

Categor - "Galois Deformations"

3/13/12

Let G be a finitely generated group. Fix $N \in \mathbb{Z}_{>0}$

Problem: Describe $\text{Hom}(G, \text{GL}(N, \mathbb{C}))$

Let $g_k = [x_{ij}^k]_{i,j=1}^N$

Then the relations lead to a system of polynomial equations:

$$R_N, \dots, X_N = \text{Spec } R_N$$

Let x be a point of X_N , $\rho_x: G \rightarrow \text{GL}(N, \mathbb{C})$

Q: What does X_N look like in a neighborhood of x ?

We can write $\rho: G \rightarrow \text{GL}(N, R_N)$
 $\rho_x: G \rightarrow \text{GL}(N, \mathbb{C})$

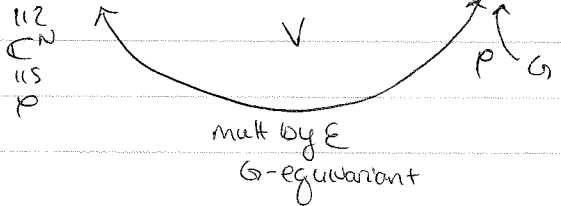
$\hat{\mathcal{O}}_{X_N, x} = \text{quotient of } \mathbb{C}\langle x_1, \dots, x_m \rangle$
 $\mathfrak{m}_x = (x_1, \dots, x_m)$
 $\dim \mathfrak{m}_x / \mathfrak{m}_x^2 = m$

Define $\tilde{\rho}_x: G \rightarrow \text{GL}(N, \mathcal{O}_{X_N, x} / \mathfrak{m}_x^2) = \text{GL}(N, \mathbb{C}\langle x_1, \dots, x_m \rangle / (x_i^2))$
 \downarrow
 $\text{GL}(N, \mathbb{C}\langle \epsilon \rangle / \epsilon^2)$

f.d. \mathbb{C} -vector space

We have the exact sequence $1 \rightarrow \epsilon \mathbb{C}\langle \epsilon \rangle / \epsilon^2 \rightarrow \mathbb{C}\langle \epsilon \rangle / \epsilon^2 \rightarrow \mathbb{C} \rightarrow 1$

$1 \rightarrow (\epsilon \mathbb{C}\langle \epsilon \rangle / \epsilon^2)^N \rightarrow (\mathbb{C}\langle \epsilon \rangle / \epsilon^2)^N \rightarrow \mathbb{C}^N \rightarrow 1$



$$1 \rightarrow \rho_x \rightarrow V \xrightarrow{\tilde{\rho}_x} \rho_x \rightarrow 1$$

$$\tilde{\rho}_x(g) = \begin{pmatrix} [A_g] & A_g B_g \\ & [A_g] \end{pmatrix} \quad \text{Think of } B_g \text{ as a map } \in \text{Hom}(\rho_x, \rho_x)$$

Compare image of g, h, gh : $\underbrace{A_g}_{\rho_x} \underbrace{B_g}_{\rho_x} \underbrace{A_h}_{\rho_x} \underbrace{B_h}_{\rho_x} = \underbrace{A_g}_{\rho_x} \underbrace{A_h}_{\rho_x} \underbrace{B_h}_{\rho_x} + \underbrace{A_g}_{\rho_x} \underbrace{B_g}_{\rho_x} \underbrace{A_h}_{\rho_x}$

$$\leadsto B_{gh} = B_h + A_h^{-1} B_g A_h$$

This is a 1-cycle inside $Z^1(G, \text{Hom}(\rho_x, \rho_x)) \dashrightarrow H^1(G, \text{Ad } \rho_x)$
 $\text{Ad } \rho_x := \text{Hom}(\rho_x, \rho_x)$

We get an equality of dimensions: $\dim m_x/m_x^2 = \dim Z^1(G, \text{Ad } \rho_x) = m$
 (all cocycles arise as above)

§ Galois

We want to take $G = G_{\bar{Q}} = \text{Aut}(\bar{Q})$

$K = \# \text{field}$, $S = \text{finite set of places of } K$, $K^S = \text{max } K \text{ ext of } K \text{ unramified outside } S$
 $G_{K,S} := \text{Gal}(K^S/K)$ (not finitely presented, but profinite)

Q: Can we repeat our previous construction for topologically finitely generated profinite group.

