— x —

p-adic Hodge theory : Keerthi

X smooth proj/$\mathbb{C}$: $H^i_B(X, \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} H^i_{dR}(X/\mathbb{C})$.

$X = \mathbb{G}_m$. $H^1_{dR}(\mathbb{G}_m) \times H_1(\mathbb{G}_m, \mathbb{Q}) \longrightarrow \mathbb{C}$.

$\left\langle \frac{dz}{z}, \bigcirc \right\rangle = 2\pi i$ period for $\mathbb{G}_m$.

X : smooth (proper) variety / $\mathbb{Q}_p$.

$Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \circlearrowright H^i_{et}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \longrightarrow H^i_{dR}(X/\mathbb{Q}_p) \leftarrow$ filtered vector space.

If $X = \mathbb{G}_m$. $H^1_{et}(\mathbb{G}_m, \mathbb{Q}_p) = \mathbb{Q}_p(-1)$.

$H^1_{dR}(\mathbb{G}_m) = \mathbb{Q}_p$, $gr^i H^1_{dR}(\mathbb{G}_m) \neq 0 \iff i = 1$

Can we find ~~the~~ a p-adic period for $\mathbb{G}_m$?

So, Ques: $H^1_{et}(\mathbb{G}_m, \mathbb{Q}_p) \otimes \mathbb{C}_p \xrightarrow{\sim} H^1_{dR}(\mathbb{G}_m) \otimes \mathbb{C}_p$??

should be Galois equivariant

$\| $

$\mathbb{C}_p(-1) \xrightarrow{\sim} \mathbb{C}_p$??

$\Rightarrow \mathbb{C}_p(-1)^{G_{\mathbb{Q}_p}} \neq 0$.

Thm (Tate):
- $\mathbb{C}_p^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$.
- $\mathbb{C}_p(i)^{G_{\mathbb{Q}_p}} = 0$ if $i \neq 0$.

Thm (Tate): A abelian variety / $\mathbb{Q}_p$.

$$H^1_{et}(A_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes \mathbb{C}_p \xrightarrow{\sim} (\omega_A \otimes \mathbb{C}_p(-1)) \oplus (Lie(\check{A}) \otimes \mathbb{C}_p)$$

$G_{\mathbb{Q}_p}$- equivariant

Thm (Faltings): $K/\mathbb{Q}_p < \infty$. X smooth proj. variety /K

$$H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p) \otimes \mathbb{C}_p \cong \bigoplus_{i+j=n} \left(H^j(X, \Omega^i_X) \otimes \mathbb{C}_p(-i)\right)$$

can recover Hodge cohomology from étale cohomology.

$$H^j(X, \Omega^i_X) = \left(H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p) \otimes \mathbb{C}_p(i)\right)^{G_K}$$

$G_K \circlearrowright B_{dR}$ = complete discrete valuation field with residue field $\mathbb{C}_p$.

$$0 \to \mathrm{Fil}^1 B_{dR} \to B^+_{dR} \xrightarrow{\theta} \mathbb{C}_p \to 0.$$

$t \in \mathrm{Fil}^1 B_{dR}$, $B^+_{dR} \subset B_{dR}$.

t: period for $\mathbb{G}_m$. (analog for $2\pi i$)

$$\mathrm{gr}^i B_{dR} \cong \mathbb{C}_p(i).$$

$$\Rightarrow \mathrm{gr}^\circ B_{dR} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i) := B_{HT} \qquad \Rightarrow \qquad B_{dR}^{G_K} = K$$

Thm (Faltings ...): $H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p) \otimes B_{dR} \xrightarrow{\sim} H^n_{dR}(X/K) \otimes_K B_{dR}$ respecting all additional structures.

We can recover $H^n_{dR}(X/K) \underset{\text{iso. as filtered } K\text{-vector spaces}}{\cong} \left(H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p) \otimes B_{dR}\right)^{G_K} := D_{dR}(H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p))$

~~In par~~ then X smooth (proper) variety $/K$ with good reduction to X_0/k. for $l \neq p$

Thm (Grothendieck): $H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_l) \xrightarrow{\sim} H^n_{\acute{e}t}(X_0/\bar k, \mathbb{Q}_l)$

$l = p$, crystalline cohomology $H^\cdot_{cris}(X/W(k))$: $W(k)$-module equipped with $\varphi: \sigma^* H^\cdot_{cris} \to H^\cdot_{cris}$.

Thm. (Berthelot - Ogus) $H^\cdot_{cris}(X_0/W(k)) \otimes_{W(k)} K \xrightarrow{\sim} H^\bullet_{dR}(X/K)$.

