Ana - "Local Langlands & local-global" 3/15/12

Fix ℓ-prime numbers $\neq p$, $\iota_\ell : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

Recall: $n=1$ CFT

F any # field, $v \nmid \infty$ place of F

$W_{F_v} \overset{\text{dense}}{\subseteq} G_{F_v}$ where W_{F_v} is the Weil group

Local CFT: $\exists$ a map $\mathrm{Art}_{F_v} : F_v^\times \xrightarrow{\sim} W_{F_v}^{ab}$ uniformizer $\mapsto$ geometric Frob

Via the Artin map, we can compare characters of $F_v^\times$ & W_{F_v}

(For $v | \infty$, $\mathrm{Art}_{F_v} : F_v^\times / (F_v^\times)^\circ \xrightarrow{\sim} G_{F_v}$) rep'ns of $Gl(1)$

Global CFT: $\left\{ \begin{array}{l} \text{grossencharacters} \\ \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times \end{array} \right\}$ algebraic (condition at ∞) "integral infinitesimal character" $\xleftrightarrow{1-1}$ $\left\{ \begin{array}{l} \text{continuous } \ell\text{-adic} \\ \text{characters} \\ Gal(\overline{F}/F) \to \overline{\mathbb{Q}}_\ell^\times \end{array} \right\}$ "de Rham at ℓ"

(r_ℓ or π_ℓ)

We can compare via global Artin map: $\mathrm{Art}_F = \prod_{\substack{v \\ \text{places of } F}} \mathrm{Art}_{F_v}$ $\quad \mathbb{A}_F^\times / F^\times (F_\infty^\times)^\circ \xrightarrow{\sim} G_F^{ab}$

If we write $\chi \longmapsto \pi_\ell(\chi)$, then $\pi_\ell(\chi)|_{W_{F_v}} \simeq \pi_\ell(\chi_v)$

We have local-global compatibility by construction.

(ex: $(\mu_{\ell^n})_n \circlearrowleft Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ $\quad \mathrm{Frob}_p$ acts via $\frac{1}{p}$)

Gl_2 - cuspidal automorphic rep'ns of $Gl_2(\mathbb{A})$

$\updownarrow$

2 dim'l Galois rep'ns

For Gl_n, the conjecture due to Langlands & Fontaine-Mazur:

$\left\{ \begin{array}{l} \text{cuspidal aut. rep'ns} \\ \text{of } Gl_n(\mathbb{A}_F), \\ \text{algebraic (condition on } \pi_\infty \\ \text{to have integral infinitesimal} \\ \text{character} \end{array} \right\} / \sim$ $\xleftrightarrow{1-1}$ $\left\{ \begin{array}{l} \text{continuous irreps} \\ R : Gal(\overline{F}/F) \to Gl_n(\overline{\mathbb{Q}}_\ell) \\ \bullet \text{ unramified at almost all primes} \\ \bullet \text{ de Rham at } \ell \end{array} \right\} / \sim$

$\pi \longrightarrow R_\ell(\pi)$

Known: • F - CM field

• $\pi^\vee \simeq \pi \circ c$ (π descends to an aut rep of unitary group)

• π reg. algebraic ((condition on π_∞)

<u>Note</u>. Π regular algebraic - needed in order to "see" Π in cohomology of Shimura variety. (For Gl_2, $\Rightarrow$ m.f. has wt ≥ 2 ; holomorphic)

K - p-adic field

<u>WD-rep'n</u> of W_K over $\mathbb{C}$ is a triple (V, r, N)
- V: n-dim'l ~~rep~~ vector space over $\mathbb{C}$
- r: rep'n of W_K over $\mathbb{C}$
- N: nilpotent operator on V

(in an ℓ-adic Gal rep'n, ℓ-part of tame inertia acts unipotently $\Rightarrow$ N encodes action of ℓ-part of tame inertia)

