Local Models - II : Keerthi

$Y = Y(N)$ parametrizes (E, α) $\quad \alpha : \left(\mathbb{Z}/N\mathbb{Z}\right)^2 \xrightarrow{\sim} E[N]$

Fix prime p, $p \nmid N$. Want to study $Y(\mathbb{F}_q)$, $q = p^r$.

Fix $E_0/\mathbb{F}_q$, $\quad Y(\mathbb{F}_q)_{E_0} = \{ (E, \alpha) :$ E isogenous to E_0 over $\mathbb{F}_q \}$.

$x \in Y(\mathbb{F}_q)_{E_0}$, $\quad x \rightsquigarrow (E_x, \alpha_x)$, $\quad f : E_0 \longrightarrow E_x$ isogenous to $\mathbb{F}_q$.

E : an elliptic curve$/\mathbb{F}_q$ $\quad H^1_{ét}(E_{\overline{\mathbb{F}}_p}, \mathbb{Z}_\ell) \circlearrowleft \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \ni \Phi_q$, $\ell \neq p$

$$= \varprojlim H^1_{ét}(E_{\overline{\mathbb{F}}_p}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

$$H^1_{ét}(E_{\overline{\mathbb{F}}_p}, \mathbb{Z}/\ell^n\mathbb{Z})^\vee = E[\ell^n].$$

$\ell = p$, $\quad H^1_{cris}(E/\mathbb{Z}_q) = \varprojlim \mathbb{D}(E)\left(\mathbb{Z}_q/p^n\mathbb{Z}_q\right)$.

$H^1_{cris}(E/\mathbb{Z}_q)\left[\frac{1}{p}\right]$ is an F-isocrystal over $\mathbb{F}_q$.

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[rr, bend right, "p"'] & E^{(p)} \arrow[r, "V"] & E \end{tikzcd} \qquad \begin{tikzcd} H^1_{cris}(E) \arrow[dr, bend right, "V"'] & H^1_{cris}(E^{(p)}) \arrow[l, "F"'] \arrow[d, phantom, "\shortparallel"] \\ & \sigma^* H^1_{cris}(E) \arrow[ul, "F"] \end{tikzcd}$$

$$\langle \sigma \rangle = \mathrm{Gal}(\mathbb{Z}_q/\mathbb{Z}_p).$$

We get $FV = VF = p$.

$$H^p = H^1_{et}(E_{0\,\overline{\mathbb{F}}_p}, \mathbb{A}_f^{(p)}) \; ; \; H_p = H^1_{ét}(E_0/\mathbb{Z}_q)\left[\frac{1}{q}\right]$$

$$\begin{tikzcd} f^*\left(H^1_{ét}(E_x, \hat{\mathbb{Z}}^{(p)})\right) \subseteq H^p. \\ \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)\text{-stable } \hat{\mathbb{Z}}^{(p)} \text{ lattice} \arrow[u] \end{tikzcd}$$

$f^*\left(H^1_{cris}(E_x/\mathbb{Z}_q)\right) \subseteq H_p$ $\qquad$ F, V stable $\mathbb{Z}_q$-lattice

- $L^p_{x,f} \subset H^p$ Galois stable lattice
- $L_{p,x,f} \subset H_p$ (F, V)-stable $\mathbb{Z}_q$-lattice
- $\alpha_{x,f} : \left(\mathbb{Z}/N\mathbb{Z}\right)^2 \xrightarrow{\sim} L^p_{x,f} / N L^p_{x,f}$ of $\mathrm{Gal}\left(\overline{\mathbb{F}}_q / \mathbb{F}_q\right)$-modules

- $Y^p = \left\{ \begin{array}{l} (L^p, \beta) : \ L^p \subseteq H^p \text{ is a Galois stable } \text{~~lattice~~ } \hat{\mathbb{Z}}^{(p)}\text{-lattice.} \\ \beta : \left(\mathbb{Z}/N\mathbb{Z}\right)^2 \xrightarrow{\sim} L^p / N L^p \quad (\text{Galois equivariant}) \end{array} \right\}$

