Calegari – "Taylor–Wiles Method" 3/13/12

Goal: Let $\rho: G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O}_\lambda)$ to be a rep'n arising from an elliptic curve of conductor N = squarefree. Show ρ is modular (i.e. $\rho = \rho_{f,\lambda}$)

modular form of level $\Gamma_0(N)$, $k=2$

L = has the eigenvalues of all cuspidal f

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "\rho"] \arrow[dr] \arrow[ddrr, bend right, "\bar{\rho}"'] & \bigoplus_{i=1}^{*} GL_2(L_\lambda) & \\
 & \bigoplus_{i=1}^{*} GL_2(\mathcal{O}_\lambda) \arrow[u, phantom, sloped, "\subseteq"] \arrow[d] & \\
 & \bigoplus_{i=1}^{*} GL_2(k) \arrow[r] & GL_2(k)
\end{tikzcd}$$

indexed over all cuspidal f_i (s.t. $\bar{\rho}_{f_i} = \bar{\rho}$) ← absolutely irreducible

Note:

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "\rho"] \arrow[dr, bend right, "\bar{\rho}\oplus\cdots\oplus\bar{\rho}"'] & \bigoplus GL_2(\mathcal{O}_\lambda) \arrow[d] \\
 & \bigoplus GL_2(k)
\end{tikzcd}$$

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "\rho"] \arrow[d, "\bar{\rho}"'] & GL_2(\oplus \mathcal{O}_\lambda) \arrow[d] \\
GL_2(k) \arrow[r] & GL_2(\oplus k)
\end{tikzcd}$$

image is $GL_2(S)$ for some local $S \hookrightarrow \oplus \mathcal{O}_\lambda$

$tr(\rho(Frob_p)) = T_p$ $\mathbb{T} = \mathbb{Z}[T_1, T_2, \ldots]$ acting on modular forms

$\mathbb{T} \subseteq End(\text{modular forms})$
given by Hecke operators

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & GL_2(S) \arrow[d] \\
 & GL_2(k)
\end{tikzcd}$$

S is generated by traces of $\bar{\rho}(g)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ← image of complex conjugation

trace $= 2a$

$\bar{\rho}$ irred

S = image of T_p in $\prod \mathcal{O}_\lambda$

$$\begin{tikzcd} \mathbb{T} \arrow[r] \arrow[dr] & \mathbb{T}/m \arrow[d, equal] \\ & k \end{tikzcd}$$

image $S = \mathbb{T}_m$ (completion)

ex: $\Gamma_0(11)$, cuspidal space is 1-dim'l

$$f = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \qquad \mathbb{T} = \mathbb{Z}, \ \mathbb{T}_m = \mathbb{Z}_\ell$$

$N=37$, dim $=2$ f, g coefficients in $\mathbb{Z}$ $f \equiv g \bmod 2$

~~$\sigma f, f$ coeff in $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$~~
~~$\sigma f \equiv f \bmod 2$~~ $\Big\}$ $N=23$

Hecke algebra $\mathbb{Z}\oplus\mathbb{Z}$ s.t. $a \equiv b \bmod 2$

$$\mathbb{T}_m = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad a \equiv b$$

$$\mathbb{Z}_2[x]/x(2-x)$$

$\mathbb{T}_m/m = k$ $\quad \overline{\rho}_m : G_{\mathbb{Q}} \longrightarrow GL_2(k)$ absolutely irred.

have a deformation

$$\begin{tikzcd} G_{\mathbb{Q},N\ell} \arrow[r, "\rho"] \arrow[dr] & GL_2(R) \arrow[d] \\ & GL_2(\mathbb{T}_m) \end{tikzcd} \qquad R \twoheadrightarrow \mathbb{T}_m$$

Assumptions:

$$\overline{\rho}|_{D_\ell} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

ramified, unramified (ordinary)

$x \mid N$ $\quad \overline{\rho}|_{D_x} = \begin{pmatrix} a & * \\ 0 & * \end{pmatrix}$ ramified

$\Big\} \Longrightarrow \rho$ has the same properties ($\ell \neq 2$) $(\ell, N) = 1$

We want to now cut R down to only record deformations which could possibly be of the form

Construct $R \twoheadrightarrow R^{min} \twoheadrightarrow \mathbb{T}_m$

TW: prove $R^{min} \cong \mathbb{T}_m$.

ρ_E coming from E, $\overline{\rho}_E = \overline{\rho}_m$; deduce ρ_E is a quotient of R^{min}

residually modular

Note: Have to make assumption that ρ_E is residually modular

From before, ρ_E is a quotient of R^{min} $\Rightarrow$ ρ_E is a quotient of $\mathbb{T}_m$
$\Rightarrow$ it is modular.

