Galois Deformations - II Calegari

$\bar{\rho} : G_{\mathbb{Q}} \to GL_2(k)$ modular. $\mathbb{T}_m$ = Hecke ring (modular deformations min deform of $\bar{\rho}$)

R^{min} = min def that "look" modular.

$$R^{min} \twoheadrightarrow \mathbb{T}_m.$$

$$m_R/(m_R^2, p) \subseteq H^1_\Sigma(\mathbb{Q}, \mathrm{Ad}\,\bar{\rho})$$

Auxillary TW prime

$q \equiv 1 \ (p^m)$

$\Delta = \mathbb{Z}/p^m\mathbb{Z}$.

$$\begin{tikzcd}
W(k)[[X]] \arrow[r] \arrow[dr, bend right, two heads] & R^{min} \arrow[r, two heads] & \mathbb{T}_m \\
 & R_q \arrow[u, two heads, "\text{quotient by } \mathfrak{a}"'] \arrow[r, two heads] & \mathbb{T}_{q,m} \arrow[u] \\
W(k)[\Delta] \arrow[ur] \arrow[urr, bend right] \arrow[d] & & \\
W(k) & &
\end{tikzcd}$$

quotient by $\mathfrak{a}$ = augmentation ideal.

free module / $W(k)[\Delta]$.

$$\begin{tikzcd}
\text{smooth} \arrow[r] & W(k)[[X]] \arrow[r, two heads] \arrow[dr, two heads] & R^{min} \arrow[r, two heads] & \mathbb{T}_m \\
 & & R_\infty \arrow[u, "T=0"] \arrow[r] & \mathbb{T}_\infty \arrow[u, "T=0"] \\
 & W(k)[[T]] \arrow[urr, bend right, "\text{free}"'] & &
\end{tikzcd}$$

$\dim \mathbb{T}_\infty \geq 2$.

$\dim R_\infty \leq 2$.

$R_\infty = T_\infty$, $R^{min} = \mathbb{T}_m$

<u>Kisin-Taylor-Wiles:</u>

<u>Thm:</u> $p = \text{odd}$ If $K/\mathbb{Q}$ abelian extn. of degree p and $K/\mathbb{Q}$ unr., then K does not exist.

<u>Pf:</u> "modular" reps ~~of~~ for $GL_1/\mathbb{Q}$ are dirichlet characters.

$$\bar{\rho} : G_{\mathbb{Q}} \to \mathbb{F}_p^\times \quad \text{trivial repr.}$$

R^{min} = deformations which are unramified everywhere

$$m_R/(m_R^2, p) \subseteq H^1_\Sigma(\mathbb{Q}, \mathbb{F}_p)$$

for $q \neq p$. $\Sigma_q \subseteq H^1(\mathbb{Q}_q, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } q \equiv 1 \ (p). \\ \mathbb{F}_p^2 & \text{if } q \equiv 1 \ (p). \end{cases}$

$$\Sigma_q = H^1(G_k, \mathbb{F}_p) \leftarrow \text{unr. classes}, \quad k = \mathbb{F}_q.$$

$q = p$, ~~Σ_p~~ $\dim H^1(G_p, \mathbb{F}_p) = 2$

$$\Sigma_p = H^1(G_{\mathbb{F}_p}, \mathbb{F}_p) = 1 \quad (\subseteq H^1(G_p, \mathbb{F}_p))$$

$$\Sigma_\infty = H^1(G_{\mathbb{R}}, \mathbb{F}_p) = 0 \quad \text{since } p = \text{odd}.$$

$(\mathbb{F}_p)^* = \mu_p \leftarrow$ dual of $\mathbb{F}_p$

$$\Sigma_q^* = \begin{cases} H^1(G_{\mathbb{F}_q}, \mu_p) & \text{if } p \neq q \\ & \Sigma_p^* \subseteq H^1(G_{\mathbb{Q}_p}, \mu_p)^\perp \quad (\dim = 1 \text{ by Tate Duality}) \end{cases}$$

$$\frac{|H^1_\Sigma(G, \mathbb{F}_p)|}{|H^1_{\Sigma^*}(G, \mu_p)|} = \frac{|H^0(\mathbb{Q}, \mathbb{F}_p)|}{|H^0(\mathbb{Q}, \mu_p)|} \prod_v \frac{|\Sigma_v|}{|H^0(G_v, \mathbb{F}_p)|}$$

$$= \frac{p}{1} \cdot \frac{|\Sigma_p|}{|H^0(G_p, \mathbb{F}_p)|} \cdot \frac{|\Sigma_\infty|}{|H^0(G_{\mathbb{R}}, \mathbb{F}_p)|}$$

$$= \frac{p}{1} \cdot \frac{p}{p} \cdot \frac{1}{p} = 1.$$

Assume $\dim H^1_\Sigma(G, \mathbb{F}_p) = 1$.

$$\begin{array}{ccccc} \mathbb{Z}_p[[X]] & \twoheadrightarrow & R^{min} & \twoheadrightarrow & \mathbb{T}_m \ (= \mathbb{Z}_p) \\ & \searrow & \uparrow & & \\ & & R^q & \twoheadrightarrow & \mathbb{T}_{q,m} \end{array}$$

want $q \equiv 1 \pmod{p^n}$

$$(\Sigma \cup \{q\})_v = \Sigma_v \quad \text{if } v \neq q$$

$$(\Sigma \cup \{q\})_q = H^1(G_q, \mathbb{F}_p).$$

On the dual side: $(\Sigma \cup \{q\})^*_q = 0 \subseteq H^1(G_q, \mu_p)$.