<u>Thm</u>: (HT, Henniart) We have a bijection

$$\left\{\begin{matrix}\text{irred smooth} \\ \text{rep'ns of} \\ Gl_n(K) \text{ over } \mathbb{C}\end{matrix}\right\}/\sim \quad \xleftrightarrow[\text{rec}]{1-1} \quad \left\{\begin{matrix}\text{Frobenius semisimple } (r \text{ is semisimple}) \\ n\text{-dim'l WD-rep'n of } W_K \\ \text{over } \mathbb{C}\end{matrix}\right\}/\sim$$

The statement of local-global compatibility:

y a place above p

$$WD\left(R_\ell(\Pi)\Big|_{Gal(\bar{F}_y/F_y)}\right)^{F\text{-}ss} \simeq \iota_\ell^{-1}\left(rec\left(\Pi_y \otimes |\det|^{\frac{1-n}{2}}\right)\right)$$

Desirable properties of $r_\ell(\pi) \simeq \iota_\ell^{-1} rec(\pi \otimes |\det|^{\frac{1-n}{2}})$
- for $n=1$, get it via local cft
- $\pi \subseteq Ind_P^G(\pi_1 \times \dots \times \pi_k)$, then $r_\ell(\pi)^{ss} = r_\ell(\pi_1)^{ss} \oplus \dots \oplus r_\ell(\pi_k)^{ss}$

  π is an irreducible constituent of

  (up to ss without keeping track of N, monodromy)

$n=3$, Gl_3, B - Borel $= \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

Let π_1 be a discrete series subquotient of $Ind_B^G\begin{pmatrix}\chi & & \\ & \chi|\det| & \\ & & \chi|\det|^2\end{pmatrix}$ where χ is an unramified character.

Then semisimplification should be: $r_\ell(\pi_1)^{ss} = r_\ell(\chi) \oplus r_\ell(\chi|\det|) \oplus r_\ell(\chi|\det|^2)$

and N should be $\begin{pmatrix} 0 & & 0 \\ 1 & 0 & \\ 0 & 1 & 0\end{pmatrix}$ takes N N

(consequence of weight-monodromy conjecture)

If Π aut rep'n w/ $\Pi_y \simeq \pi_1$, want $R_\ell(\Pi_y)$ irred.
If N has the above form, proves irreducible.

(need to be careful between indecomposable vs irred)

ex: χ_1, χ_2 unramified characters of $K^\times = F_y^\times$

$Sp_2(\chi_1)$ - Steinberg rep'n (subquotient $\mathrm{Ind}_B^{GL_2}(\chi_1\ \chi_1|\det|)$)

$\pi_2 = \mathrm{Ind}_{(\,\llcorner\,)}^{GL_3}(Sp(\chi_1)\otimes\chi_2)$ ← irreducible

Assume

$\chi_1 \neq \chi_2|\det|$, $\chi_2 = \chi_1|\det|^2$

$r_\ell(\pi_2) = r_\ell(\chi_1|\det|) \oplus r_\ell(\chi_1) \oplus r_\ell(\chi_2)$

$N = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & \\ & & \end{pmatrix}$ | π_3 - supercuspidal
how do we find $r_\ell(\pi_3)$?

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use Shimura variety of HT-type for $n=3$
→ unitary similitude group $G(\mathcal{U}(1,2)\times\mathcal{U}(0,3)^{d-1}) =: G$

π_1, π_2 have Iwahori level at y

$\mathcal{U} \subseteq G(\mathbb{A}^\infty)$ - level

$\mathcal{U}_y = \{M \in GL_2(\mathcal{O}_{F_y}) : M \equiv \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \bmod y\}$

$X_{\mathcal{U}}$ for such $\mathcal{U}$ having Iwahori level at y

cohomology:

$$H(X) = \varinjlim_{\mathcal{U}} H(X_{\mathcal{U}} \times_F \bar{F}, \bar{\mathbb{Q}}_\ell)$$

$G(\mathbb{A}^{\infty,y})$ ↻, Gal(↻

← ℓ-adic sheaf encodes weight

$\left(H(X) = (-1)^i H^i(X) \right.$ Grothendieck Group $\left.\right)$

can write $H(X) = \bigoplus_{\pi^{\infty,y}} \pi^{\infty,y} \otimes R_\ell(\pi^{\infty,y})$

