Calegari - "Galois Deformations" 3/13/12

Let G be a finitely generated group. Fix $N \in \mathbb{Z}_{>0}$

Problem: Describe $\mathrm{Hom}(G, GL_N(\mathbb{C}))$

Let $g_k = [x_{ijk}]_{i,j=1}^{N}$

Then the relations lead to a system of polynomial equations:

$R_N, \dots \quad X_N = \mathrm{Spec}\, R_N.$

Let x be a point of X_N, $\rho_x : G \longrightarrow GL_N(\mathbb{C})$

Q: What does X_N look like in a neighborhood of x?

We can write
$$\begin{tikzcd} \rho : G \arrow[r] \arrow[dr, "\rho_x"'] & GL_N(R_N) \arrow[d] \\ & GL_N(\mathbb{C}) \end{tikzcd}$$

$\hat{\mathcal{O}}_{X_N,x}$ = quotient of $\mathbb{C}[[x_1, \dots, x_m]]$

$m_x = (x_1, \dots, x_m)$

$\dim m_x / m_x^2 = m$

Define
$$\begin{tikzcd} \tilde{\rho}_x : G \arrow[r] \arrow[dr] & GL_N(\mathcal{O}_{x,x}/m_x^2) = GL_N(\mathbb{C}[[x_1, \dots x_m]]/x_ix_j) \arrow[d] \\ & GL_N(\mathbb{C}[\varepsilon]/\varepsilon^2) \end{tikzcd}$$

f.d. $\mathbb{C}$-vectorspace

We have the exact sequence
$$1 \longrightarrow \varepsilon\, \mathbb{C}[\varepsilon]/\varepsilon^2 \longrightarrow \mathbb{C}[\varepsilon]/\varepsilon^2 \longrightarrow \mathbb{C} \longrightarrow 1$$
$$1 \longrightarrow (\varepsilon\, \mathbb{C}[\varepsilon]/\varepsilon^2)^N \longrightarrow \underbrace{(\mathbb{C}[\varepsilon]/\varepsilon^2)^N}_{V} \longrightarrow \mathbb{C}^N \longrightarrow 1$$

$(\varepsilon\, \mathbb{C}[\varepsilon]/\varepsilon^2)^N \cong \mathbb{C}^N \cong \rho$; $\rho \curvearrowleft G$ on $\mathbb{C}^N$

mult by ε

G-equivariant

$$1 \longrightarrow \rho_x \longrightarrow \underset{\hat{\rho}_x}{V} \longrightarrow \rho_x \longrightarrow 1$$

$$\hat{\rho}_x(g) = \begin{pmatrix} \boxed{A_g} & A_g B_g \\ & \boxed{A_g} \end{pmatrix}$$ Think of B_g as a map $\in \mathrm{Hom}(\rho_x, \rho_x)$

Compare image of g, h, gh: $\underbrace{A_{gh}}B_{gh} = \underbrace{A_g A_h}B_h + A_g B_g A_h$

$$\rightsquigarrow \quad B_{gh} = B_h + A_h^{-1} B_g A_h$$

This is a 1-cocycle inside $Z^1(G, \mathrm{Hom}(\rho_x, \rho_x)) \dashrightarrow H^1(G, \mathrm{Ad}\,\rho_x)$

$\mathrm{Ad}\,\rho_x := \mathrm{Hom}(\rho_x, \rho_x)$

We get an equality of dimensions: $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim Z^1(G, \mathrm{Ad}\,\rho_x) = m$
(all cocycles arise as above)

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§ <u>Galois</u>

We want to take $G = G_{\mathbb{Q}} = \mathrm{Aut}(\overline{\mathbb{Q}})$

K = # field, S = finite set of places of K, ~~⊗~~ K^S = max'l ext of K unramified outside S

$G_{K,S} := \mathrm{Gal}(K^S/K)$ (not finitely presented, but profinite)

<u>Q</u>: Can we repeat our previous construction ~~⊗~~ for topologically finitely generated profinite group.

