~~Kisin~~ Kisin - Taylor - Wiles : Calegari

X/F Smooth proper alg. variety / # field. $\rightsquigarrow$ attach motivic L-function

in terms of $\# X(\mathbb{F}_q)$ or $H^*_{ét}(X_{\bar F}, \bar{\mathbb{Q}}_p) \circlearrowleft \mathrm{Gal}(\bar F/F)$.

Conj: $L(X, s)$ has "good" analytic properties.

$V \subseteq H^*_{ét}(X/\bar F, \bar{\mathbb{Q}}_p)$ is an irreducible representation.

Step 0: Identify the group G for which automorphic forms are defined.

$$\rho: G_F \to GL(V) \cong GL_N(\bar{\mathbb{Q}}_p).$$

$\overset{?}{\exists}\ \pi$ for $GL(n)/F$ associated to ρ. $\overset{???}{\longrightarrow}$ $\overset{?}{\exists}\ r(\pi)_\wp : G_F \to GL_N(\bar{\mathbb{Q}}_p)$.

(via Local Langlands)

The Frobenius has n-eigenvalues $\rightsquigarrow$ Satake parameters for the representation

i.e. $r(\pi)_\wp = \rho$.

V may have extra symmetries, which can lead to an alternate choice of G.

X = curve, $V = H^1_{ét}(X/\bar F, \bar{\mathbb{Q}}_p)$ $2g$-dim'l.

$$V \times V \longrightarrow \mathbb{Q}_p(1) \qquad \text{by Poincare duality.}$$

$$\rho: G_F \longrightarrow GSp_{2g}(\bar{\mathbb{Q}}_p)$$

$\rho: G_F \to G(\bar{\mathbb{Q}}_p)$. by compactness

$$\begin{tikzcd} G_F \arrow[r, "\rho"] \arrow[dr, "\bar\rho"'] & G(\mathcal{O}_\wp) \arrow[d] \\ & G(k) \end{tikzcd} \qquad [k:\mathbb{F}_p] < \infty.$$

Step 1: $\bar\rho$ is modular. (i.e. $\exists\ \pi$, ρ_π, such that $\bar\rho = \bar\rho_\pi$
(Hard)

Step 2: V is unramified outside S. $\bar\rho = G_F$ acting on $\bar V$ (makes sense if $\bar V$ is irreducible)

$R^\square$ = framed deformations of ρ satisfying local conditions at $v \in S$ and unramified outside S.

$v \in S$, $v \nmid p$ no restriction.

$v \mid p$. want to record deformations which are potentially semi-stable. "looks geometric". with added restrictions (like fixing Hodge-Tate wts, or semi-stable after a fixed E_v/F_v)

But we cannot put any deformation condition:
fr ex: If we ~~argue~~ want ~~that~~ the eigenvalues of Frob $\in \mathbb{Z}$.

then we have the following:

$$I \longrightarrow \mathbb{Z}_p[[T]] \longrightarrow \underset{\substack{T\mapsto a \\ a\in\mathbb{Q}}}{\oplus} \text{ quot.}$$

$I = (0)$, then you don't cut anything out.

Step 3: Construct a map:

$$R^{\square} \longrightarrow \mathbb{T}^{\square}_{m}$$

$\oplus\, \pi^K$ ← level
autom orphic for GSp_4
$\mathbb{T}^{\square}_m$.

needs Galois representations associated to π.
• π satisfy (some form) of local-global compatibility.

When can we solve this problem?
GL_2/F^+, F^+ = totally real.
π_∞ = discrete series $\Longleftrightarrow$ V (distinct Hodge-Tate wts = regular)

dim $V=3$, $F=\mathbb{Q}$. $\mathrm{im}(\rho_V) = GL_3$ $GL_3/\mathbb{Q}$ ← no ideas about it as π_∞ is not discrete series.

dim $V=2$, F totally real, $\mathrm{Im}(\rho) = \tilde{A}_5$? should come from some Maaß form.

X/F curve of genus >1 End $(J(X)) = \mathbb{Z}$. $H^1(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$.
If $g \geq 3$, no idea.