(supercuspidal, Galois)

irred adm rep'ns of $G(\mathbb{A}^{\infty,y})$

here we see $R_\ell(\Pi)$ Π aut rep'n of $GL_n(\mathbb{A}_F)$

$(\Pi_y = \pi_1 \text{ or } \pi_2)$

Bottomline: want to compute "$\pi^{\infty,y}$"-part of $H(X)$

- use integral model $/\mathcal{O}_K$, where $K = F_y$

Assumptions: $F = F^+E$

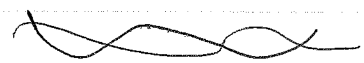
p-splits in E

y-preferred place above p.

note: for $n=2$, Iwahori level parametrizes deg-$k(y)$-isogenies
$A_1 \longrightarrow A_2$ of abelian varieties

This is the same as chains of two isogeny deg $k(y)$ $\quad g_1 \longrightarrow g_2 \longrightarrow g_1/g_1[p]$

where $g_i = A_i[p^\infty]$

in the special fiber (char p) 

For $n=3$, Iwahori level parametrizes chains of deg$(k(y))$-isogenies

$$g_1 \xrightarrow{f_1} g_2 \xrightarrow{f_2} g_3 \xrightarrow{f_3} g_1/g_1[\varpi]$$

where ϖ is a uniformizer of K

g_1: 1-dim'l p-divisible group w/ $\mathcal{O}_K$-action

There are supersingular points, i.e. $A[\varpi](\overline{\mathbb{F}}_p) = \{0\}$
p-divisible group - connected height 3

use parameters for maps induced on Lie algebras, call param X_1, X_2, X_3
from f_1, f_2, f_3

locally, X_U looks like $\mathcal{O}_K[[X_1, X_2, X_3]]$

For $n=3$, at ss point X_U "looks like" $\mathcal{O}_K[[X_1,X_2,X_3]]\,/\,X_1X_2X_3-\varpi$

$\Rightarrow$ stratification of Y - special fibre of X_U

$Y = Y_1 \cup Y_2 \cup Y_3$ where Y_1 - 1st isogeny map induces 0 map on Lie algebra.

ss-loci: $Y_1 \cap Y_2 \cap Y_3$ - smooth 1-dim'l $\}\ Z^0$

Z^1: $\{\ Y_1 \cup Y_2 \sqcup Y_2 \cap Y_3 \sqcup Y_1 \cap Y_3$
étale height ≤ 1
conn ≥ 2
smooth 1-dim'l

$Y_1 \cup Y_2 \cup Y_3$
étale height ≤ 2
conn ≥ 1
sm, 2 dim'l $\}\ Z^2$

spectral sequence $\Rightarrow H^{i+j}(X)$

for $H^2(X)$ $\quad E_1^*$-page $\qquad$ N nonzero because it is a Tate twist

$H^0(Z^0)(-2) \xrightarrow{N}$

$H^1(Z^1)(-1)$

$H^0(Z^0)(-1) \oplus H^2(Z^2) \xrightarrow{N}$

$\xrightarrow{N} H^2(Z^1) \qquad H^0(Z^0)$

Recall π_1 from $\begin{pmatrix} \chi & & \\ & \chi|\det| & \\ & & \chi|\det|^2 \end{pmatrix}$

π_1 part of spectral sequence

$$\begin{tikzcd}[column sep=small, row sep=small]
r_\ell(\chi)(-2) \arrow[dr, bend left, "N"] & & \\
0 & r_\ell(\chi)(-1) \arrow[dr, bend left, "N"] & \\
& 0 & r_\ell(\chi)
\end{tikzcd}$$

$r_\ell(\pi_1) = r_\ell(\chi)(-2) \oplus r_\ell(\chi)(-1) \oplus r_\ell(\chi)$

$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\pi_2 \quad \chi_1, \ \chi_1|\det|, \ \chi_2$

$$\begin{tikzcd}[column sep=small, row sep=small]
0 & & & \\
& r_\ell(\chi_1)(-1) \arrow[ddrr, bend left] & & \\
& & r_\ell(\chi_2) & \\
& & & r_\ell(\chi_1) \\
& & & 0
\end{tikzcd}$$