If we think of $g_k \longrightarrow [x_{ijk}]$, we run into problem that relations involve "infinite products of matrices"

<u>Salvage</u>: think of x_{ijk} being "small," i.e. nilpotent or topologically nilpotent.
we can't always do this, instead

<u>start</u>: $\rho_x : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$, $g_k \longrightarrow \rho_x(g_k) + [x_{ijk}]$ ← think infinitesimally

will allow us to reconstruct $\widehat{\mathcal{O}}_{X,x}$

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exchange $\mathbb{C}$ for something p-adic

Start with $\bar{\rho}: G \longrightarrow GL_n(k)$ where k = finite field

G = profinite ~~field~~ group (top'ly f.g.) group G

Deformation: $\tilde{\rho}(g_i) = \langle \bar{\rho}(g_i) \rangle + [x_{ijk}] \longrightarrow W(k)[[x_1, \ldots, x_m]]/\circledast$

Still have a problem: if $G = G_{K,S}$, is this topologically finitely generated?
rep'ns of G are going to factor through a smaller group

← pro-p extension
L = fixed field of $\bar{\rho}$
finite
K

$W(k)[[x_1 \ldots]] / (x_1^k \quad) p^k)$

Claim: $G_{K,S}^{(p)}$ is topologically finitely generated.

We can abuse: if Γ = finite p-group, then we can take the quotient

$\longrightarrow \Gamma/[\Gamma,\Gamma]\Gamma^p \cong (\mathbb{Z}/p\mathbb{Z})^m$ $\quad [\Gamma,\Gamma]\Gamma^p = \Phi(\Gamma)$ "Frattini subgroup"

Γ is generated by n elements.

To use this, we need to show that that the largest exp-p abelian extension of L is finite, which follows from CFT, (lies in some rayclass group)

Given $\bar{\rho}: G_{K,S} \longrightarrow GL_N(k)$

- there exists a universal deformation $\rho: G_{K,S} \longrightarrow GL_N(R^{\square})$
  where $R^{\square}$ = $W(k)$-algebra, complete noetherian.
  $W(k)[[X_1, \dots, X_m]]/(\ )$ where $m = \dim Z^1(G_{K,S}, \mathrm{Ad}\,\bar{\rho})$

Historical view of deformations (not usually way Galois deformations are seen)

Again consider $\mathrm{Hom}(G, GL_N(\mathbb{C}))$. We want rep'ns up to conjugation.

$\mathrm{Hom}(G, GL_N(\mathbb{C})) /\!/ PGL_N(\mathbb{C})$ $\qquad X_N /\!/ PGL_N(\ )$

Start with $\bar{\rho}: G_{K,S} \longrightarrow GL_N(k)$ which is absolutely irreducible
(no extra automorphisms)

There exists a universal deformation ring (Schur's Lemma $\mathrm{End}\,\bar{\rho} = k$)
R which records deformation "up to conjugation."

<u>Previous thing</u>: $R^{\square}$ is called universal <u>framed</u> deformation
↑ choosing a basis.

If $\mathrm{End}\,\bar{\rho} = k$, $R^{\square} \cong R[[x_i]]_{i=1}^{N^2-1}$

$\dim m_R/(m_R^2, p) = \dim H^1(G_{K,S}, \mathrm{ad}\,\bar{\rho})$
adding mod coboundaries

<u>Q</u>: What can one say about relations?
⟶ what are the obstructions to lifting?

$$\begin{tikzcd} & GL_N(R/I^2) \arrow[d] \\ \rho: G \arrow[ur, "?"] \arrow[r] & GL_N(R/I) \end{tikzcd}$$

There is always a set-theoretic lift, call it $\tilde{\rho}$

$c(\sigma,\tau) := \rho^{\mu}(\sigma\tau)\left(\rho^{\mu}(\sigma)\rho^{\mu}(\tau)\right)^{-1} \in M_N(IR/I^2) \twoheadrightarrow M_n(k) = \mathrm{Ad}\,\bar{\rho}$

$H^2(G, \mathrm{Ad}\rho)$ — exactly measures the failure of smoothness of deformation

$R^{\square}$ or R

$= W(k)[[X_1, \ldots, X_m]]/I$ where $m = \dim Z^1(G, \mathrm{Ad}\,\bar{\rho})$ or $\dim H^1(G, \mathrm{Ad}\bar{\rho})$

$\dim I/mI = \dim H^2(G, \mathrm{Ad}\rho)$

Example: $G = \mathbb{Z}_p^R \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p^{a_i}\mathbb{Z}$ (f.g. $\mathbb{Z}_p$-module)

$N = 1$

$\bar{\rho} = \mathrm{id} \bmod p$

1 top'ly generates $\mathbb{Z}_p$,

$1 + x_i$ ← pronilpotent

$R = W(k)[[x_1, \ldots, x_r, y_1, \ldots, y_k]] \big/ \left((1+y_i)^{p^{a_i}} - 1\right)$

$H^1(G, \mathbb{F}_p)$ has dim $= R + k$ (# generators)

$H^2(G, \mathbb{F}_p)$ has dim $= k$ (# relations)