Local models: Keerthi

Local structure of $Y_0(N)_{\mathbb{Z}_{(p)}}$, $p \nmid N$.

$Y_0(N)$ parametrizes (E, C_N), $C_N \subseteq E[N]$ cyclic subgroup of order N

$x \in Y_0(N)(\mathbb{F}_p) \rightsquigarrow (E_0, C_0)/_{\mathbb{F}_p}$

$\hat{\mathcal{O}}_{Y_0(N),x}$: the deformation ring for the pair (E_0, C_0) ↪ this doesnot deform.

R : Artin local ring with residue field $\mathbb{F}_p$.

E_R lift of E_0 to R, there is a ! cyclic subgroup $C \subset E_R[N]$ lifting C_0.

$\hat{\mathcal{O}}_{Y_0(N),x}$ = deformation ring for E_0

General considerations show that

$$\hat{\mathcal{O}}_{Y_0(N),x} = \mathbb{Z}_p[[T]].$$

Complex analytic picture:

$$\begin{array}{l} \mathcal{E} \leftarrow \text{family of elliptic curves} \\ \pi \downarrow \\ \Delta \quad \text{open disc} \end{array}$$

$$\underset{\substack{\uparrow \\ \text{sheaves over } \Delta}}{\mathcal{E} =} \omega_{\mathcal{E}} \backslash H_1^{dR}(\mathcal{E}/\Delta) / R_1\pi_* \mathbb{Z} = \omega_{\mathcal{E}} \backslash \Delta \times \mathbb{C}^2 / \underline{\mathbb{Z}}^2$$

Locally on $\mathcal{E}$, we can trivialize $H_1^{dR}(\mathcal{E}/\Delta) \simeq \Delta \times \mathbb{C}^2$.
$R_1\pi_*\mathbb{Z} \xrightarrow{\sim} \underline{\mathbb{Z}}^2$.

Giving $\omega_{\mathcal{E}} \subset \Delta \times \mathbb{C}^2 \iff \Delta \underset{\text{"period"}}{\longrightarrow} \mathbb{P}^1_{\mathbb{C}}$. Moduli of elliptic curves ~~loot~~ locally looks like $\mathbb{P}^1$.

We want to do the same thing over $\mathbb{Z}_p$.

$$\widetilde{Y_0(N)}_{\mathbb{Z}_p} = \{(E, C_N, \eta)/_R \;,\; R\text{ - }\mathbb{Z}_p \text{ alg.}$$

$Y_0(N)_{\mathbb{Z}_p} \leftarrow \widetilde{Y_0(N)}_{\mathbb{Z}_p}$ ← $GL_2/_{\mathbb{Z}_p}$ - torsor

$\eta : H^1_{dR}(E) \xrightarrow{\sim} R^2$.

$H^1_{dR}(E) \supseteq \omega_E$

$(E, C_N, \eta) \mapsto \eta(\omega_E) \in R^2$.

$\widetilde{Y_0(N)}_{\mathbb{Z}_p} \to \mathbb{P}^1_{\mathbb{Z}_p}$: smooth of rel. dim. $4 = \dim GL_2$

Why is $\widetilde{Y_0(N)}_{\mathbb{Z}_p} \longrightarrow \mathbb{P}^1_{\mathbb{Z}_p}$ smooth?

$R \twoheadrightarrow R_0$ is a surj with $I = \ker(R \twoheadrightarrow R_0)$ satisfying $I^2 = 0$.

$$\begin{tikzcd}
\widetilde{Y_0(N)}_{\mathbb{Z}_p}(R) \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{Z}_p}(R) \arrow[d] \\
\widetilde{Y_0(N)}_{\mathbb{Z}_p}(R_0) \arrow[r] \arrow[ur, dashed] & \mathbb{P}^1_{\mathbb{Z}_p}(R_0)
\end{tikzcd}$$

Grothendieck-Messing theory:

R_0 : $\mathbb{Z}_p$-alg in which p is nilpotent. A_0/R_0 : abelian scheme

then : ① $H^1_{dR}(A_0/R_0)$ propagates to a crystal over R_0.

If $R \to R_0$ is a nilpotent (PD) thickening ($\frac{a^n}{n!}$ makes sense f. $a \in \ker$)

then there is a canonical locally free R-module $\mathbb{D}(A_0)(R)$, such that $R' \to R$, then
$$\begin{tikzcd}
R' \arrow[r] \arrow[dr] & R \arrow[d] \\
& R_0
\end{tikzcd}$$

$\mathbb{D}(A_0)(R') \otimes R = \mathbb{D}(A_0)(R)$.

② If A is a lift of A_0 over R, then there is a canonical identification. $H^1_{dR}(A/R) = \mathbb{D}(A_0)(R)$.

$$\begin{array}{ccc} H^1_{dR}(A/R) & = & \mathbb{D}(A_0)(R) \\ \cup| & & \cup \\ \omega_A & \longrightarrow & \omega \end{array} \leftarrow \text{direct summand}$$

③ $\left\{ A/R \text{ lifting } A_0 \right\} \longrightarrow \left\{ \omega \subset \mathbb{D}(A_0)(R) \text{ direct summand lifting } \omega_{A_0} \right\}$

$A \longmapsto \omega_A$ bij

If $\ker(R \twoheadrightarrow R_0)^2 = 0$, then $(E_0, C_0, \eta_0) \in \widetilde{Y_0(N)}(R_0)$

$\eta_0(\omega_{E_0}) \subset R^2$ direct summand

$\qquad F_0$

$\leftarrow$ this gives me an elliptic curve

$F \in \mathbb{P}^1(R)$ for each choice of identification $R^2 = \mathbb{D}(E_0)(R)$.