We know something about $m_{R^{min}} / (m_{R^{min}}^2, \ell)$

$\leq H^1_{\Sigma}(G_{\mathbb{Q}}, Ad\,\bar{\rho})$

local conditions corresponding to $(\bar{\rho}|_{D_\ell})$

Suppose $\dim H^1_{\Sigma}(G_{\mathbb{Q}}, Ad\,\bar{\rho}) = 0$

$\Rightarrow R^{min}$ = quotient of $W(k)$

$$W(k) \twoheadrightarrow R^{min} \twoheadrightarrow \mathbb{T}_m$$
$$= \qquad\qquad = \qquad\qquad$$

torsion free

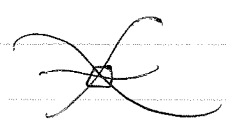
Thm: $\dim H^1_{\Sigma}(G_{\mathbb{Q}}, Ad\,\bar{\rho}_{E,\ell}) = 0$ all but finitely many ℓ

Doesn't work so well (needs E already modular)

Assume $\dim H^1_{\Sigma}(G_{\mathbb{Q}}, Ad\,\bar{\rho}) = 1$

$$W(k)[[X]] \twoheadrightarrow R^{min} \longrightarrow \mathbb{T}_m$$

$W(k)[[X]] / X(X-p)(X-2p)$



$$\begin{tikzcd} R^{min} \arrow[r, two heads] & \mathbb{T}_m \\ R^{min}_{Q_N} \arrow[u] \arrow[r, two heads] & \mathbb{T}_{Q_N, m} \arrow[u, two heads] \end{tikzcd}$$

such that T.S stays the same

auxiliary primes

allow ramification

$$\frac{|H^1_\Sigma(\mathrm{Ad}\,\bar\rho)|}{|H^1_{\Sigma^*}(\mathrm{Ad}\,\bar\rho)^*|} = 1$$

Choose an auxiliary prime q, $\dfrac{|H^1_{\Sigma\cup q}(\mathrm{Ad}\,\bar\rho)|}{|H^1_{(\Sigma\cup\{q\})^*}(\mathrm{Ad}\,\bar\rho)^*|} = \ell$

$$\text{Choose } q \text{ s.t. } H^1_{\Sigma^*}((\mathrm{Ad}\,\bar\rho)^*) \xleftarrow{\ \ } H^1_{(\Sigma\cup q)^*}((\mathrm{Ad}\,\bar\rho)^*)$$

dim >

$$\downarrow$$

$$H^1(\mathbb{Q}_q, \quad)$$

ex. $H^1_\Sigma(\mathbb{Q}, \mathbb{F}_p) \longrightarrow H^1(\mathbb{Q}_q, \mathbb{F}_p)$

this is surjective for a positive density of q

What do we understand about $\mathbb{T}_{Q_N, m}$ $\quad \Gamma_0(N) \cap \Gamma_1(q)$

$\bar\rho(\mathrm{Frob}_q) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \bmod \ell$ at q, $\Gamma(1)$

if we change $\Gamma_0(q)$, expect to get $\begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}$

~~get new stuff if $\alpha/\beta = q^{\pm 1} \bmod \ell$~~ $\quad \varepsilon(\mathrm{Frob}_q) = q$.

if we change $\Gamma_1(q)$, expect to get $\begin{pmatrix} \alpha\psi & 0 \\ 0 & \beta\psi \end{pmatrix}$

get new stuff if $q \equiv 1 \bmod \ell$ $\qquad \psi \equiv 1 \bmod \ell$ $\quad$ tamely ramified

Choose q s.t. $q \equiv 1 \bmod \ell$ not $\frac{\alpha}{\beta} = q^{\pm 1} \bmod \ell$

in fact, choose $q \equiv 1 \bmod \ell^n$ and if $\varphi(\mathrm{Frob}_q) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \bmod \ell$, then $\frac{\alpha}{\beta} = q^{\pm 1} \bmod \ell$

and $\dim H^1_{\Sigma\cup\{q\}} = \dim H^1_\Sigma$

(can have this by Chebotarev)

$H(X)^2_m \longrightarrow H(X_0(q))_m$

$T_q \qquad\qquad (U_q-\alpha)(U_q-\beta)$ don't get more stuff

Consider the map $X_1(q) \longrightarrow X_0(q)$

$\Delta = (\mathbb{Z}/q\mathbb{Z})^\times$

In fact, can take $\Delta = (\mathbb{Z}/\ell^n\mathbb{Z})$, $X_H(q) \longrightarrow X_0(q)$

$$\begin{tikzcd}
H^1(X_H(q),\mathbb{Z})_m \\
H^1(X_0(q),\mathbb{Z})_m \arrow[u]
\end{tikzcd}$$

free over $\mathbb{Z}_\ell[\Delta]$

Suppose was free over $\mathbb{T}_m$ and free over $\mathbb{T}_{Q,m}$, then we could show $\mathbb{T}_{Q,m}$ thickens $\mathbb{T}_m$.

$$\begin{tikzcd}
W(k)[[X]] \arrow[r] & R^{min} \arrow[r] & \mathbb{T}_m \\
\mathbb{Z}_p[\Delta] \arrow[r, bend left] \arrow[rr, bend right] & R^{min}_Q \arrow[r] \arrow[u] & \mathbb{T}_{Q,m} \arrow[u, two heads]
\end{tikzcd}$$

Suppose we have $q \equiv 1 \bmod \ell^\infty$

need to approximate

$$\begin{tikzcd}
W(k)[[X]] \arrow[r] \arrow[dr, "\sim"] & R^{min} \arrow[r, "\sim"] \arrow[dr, phantom, "\shortparallel"] & \mathbb{T}_m \\
W(k)[[T]] \arrow[r] & R^{min}_Q \arrow[u] \arrow[r, "\sim"] & \mathbb{T}_{Qm} \arrow[u]
\end{tikzcd}$$

Take $q \equiv 1 \bmod^k$ $\quad R_Q /\!- \quad W[[T]]/(T^n, p^n)$

get R^{min}_∞, $\mathbb{T}_\infty$