- $Y_p = \left\{ L_p \subseteq H_p : (F, V)\text{-stable } \mathbb{Z}_q\text{-lattices} \right\}$

$I :=$ alg group $/\mathbb{Q}$, $I(R) = \left(\mathrm{End}(E_0) \otimes R\right)^\times$

In particular, $I(\mathbb{Q}) = \left(\mathrm{End}(E_0) \otimes \mathbb{Q}\right)^\times = \overset{\mathrm{Aut}}{\cancel{\mathrm{End}}}{}^{\circ}(\mathbb{Q})$

$I \circlearrowright H^p \times H_p$.

$$Y(\mathbb{F}_q)_{E_0} \longrightarrow I(\mathbb{Q}) \backslash Y_p \times Y^p.$$

$$x \longmapsto \left[\left(L^p_{x,f}, \alpha_{x,f}, L_{p,x,f}\right)\right].$$

<u>Thm</u>: This map is a bijection.

<u>Pf</u>: (surj): Suppose we have $(L^p, \beta, L_p) \in Y^p \times Y_p$. Upto scaling, we can assume $m_0(-) \subset L^p \subset H^1_{\acute{e}t}(E_0/\overline{\mathbb{F}}_q, \hat{\mathbb{Z}}^{(p)})$ and

$$p^r H^1_{cris}(-) \subset L_p \subset H^1_{cris}(E_0/\mathbb{Z}_q).$$

$$L^p \longrightarrow \bar{L}^p \subset H^1_{\acute{e}t}(E_0/\overline{\mathbb{F}}_q, \mathbb{Z}/m_0\mathbb{Z})$$

$$\Big\Updownarrow$$

subgroup $C^p \subseteq E_0[m_0]$

Galois stable $\rightsquigarrow E_0/C^p$.

$$\begin{tikzcd} E_0[p^r] \arrow[r, rightsquigarrow] & \dfrac{\mathbb{D}(E_0)(\mathbb{Z}_q)}{p^r \mathbb{D}(E_0)(\mathbb{Z}_q)} \\ \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} \\ C_p \arrow[r, rightsquigarrow] & L_p / p^r \mathbb{D}(E_0)(\mathbb{Z}_q) \end{tikzcd}$$

group scheme over E_0.

$E = E_0 / C_p C^p$.

this gives rise to (L^p, β, L_p)

<u>Injectivity</u>: Suppose x & y have the same image $\Rightarrow$ there exist isogenies $f : E_0 \to E_x$, $g : E_0 \to E_y$, such that such that $(L^p_{x,f}, \alpha_{x,f}, L_{p,x,f})$ differs from

$(L^p_{y,g}, \beta_{y,g}, L_{p,y,g})$ by an element $h \in I(\mathbb{Q})$.

Find $h_0 : E_0 \to E_0$ s.t. $h = m^{-1} h_0$ for some $m \in \mathbb{Z}_{>0}$.
Replace f by fh_0, g by mg. Now we can assume that the tuples are actually equal to (L^p, β, L_p)

Claim: There exist an isomorphism $i : (E_x, \alpha_x) \to (E_y, \alpha_y)$ such that the diagram

$$\begin{tikzcd} E_0 \arrow[r, "f"] \arrow[dr, "g"'] & E_x \arrow[d, "i", "\cong"'] \\ & E_y \end{tikzcd}$$ commutes.

Definitely, we can find i' : $$\begin{tikzcd} E_0 \arrow[r, "f"] \arrow[dr, "mg"'] & E_x \arrow[d, "i'"] \\ & E_y \end{tikzcd}$$ commutes for $m >> 0$.

But $i'(E_x[m]) = 0$. $\Rightarrow$ $$\begin{tikzcd} E_x \arrow[r, "i'"] \arrow[dr, "m"'] & E_y \\ & E_x \arrow[u, "i"'] \end{tikzcd}$$

$\Rightarrow x = y \in Y(\overline{\mathbb{F}}_q)_{E_0}$.