$\downarrow$

$F_0 \in \mathbb{P}^1(R_0)$

$Y_0(N)_{\mathbb{Z}_p}$ parametrizes $(E, C_N, \alpha : E \to E')$ p-divisible

$$\widetilde{Y_0(Np)}_{\mathbb{Z}_p} = \left\{ (E, C_N, \alpha : E, E', \eta, \eta') \right\}/R$$

$$H^1_{dR}(E') \xrightarrow{\alpha^*} H^1_{dR}(E)$$

$\swarrow \leftarrow \mathcal{G}$ torsor

$$Y_0(Np)_{\mathbb{Z}_p} \qquad \mathcal{G}(R) = \left\{ \begin{matrix} R^2 \xrightarrow{\left[\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right]} R^2 \\ \downarrow\wr \qquad \downarrow\wr \\ R^2 \xrightarrow[\left[\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right]]{} R^2 \end{matrix} \right\} \qquad \eta' \downarrow \qquad \downarrow \eta$$

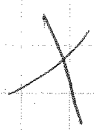
$$R^2 \xrightarrow[\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}]{} R^2 \longrightarrow M_0^{loc}(Np)$$

smooth of dim 4.

$$M_0^{loc}(Np) = \left\{ \begin{matrix} F' \to F \\ \cap \quad\ \cap \\ R^2 \xrightarrow[\left[\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right]]{} R^2 \end{matrix} \leftarrow \text{direct summands} \right\} \subseteq \mathbb{P}^1_{\mathbb{Z}_p} \times \mathbb{P}^1_{\mathbb{Z}_p}$$

$$= \left\{ ([x_1 : x_2], [y_1 : y_2]) : p x_1 y_2 = x_2 y_1 \right\}$$

$$\cup\!\!\mid \qquad\qquad ([1 : x], [y : 1])$$



$$\operatorname{Spec} \mathbb{Z}_p[x,y]/(xy-p) \qquad (x, y)$$

If a point $(E, C_N, \alpha : E \to E')/\mathbb{F}_p$ corresponds to the origin in $\operatorname{Spec} \mathbb{F}_p[x,y]/(xy)$

$$\begin{array}{ccccccccc} 0 & \to & \omega_{E'} & \to & H^1_{dR}(E') & \to & \omega_{E'}^{\otimes -1} = \mathrm{Lie}(E') & \to & 0 \\ & & \downarrow 0 & & \downarrow \alpha^* & & \downarrow 0 & & \\ 0 & \to & \omega_E & \to & H^1_{dR}(E) & \to & \omega_E^{\otimes -1} & \to & 0 \\ & & & & & & \mathrm{Lie}(E) & & \end{array}$$

this map is induced from dual $\alpha^\vee : E' \to E$.

$\Rightarrow$ α, $\hat{\alpha}$ are purely inseparable $\Leftrightarrow$ α is Frob. and E is supersingular.

In general (G, X) : Shimura datum

$\rightsquigarrow$ conjugacy class of $\{\mu : \mathbb{G}_m \to G\}$

$\rightsquigarrow$ a parabolic $P_\mu \subseteq G$. $\left(G = GL_2 \quad \mu(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \right)$

$K \subset G(\mathbb{A}_f)$ cpt open, $K_p \subset G(\mathbb{Q}_p)$ is a parahoric subgrp

$\rightsquigarrow$ $K_p = \mathcal{G}(\mathbb{Z}_p)$, $\mathcal{G}$ is a smooth $\mathbb{Z}_p$-grp scheme with $\mathcal{G}_{\mathbb{Q}_p} = G_{\mathbb{Q}_p}$.

$$\begin{tikzcd} & \tilde{S}_K(G,X) \arrow[dl, "\mathcal{G}\text{-torsor}"'] \arrow[dr, "\text{smooth}"] & \\ S_K(G,X)_{\mathcal{O}_p} & & M^{loc}_{K_p}(G_{\mathbb{Q}_p}, \{\mu\}) \end{tikzcd}$$

$\leftarrow$ local model.

$\uparrow$ integral model for $Sh(G,X)$.

ex: $G = GSp_4$. $Sh_K(G,X)$ = parametrizes polarized abelian surfaces with K-level structure.

$V = \bigoplus_{i=1}^{4} \mathbb{Q} e_i$, symplectic form given by $\begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$

$$\begin{array}{ccccccccccc} \Lambda_0 & \subseteq & \Lambda_1 & \subseteq & \Lambda_2 & \subseteq & \Lambda_3 & \subseteq & \Lambda_4 & \subset & \mathbb{Q}_p^4. \\ \| & & \| & & \| & & \| & & \| & & \\ \langle e_i \rangle & & \langle p^{-1}e_1, e_2 \rangle & & \langle p^{-1}e_1, p^{-1}e_2, e_3 \rangle & & \langle p^{-1}e_1, \dots, e_4 \rangle & & \langle p^{-1}e_i \rangle. & & \end{array}$$

$K_p = \mathrm{Stab}(\Lambda_\cdot) \subseteq GSp_4(\mathbb{Z}_p)$

$$S_K(GSp_4) = \left\{ \begin{tikzcd} A_0 \arrow[r, "p\text{-isogeny}", "\alpha_1"'] \arrow[d, "\cong"'] & A_1 \arrow[r, "p\text{-isogeny}", "\alpha_2"'] & A_2 \arrow[d, "\cong"] \\ \check{A}_0 & \check{A}_1 \arrow[l] & \check{A}_2 \arrow[l, "\alpha_2^\vee"] \end{tikzcd} \right.$$

Going round gives mult. by p.