Step 4: (Kisin-Taylor-Wiles) Local deformation conditions make $R^{\square}$ an algebra over $R^{\square}_v$, $v \in S$, so this makes $R^{\square}$ an ~~and~~ algebra over $\bigotimes_{v\in S} R^{\square}_v =: R^{\square}_{loc}$.

$$\begin{tikzcd}
R^{\square}_{loc}[[x_1,\dots,x_m]] \arrow[r] & R^{\square} \arrow[r, two heads] & \mathbb{T}^{\square} \\
& R^{\square}_\infty \arrow[u, "0=T_i"] \arrow[r] & \mathbb{T}^{\square}_\infty \arrow[u, "T_i=0"'] \\
W(k)[[T_1,\dots,T_d]] \arrow[ur, bend right] \arrow[urr, bend right=40, "\text{free}"'] & &
\end{tikzcd}$$

In Taylor-Wiles, needed freeness of $\mathbb{T}_Q$ over $W(k)[\Delta]$.

– needed to assume that relevant π only contributed to cohomology in ~~other~~ 1 degree
in other degrees
– vanishing of cohomology (mod p), [This was not an issue for modular curves]

Transfer to an inner form, where the locally symmetric space is finite.

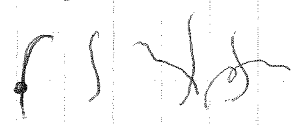
- Need to assume Π_∞ = discrete series

Want to compare sizes of $\mathbb{T}^\square_\infty$ and $R^\square_{loc}[[x_1, \dots, x_m]]$.

$\mathrm{Spec}\left(\mathbb{T}^\square_\infty\left[\frac{1}{p}\right]\right)$, all components have dim $\geq m$

closed $\subseteq$

and $\mathrm{Spec}\left(R^\square_\infty\left[\frac{1}{p}\right]\right)$ have dim $= n$.



$S = \{q, p\}$.

ex: If X = modular curve

$\boxed{\text{Ihara's lemma}}$

Fricke involution.

$$\begin{array}{c} H^1(X,\mathbb{Z})_m \\ \oplus \\ H^1_p(X,\mathbb{Z})_m \end{array} \hookrightarrow H^1(X_0(q),\mathbb{Z})_m \twoheadrightarrow \begin{array}{c} H^1(X,\mathbb{Z})_m \\ \oplus \\ H^1(X,\mathbb{Z})_m \end{array}$$

$$\bar\rho = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} q+1 & T_q \\ T_q & q+1 \end{pmatrix} \qquad \det = (q+1)^2 - T_q^2.$$

So if $\bar\rho = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ can see new stuff in the cohomology of $X_0(q)$

~~The result~~

Back to Step 1: To show $\bar\rho$ is modular.

Base change + automorphic induction $\rightsquigarrow \rho : G_F \to GL_2(\bar{\mathbb{Q}}_p)$

So if F'/F, such that $\rho : G_{F'} \to GL_2(\bar{\mathbb{Q}}_p)$.

~~So is $\bar\rho$~~

$\rho \rightsquigarrow \pi \quad GL_2/F$

$\downarrow BC(\pi)$. Is $\rho|_{G_{F'}}$ is modular?

$\rho|_{G_{F'}} \rightsquigarrow \pi'$

If F'/F Galois cyclic extension. $\chi : G_{F'} \to \mathbb{Q}_p^\times$.

$$\rho : \mathrm{Ind}_{G_{F'}}^{G_F} \chi \longrightarrow GL_d(\bar{\mathbb{Q}}_p)$$

If $\chi : G_{F'} \to \mathbb{F}_p^\times$, $\bar\rho : \mathrm{Ind}_{G_{F'}}^{G_F} \chi \to GL_d(\mathbb{F}_p)$

E modular $\rightsquigarrow \bar\rho_{E,p}$ modular. $E'[p] = E[p] \rightsquigarrow \bar\rho_{E,p}$.

$E'[\lambda] = \mathrm{Ind}\, \chi$.

$\mathrm{Ind}_{mod}\, \bar\chi \to E'$ mod $\to E'[p]$ modular $\to E(p)$ modular $= \bar\rho_{E,p}$ modular $\Rightarrow E$.

Find ~~modular~~ elliptic curves with prescribed p and ℓ torsion.
So find points on $X(p\ell)^{\text{twist}}$. At least try to find points over some extn. F'.

$\checkmark$ "modular" over F'/F.