finite places of F^+ that split in F and at which $\overline{r}$ is ramified, then one can associate to $\overline{r}$ a maximal ideal $\mathfrak{m}^\Sigma(\overline{r})$ of $\mathbb{T}^\Sigma$ with residue field k_E (using the characteristic polynomials of Frobenius elements).

Definition 3.2.1 (Gee-Geraghty)**.** The representation $\overline{r}$ is modular-ordinary if there exist (U^p, Σ) as above and a Serre weight σ for $G(\mathcal{O}_{F^+,p})$ such that:

$$\mathrm{Hom}_{G(\mathcal{O}_{F^+,p})}\left(\sigma, S(U^p, k_E)_{\mathfrak{m}^\Sigma(\overline{r})}\right)^{\mathrm{ord}} \neq 0$$

where $S(U^p, k_E)_{\mathfrak{m}^\Sigma(\overline{r})}$ is the localization of the $\mathbb{T}^\Sigma$-module $S(U^p, k_E)$ at $\mathfrak{m}^\Sigma(\overline{r})$.

In fact, in the above definition one has to assume that U_v is hyperspecial maximal at places v of F^+ that are inert in F, but we ignore this technical point in the sequel.

For each $v|p$ in F^+ we choose one of the two places above v in F, call it $\widetilde{v}$, and set $\overline{r}_{\widetilde{v}} := \overline{r}|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_{\widetilde{v}})}$ (this choice won't matter). If $\overline{r}$ is modular-ordinary, then all $\overline{r}_{\widetilde{v}}$ are upper triangular (this is due to Gee and Geraghty). We assume moreover that $\overline{r}_{\widetilde{v}}$ is generic for all $\widetilde{v}$. We have then defined $C(\overline{r}_{\widetilde{v}})$, $W_{C(\overline{r}_{\widetilde{v}})}$, $\Pi(\overline{r}_{\widetilde{v}})^{\mathrm{ord}}_{w_{\widetilde{v}}}$ and $\sigma(\overline{r}_{\widetilde{v}})_{w_{\widetilde{v}}}$ for $w_{\widetilde{v}} \in W_{C(\overline{r}_{\widetilde{v}})}$, etc.

Before stating the main theorem, I need one more definition. If Π is an admissible smooth representation of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ over k_E, I denote Π^{ord} the maximal subrepresentation of Π such that its irreducible constituents are subquotients of principal series for $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. Recall also that an injection $\pi \hookrightarrow \Pi$ of smooth representations of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ is said to be *essential* if , for any $0 \neq \pi' \subseteq \Pi$, one has $\pi \cap \pi' = 0$.

Theorem 3.2.2. *Assume that $\overline{r}$ is absolutely irreducible, modular-ordinary (+ some small technical assumptions to make all modularity lifting theorems work in that context) and that $\overline{r}_{\widetilde{v}}$ is inertially generic for all $\widetilde{v}$. Then there exit (U^p, Σ) as above and, for each $w = (w_{\widetilde{v}})_{v|p} \in \prod_{v|p} W_{C(\overline{r}_{\widetilde{v}})}$, an integer $d_w \in \mathbb{Z} > 0$ such that we have an essential injection of admissible smooth representations of $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ over k_E:*

$$\bigoplus_{w \in \prod_{v|p} W_{C(\overline{r}_{\widetilde{v}})}} \left(\bigotimes_{v|p} \Big(\Pi(\overline{r}_{\widetilde{v}})^{\mathrm{ord}}_{w_{\sigma_{\widetilde{v}}}} \otimes (\omega^{n-1} \circ \det) \Big) \right)^{\oplus d_w} \hookrightarrow S(U^p, k_E)[\mathfrak{m}^\Sigma(\overline{r})]^{\mathrm{ord}}$$

where $S(U^p, k_E)[\mathfrak{m}^\Sigma(\overline{r})] \subset S(U^p, k_E)$ denotes the $\mathfrak{m}^\Sigma(\overline{r})$-eigenspace.