Now, $\dfrac{|H^1_{\Sigma \cup \{q\}}(\mathbb{F}_p)|}{|H^1_{\Sigma \cup \{q\}}(\mu_p)|} = p$ (p appeared because of the ramified extn. at q).

$$H^1_{\Sigma^*}(\mathbb{Q}, \mu_p) \longleftrightarrow \begin{pmatrix} \mu_p & * \\ 0 & 1 \end{pmatrix}$$

$[c] \in H^1_{\Sigma^*}$

$$\begin{tikzcd} L_c \arrow[d, dash, "p"'] \\ \mathbb{Q}(\zeta_p) \arrow[d, dash] \\ \mathbb{Q} \end{tikzcd}$$

L_c } unr. at all primes.

Frob $\equiv 1$ in $(\mathbb{Z}/p\mathbb{Z})^\times$.

$[c] \in \Sigma^*$.

does $[c] \in H^1_{(\Sigma \cup \{q\})^*}(\mathbb{Q}, \mu_p)$

$q \equiv 1 \pmod p$.

$$[c] \mapsto H^1(G_q, \mu_p), \quad \mapsto 0 \iff \mathrm{Frob}_q = 1 \text{ in } \mathrm{Gal}(L_c/\mathbb{Q}).$$

Chebotarev thm: $\Rightarrow [c] \to 0 \quad H^1(\mathbb{Q}_q, \mu_p)$.

Choose such a $q \equiv 1 \pmod{p^n}$. $\Delta = (\mathbb{Z}/p^n\mathbb{Z})^\times$

$$\begin{tikzcd} \mathbb{Z}_p[[x]] \arrow[r, two heads] \arrow[dr] & R^{min} \arrow[r] & (\mathbb{T}_m = \mathbb{Z}_p) \\ & R^q \arrow[u] \arrow[r] & \mathbb{T}_{m,q} \cong \mathbb{Z}_p[\Delta] = \mathbb{Z}_p[(\mathbb{Z}/q\mathbb{Z})^\times \otimes \mathbb{Z}_p] \arrow[u, dashed] \\ \mathbb{Z}_p[\Delta] \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

patch together $\Rightarrow$

$$\begin{tikzcd} \mathbb{Z}_p[[x]] \arrow[r, two heads] & R_\infty \arrow[r, two heads] \arrow[d] & \mathbb{T}_\infty = \mathbb{Z}_p[[T]] \arrow[d, "T=0"] \\ & R_p \arrow[r, equal] & \mathbb{Z}_p \end{tikzcd}$$

We know that $R^{min} \cong \mathbb{T}_m \qquad GL_2/\mathbb{Q}$.

Assumed $\rho_{E,\ell}$ "no more" ramified than $\rho_{E,\ell}$, E elliptic curve.

This is not true in general.

$R_q \overset{?}{\simeq} \mathbb{T}_{q,m}$.

$$\begin{tikzcd}
 & R^{min} \arrow[r, "\sim"] & \mathbb{T}_m \\
W(k)[[X_1, \dots, X_m, X_{m+1}]] \arrow[r] & R_q \arrow[r] \arrow[u] & \mathbb{T}_{q,m} \arrow[u] \\
W(k)[[T_1, \dots, T_m]] \arrow[r] \arrow[rr, bend right, "\text{free}"'] & R_\infty \arrow[r] \arrow[u] & \mathbb{T}_\infty \arrow[u, two heads]
\end{tikzcd}$$

Γ . . . $\bar{\rho}(\mathrm{Frob}_q) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$; $q \not\equiv 1 \pmod p$; $\alpha\beta = q$.

$|$

$\Gamma_0(q)$ If the elliptic curve has multiplicative reduction, then.

$$\rho : G_{\mathbb{Q}_q} \to GL_2(\overline{\mathbb{Q}}_p), \quad \rho|_I = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

$$\rho(\mathrm{Frob}_q) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} (\pm 1) . \leftarrow \text{Tate curve}$$

So let's assume $\alpha = q$, $\beta = 1$.

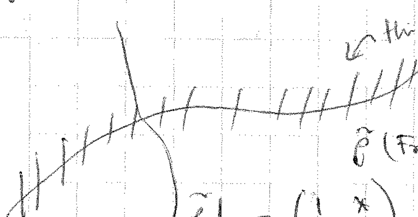
$$\rho : G_{\mathbb{Q}} \to GL_2(R_q)$$

$$\rho : G_{\mathbb{Q}_q} \to GL_2(R_q)$$

$\rho|_{G_{\mathbb{Q}_q}}$ local deformation of $\bar{\rho}$ restricted to $G_{\mathbb{Q}_q}$.

Assume the local deformation ring is R_q^{loc}. and we have a map $R_q^{loc} \to R_q$

Spec R_q^{loc}.



thickening here.

$\tilde{\rho}(\mathrm{Frob}_q) = \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\beta} \end{pmatrix}$

$\tilde{\rho}|_I = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $* \neq 0$.

So $\tilde{\rho}(\mathrm{Frob}) - \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{tikzcd}
W(k)[[X_1, \dots, X_{m+1}]] \arrow[r, two heads] \arrow[d] & R_q \\
R_q^{loc}[[X_1, \dots, X_m]] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & R_q^{loc}[[X_1, \dots, X_{m+1}]] \arrow[r] & R_\infty \arrow[r] & T_\infty \\
\mathbb{Z}_p[[T_1, \dots, T_m]] \arrow[urr] & \mathbb{Z}_p[[T_1, \dots, T_{m-1}, Y, Z]]/(YZ) \arrow[l] & &
\end{tikzcd}$$

$\operatorname{Spec} \mathbb{T}_m\left[\frac{1}{p}\right] \hookrightarrow \operatorname{Spec} R_\infty\left[\frac{1}{p}\right]$.

$\dim = n$