If we think of $g_k \rightarrow [x_{ijk}]$, we run into problem that relations involve "infinite products of matrices"

Soluce: think of x_{ijk} being "small", i.e. nilpotent or topologically nilpotent
 we can't always do this, instead

start: $\rho_x: G_{\bar{Q}} \rightarrow \text{GL}(n(\mathbb{C}))$, $g_k \rightarrow \rho_x(g_k) + [x_{ijk}]$ think infinitesimally
 will allow us to reconstruct $\hat{G}_{x,x}$

exchange \mathbb{C} for something p-adic

Start with $\bar{\rho}: G \rightarrow \text{GL}(n(\mathbb{F}))$ where $\mathbb{F} = \text{finite field}$

$G = \text{profinite group (top'ly f.g.) group } G$

Deformation: $\bar{\rho}^{\epsilon}(g_j) = \langle \bar{\rho}(g_j) \rangle + [x_{ijk}] \rightarrow \text{GL}(n(\mathbb{F})) \langle [x_1, \dots, x_m] \rangle / \ast$

Still have a problem: if $G = G_{K,S}$, is this topologically finitely generated?
 reps of G are going to factor through a smaller group

\swarrow \leftarrow pro-extension
 $L = \text{fixed field of } \bar{\rho}$
 \swarrow finite
 K

$\text{GL}(n(\mathbb{F})) \langle [x_1, \dots, x_m] \rangle / \langle (x_i^k) \rangle$

Claim: $G_{K,S}^{(0)}$ is topologically finitely generated.

We can abuse: if $\Gamma = \text{finite } p\text{-group}$, then we can take the quotient

$\rightarrow \Gamma / [\Gamma, \Gamma] \Gamma^p \cong (\mathbb{Z}/p\mathbb{Z})^m$ $[\Gamma, \Gamma] \Gamma^p = \Phi(\Gamma)$
"Frothini subgroup"

Γ is generated by n elements.

To use this, we need to show that the largest exp- p abelian extension of L is finite, which follows from CRT, (lies in some ray class group)

Given $\bar{\rho}: G_{K,S} \rightarrow GL_n(\mathbb{F})$

there exists a universal deformation $\rho: G_{K,S} \rightarrow GL_n(R^\square)$

where $R^\square = W(\mathbb{F})$ -algebra, complete noetherian.

$W(\mathbb{F})[[x_1, \dots, x_m]] / \mathfrak{I}$ where $m = \dim Z^1(G_{K,S}, \text{Ad } \bar{\rho})$

Historical view of deformations (not usually way Galois deformations are seen)

Again consider $\text{Hom}(G, GL_n(\mathbb{C}))$. We want reps up to conjugation.

$\text{Hom}(G, GL_n(\mathbb{C})) // PGL_n(\mathbb{C})$ $X // PGL_n(\mathbb{C})$

Start with $\bar{\rho}: G_{K,S} \rightarrow GL_n(\mathbb{F})$ which is absolutely irreducible (no extra automorphisms)

There exists a universal deformation ring R which records deformation "up to conjugation." (Schur's Lemma $\text{End } \bar{\rho} = \mathbb{F}$)

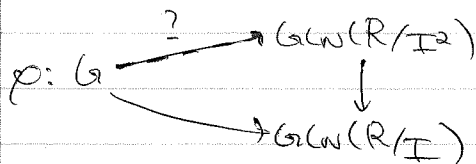
Previous thing: R^\square is called universal framed deformation
 \uparrow choosing a basis.

If $\text{End } \bar{\rho} = \mathbb{F}$, $R^\square \cong R[[x_i]_{i=1}^{n^2-1}]$

$\dim_{\mathbb{F}} R^\square / (\mathfrak{m}_R^2, \mathfrak{p}) = \dim H^1(G_{K,S}, \text{ad } \bar{\rho})$
adding mod coboundaries

Q: What can one say about relations?

\rightarrow what are the obstructions to lifting?



There is always a set-theoretic lift, call it μ

$$c(\sigma, \tau) := \rho^M(\sigma\tau) (\rho^M(\sigma) \rho^M(\tau))^{-1} \in M_n(\mathbb{R}/\mathbb{I}^2) \rightarrow M_n(\mathbb{k}) = \text{Ad } \bar{\rho}$$

$H^2(G, \text{Ad } \rho)$ — exactly measures the failure of smoothness of deformation

\mathbb{R}^D or \mathbb{R}

$$= W(\mathbb{k}) \langle x_1, \dots, x_m \rangle / \mathbb{I} \quad \text{where } m = \dim Z'(G, \text{Ad } \bar{\rho}) \text{ or } \dim H^1(G, \text{Ad } \bar{\rho})$$

$$\dim \mathbb{I}/m\mathbb{I} = \dim H^2(G, \text{Ad } \rho)$$

Example: $G = \mathbb{Z}_p^R \oplus \bigoplus_{i=1}^k \mathbb{Z}/p^{a_i}\mathbb{Z}$ (f.g. \mathbb{Z}_p -module)

$N = 1$

$\bar{\rho} = \text{id mod } p$

l top'ly generators \mathbb{Z}_p ,

$$R = W(\mathbb{k}) \langle x_1, \dots, x_r, y_1, \dots, y_\ell \rangle / \left((1+y_i)^{p^{a_i}} - 1 \right)$$

$l+x_i$ ← pronilpotent

$H^1(G, \mathbb{F}_p)$ has $\dim = R + \ell$ (#generators)

$H^2(G, \mathbb{F}_p)$ has $\dim = \ell$ (#relations)