Before sketching the proof, let me make some comments. (i) The $G(F^+\otimes_{\mathbb{Q}}\mathbb{Q}_p)$-representation $S(U^p,k_E)[\mathfrak{m}^\Sigma(\overline{r})]$ is the one we *really* would like to understand (and next $S(U^p,k_E)_{\mathfrak{m}^\Sigma(\overline{r})}\dots$), but we don't even know if it only depends on the $\overline{r}_{\tilde{v}}$ for $v|p$ (apart from multiplicities issues coming from the size of U^p). The above theorem at least gives a (purely local) piece of it. (ii) One can prove that the $G(F^+\otimes_{\mathbb{Q}}\mathbb{Q}_p)$-representation on the left hand side doesn't actually depend on the choice of the $\widetilde{v}$. (ii) When all $\overline{r}_{\tilde{v}}$ are as generic as possible, then W is reduced to the identity element and this representation is exactly a power of $\bigotimes_{v|p}\Big(\Pi(\overline{r}_{\tilde{v}})^{\rm ord}\otimes(\omega^{n-1}\circ\det)\Big)$. In general, it doesn't seem to be known that all d_w are equal (but we conjecture it below).

By results of Gee and Geraghty, there is (U^p,Σ) as above and a unique *ordinary* character $\eta(\overline{r})_w$ such that:

$$\mathrm{Hom}_{G(\mathcal{O}_{F^+,p})}\big((\otimes_{v|p}\sigma(\overline{r}_{\tilde{v}})_{w_{\tilde{v}}}\otimes(\omega^{n-1}\circ\det)),S(U^p,k_E)[\mathfrak{m}^\Sigma(\overline{r})]\big)[\eta(\overline{r})_w]\neq 0$$

(it could be nonzero for other characters but none being ordinary). We define d_w to be the (finite positive) dimension of the above k_E-vector space. We then apply the local theorem in section 3.1 to $\mathrm{res}_{\mathcal{O}_{F^+,p}/\mathbb{Z}_p}(G\times_{\mathcal{O}_{F^+}[1/N]}\mathcal{O}_{F^+,p})\cong\prod_{v|p}\mathrm{GL}_{n/\mathbb{Z}_p}$, $\overline{\rho}:=\prod_{v|p}(\overline{r}_{\tilde{v}}\otimes\omega^{n-1})$, $\Pi:=S(U^p,k_E)_{\mathfrak{m}^\Sigma(\overline{r})}$ (this localized space is still injective as it is a direct summand in $S(U^p,k_E)$) and then we take $\mathfrak{m}^\Sigma(\overline{r})$-eigenspaces on both sides of the isomorphism given by the local theorem (which is of course $\mathbb{T}^\Sigma$-equivariant). Note that assumption (ii) in this local theorem is satisfied for our Π because of the results of Gee and Geraghty again. We get a nonzero map as in the statement which is injective in restriction to the $G(\mathcal{O}_{F^+,p})$-socle and thus which is injective. Its image lies in $S(U^p,k_E)[\mathfrak{m}^\Sigma(\overline{r})]^{\rm ord}$ by definition and it is easily checked to be essential otherwise we would find that d_w is strictly smaller than the above dimension (i.e. we would find another copy of an ordinary Serre weight in the socle).

We conjecture that the following should be true:

Conjecture 3.2.3. *Assume that $\overline{r}$ is absolutely irreducible, modular-ordinary and that $\overline{r}_{\tilde{v}}$ is generic for all $\widetilde{v}$ (= weaker than inertially generic). Then there exist (U^p,Σ) as above and an integer $d>0$ such that we have an essential injection of smooth representations of $G(F^+\otimes_{\mathbb{Q}}\mathbb{Q}_p)$ over k_E:*

$$\Big(\bigotimes_{v|p}\Big(\Pi(\overline{r}_{\tilde{v}})^{\rm ord}\otimes(\omega^{n-1}\circ\det)\Big)\Big)^{\oplus d}\hookrightarrow S(U^p,k_E)[\mathfrak{m}^\Sigma(\overline{r})]^{\rm ord}.$$

This essential injection is an isomorphism if $p\geq n$.

We also conjecture an analogous isomorphism in characteristic 0 (for all p) replacing $\Pi(\overline{r}_{\tilde{v}})^{\rm ord}$ by $\Pi(r_{\tilde{v}})^{\rm ord}$ and $S(U^p,k_E)[\mathfrak{m}^\Sigma(\overline{r})]^{\rm ord}$ by a p-adic completion $(S(U^p,\mathcal{O}_E)^\wedge\otimes_{\mathcal{O}_E}E)[\mathfrak{p}^\Sigma(\overline{r})]^{\rm ord}$.