Choose bases $$\begin{tikzcd} (\mathbb{A}_f^{(p)})^2 \arrow[r, "\sim"] & H^p \arrow[loop below, "\Phi_q"] \\ & \gamma \in GL_2(\mathbb{A}_f^{(p)}) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Q}_q^2 \arrow[r, "\sim"] & H_p \arrow[loop above, "F"] \\ & \delta \in GL_2(\mathbb{Q}_q) \end{tikzcd}$$

G_γ = stabilizer of γ inside $GL_{2, \mathbb{A}_f^{(p)}}$

If we choose different bases, then $\delta \rightsquigarrow h \delta \sigma(h)^{-1}$

I_δ : $\mathbb{Q}_p$-group, $I_\delta(R) = \{ h \in GL_2(R \otimes_{\mathbb{Q}_p} \mathbb{Q}_q) : h^{-1} \delta \sigma(h) = \delta \}$.

<u>Thm</u> (Tate): ~~$\mathrm{End}(E^0) \otimes \mathbb{Q}_\ell = \mathrm{End}_{Gal}($~~

$$\mathrm{End}(E_0) \otimes \mathbb{A}_f^{(p)} = \mathrm{End}_{Gal(\overline{\mathbb{F}}_q / \mathbb{F}_q)}(H^p)$$

$$\mathrm{End}(E) \otimes \mathbb{Q}_p = \mathrm{End}_{F,V}(H_p).$$

<u>Cor:</u> $I \otimes \mathbb{A}_f^{(p)} = I_\gamma$.

$I \otimes \mathbb{Q}_p = I_\delta$

$K^p = \ker(GL_2(\hat{\mathbb{Z}}^{(p)}) \to GL_2(\mathbb{Z}/N\mathbb{Z}))$

~~$K_p = GL_2(\mathbb{Z}_q) \subset GL_2(\mathbb{Q}_q)$~~ , $f_p =$ characteristic function of $K_p^p / \mathrm{vol}(K_p^p)$

[illegible] $f^p =$ any function on $GL_2(\mathbb{A}_f^{(p)})$.

$$O_\gamma(f^p) = \int_{I_\gamma(\mathbb{A}_f^{(p)}) \backslash GL_2(\mathbb{A}_f^{(p)})} f^p(x^{-1}\gamma x)\,dx$$

For any function ϕ on $GL_2(\mathbb{Q}_q)$

$$TO_\delta(\phi) = \int_{I_\delta(\mathbb{Q}_p) \backslash GL_2(\mathbb{Q}_q)} \phi_p(y^{-1}\delta\sigma(y))\,dy.$$

$\phi_r =$ char. function of ~~$K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p$~~ $GL_2(\mathbb{Z}_q)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_q)$

<u>Thm:</u> $\# Y(\mathbb{F}_q)_{E_0} = \mathrm{vol}\left({}_{I(\mathbb{Q})}\backslash I(\mathbb{A}_f)\right) O_\gamma(f^p) TO_\delta(\phi_r)$

<u>claim:</u> ① ~~$Y^{(p)}$~~ $Y^p = \{ g : GL_2(\mathbb{A}_f^{(p)}) / K^p \ : \ \gamma g \equiv g \pmod{K^p} \}$

② $Y_p = \{ h \in GL_2(\mathbb{Q}_q)/GL_2(\mathbb{Z}_q) \ : \ h^{-1}\delta\sigma(h) \in GL_2(\mathbb{Z}_q)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_q)$

$$\Rightarrow \ \#_{I(\mathbb{Q})}\backslash(Y^p \times Y_p) = \sum_{(g,h) \in I(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f^{(p)} \times GL_2(\mathbb{Q}_q))} f^{(p)}(g^{-1}\gamma g)\ \phi_r(h^{-1}\delta\sigma(h))$$

By Fubini, $$= \int_{I(\mathbb{Q})\backslash \cdots} f^p(g^{-1}\gamma g)\,\phi_r(h^{-1}\delta\sigma(h)$$