Let $K_0 = W(k)[\frac{1}{p}]$, Fontaine defined K_0-subalgebra $\varphi \circlearrowright B_{cris} \subseteq B_{dR}$, $\circlearrowleft$ Gal

$B_{cris}^{G_K} = K_0$, we have a natural injection.

$$B_{cris} \otimes_{K_0} K \hookrightarrow B_{dR}.$$

$$B_{cris}^{\varphi=1} \cap \mathrm{Fil}^\circ B_{dR} = \mathbb{Q}_p$$

C_{cris}: $H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p) \otimes B_{cris} \xrightarrow{\sim} H^n_{cris}(X_0/W(k)) \otimes_{W(k)} B_{cris}$.

This implies $H^n_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p) = \mathrm{Fil}^\circ\left(H^n_{cris}(X_0/W(k)) \otimes_{W(k)} B_{cris}\right)^{\varphi=1}$.

<u>Fontaine formalism</u>: V p-adic Galois representation of G_K.

$D_{cris}(V) = \left(V \otimes_{\mathbb{Q}_p} B_{cris}\right)^{G_K}$ is a K_0 vector space with a Frobenius action and $D_{cris}(V) \otimes_{W(k)} K$ is filtered.

$D_{cris}(V) \otimes_{K_0} B_{cris} \hookrightarrow V \otimes_{\mathbb{Q}_p} B_{cris}$ and is an isomorphism

$\iff \dim_{K_0} D_{cris}(V) = \dim_{\mathbb{Q}_p} V$.

In this case, we say V is crystalline.

<u>Example</u>: ① f cusp form of wt. k and level N $\quad p \nmid N$

$V_p f$ = p-adic Galois representation attached to f. is a crystalline representation.

$$D_{cris}(V_p f)^* = K_0 e_1 + K_0 e_2 . \qquad \varphi = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix}, \quad K = K_0 = \mathbb{Q}_p .$$

$$Fil^i\left(D_{cris}(V_p f)^*\right) = \begin{cases} D_{cris}(V_p f)^* & \text{if } i \leq 0 \\ \mathbb{Q}_p e_1 & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i \geq k . \end{cases}$$

② $E_q = \bar{K}^\times / q^{\mathbb{Z}}$, $q \in K$, $|z| < 1$. This has bad semi-stable reduction.

$$0 \to \mathbb{Q}_p(1) \to \underset{\substack{\shortparallel \\ V_p(E_q)}}{H^1(E_q/\bar{K}, \mathbb{Q}_p)(1)} \to \mathbb{Q}_p \to 0$$

This gives us a class in $H^1(K, \mathbb{Q}_p(1))$

Fix $(\epsilon^{(n)})$ compatible system of p-power roots of 1.
$\quad\;\, (q^{(n)})$ compatible system of p-power roots of q.

$c(g)$ satisfies $\underset{G_K}{\underset{\cap}{g}} \cdot q^{(n)} = q^{(n)} (\epsilon^{(n)})^{c(g)}$

we will need a period a satisfying

$$\chi(g) g(a) + c(g) = a \qquad \text{donot exist in } B_{cris}\,!!$$

$B_{cris} \subseteq \underset{\circlearrowleft\, G_K}{\overset{N \circlearrowleft \;\; \circlearrowleft \varphi}{B_{st}}} \underset{\uparrow \text{ not canonical}}{\subseteq} B_{dR}$ with the following: $B_{st}^{G_K} = K_0$, $B_{st} \otimes_{K_0} K \hookrightarrow B_{dR}$

The embedding depends on the choice of $\log_p$.

MF_K^{φ}: tuples $(D, \varphi, \mathrm{Fil}^{\cdot})$ $\quad \overset{\circlearrowleft \varphi}{D}$ is a K_0-vector space.

$\mathrm{Fil}^{\cdot}(D \otimes_{K_0} K)$ is a decreasing, separated, exhaustive filtration.

~~$D_{cris}(V_K) \longrightarrow$~~

$$
\begin{tikzcd}
D_{cris} : \mathrm{Rep}\, G_K \arrow[r] & MF_K^{\varphi} \\
\mathrm{Rep}^{cris}_{G_K} \arrow[u] \arrow[ur, "\text{fully faithful}"'] &
\end{tikzcd}
$$

Can we describe the essential image of $\mathrm{Rep}^{cris}_{G_K}$.

D 1-dim'l, $\quad t_H(D) = i$, $gr^i D_K \neq 0$.

$t_N(D) = v_p(a)$, where $\varphi(D) = aD$.

In general $t_H(D) = t_H(\Lambda^n D)$; $t_N(D) = t_N(\Lambda^n D)$.

Thm (Colmez-Fontaine)

D comes from a crystalline Galois representation $\iff t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for $D' \subseteq D$, D' φ-stable.