Perfectoid spaces: P. Scholze

Aim: compare objects of mixed characteristic $(0,p)$ with objects with equal characteristic (p,p).

Thm (Fontaine-Wintenberger): The absolute Galois group of $\mathbb{Q}_p(p^{1/p^\infty})$ and the absolute Galois group of $\mathbb{F}_p((t))$ are canonically isomorphic

A generalization:

Defn: A perfectoid field is a top field K complete w.r.t. non-discrete & non-archimedean norm, such that the residue field has char p and $\Phi: \mathcal{O}_K/p \to \mathcal{O}_K/p$ $x \mapsto x^p$ is surj.

ex: completions of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{F}_p((t))(t^{1/p^\infty})$, $\overline{\mathbb{Q}}_p$ but not $\mathbb{Q}_p$.

Construction (Fontaine): $\mathcal{O}_{K^\flat} = \varprojlim_{\Phi} \mathcal{O}_K/p \cong \varprojlim_{\Phi} \mathcal{O}_K$ as a set compatible with multiplication but not additions.

$$K^\flat = \operatorname{Frac}(\mathcal{O}_{K^\flat}) = \varprojlim_{x \mapsto x^p} K \text{. } \exists \text{ a map } K^\flat \to K, \ x \mapsto x^\# \text{ by}$$

projecting onto the last coordinate

$$K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}, \text{ then } K^\flat = \mathbb{F}_p((t))(t^{1/p^\infty}), \quad t \cong (p, p^{1/p}, \dots) \in \varprojlim K$$

$$t^\# = p$$

$$(1+t)^\# = \lim_{n\to\infty} (1+p^{1/p^n})^{p^n}.$$

Thm: Let K be a perfectoid field, i) $K^\flat$ perfectoid field, char $K^\flat = p$.

(ii) $L/K < \infty \to L$ perfectoid

(iii) $\{$finite extn of $K\} \cong \{$finite extn of $K^\flat\}$ via $L \mapsto L^\flat$

$\Rightarrow$ absolute Galois groups are isomorphic

Want to generalize to comparison of geometric objects.

Claim: $\mathbb{A}^1_{K^\flat}$ "=" $\varprojlim_{T \mapsto T^p} \mathbb{A}^1_K$ (T coordinate on $\mathbb{A}^1$)

$$x \longmapsto (x^\#, (x^{1/p})^\#, \dots)$$

$K^\flat = \varprojlim K$ on $K^\flat$ (resp K) valued points.

explicit version involves limits $\leadsto$ need some versions of non-archimedean analytic geometry

($\underbrace{\text{rigid analytic geometry}}$, $\underbrace{\text{Berkovich spaces}}$, $\underbrace{\text{Huber's adic spaces}}$)

rigid analytic geometry $\downarrow$ no top. in general, use Grothendieck top

Berkovich spaces $\downarrow$ affinoids are closed

Huber's adic spaces $\downarrow$ locally ringed top. spaces, affinoids are open (right object to look at)

$$\{\text{varieties over } k\} \longleftrightarrow \{\text{adic spaces}/k\}$$

$$X \longmapsto X^{ad}$$

Analog to $\{\text{varieties}/\mathbb{C}\} \leftrightarrow \{\text{complex analytic spaces}\}$.

Thm: $\left|(\mathbb{A}^1_{K^\flat})^{ad}\right| \cong \varprojlim \left|(\mathbb{A}^1_K)^{ad}\right|$ homeomorphism of top. spaces

locally ringed spaces

Q) Can one compare structure sheaves?

On RHS, get rings like $\varinjlim_n K\langle T^{1/p^n}\rangle$

Defn: A perfectoid K-alg. is a Banach K-alg. R, such that $R^\circ := \{x \in R : x \text{ powers bounded} : \{x^n, n \geq 0\} \subset R \text{ bounded}\} \subset R$ is bounded ($\Rightarrow R$ reduced) and $\Phi : R^\circ/p \to R^\circ/p$, $x \mapsto x^p$ is surj.

(R° is something like set of integral elements)

example: completion of $\lim K\langle T^{1/p^n}\rangle$. If char $k = p$, last condition $\Rightarrow R$ perfect

Thm (lifting equivalence): The functor

$$R \longmapsto R^\flat = \left(\varprojlim_{\Phi} R^\circ/p\right) \otimes_{O_{K^\flat}} K^\flat = \varprojlim_{x \mapsto x^p} R$$

induces equivalence of categories

$$\{\text{perfectoid } K\text{-alg}\} \cong \{\text{perfectoid } K^\flat \text{ algebras}\}$$

$\exists$ adic spaces associated to perfectoid alg. (cannot associate rigid analytic spaces, lacks certain finiteness conditions)

As top. spaces $\{|\cdot| : R \to \Gamma \cup \{0\}$ cont. valuations $\}/\cong$

Γ totally ordered abelian grp.

Thm: (1) $\{$perfectoid spaces$/K\} \cong \{$perfectoid spaces$/K^\flat\}$.

$X \longmapsto X^\flat$

(2) There is an homeomorphism $|X| \cong |X^\flat|$, $\mathcal{O}_X$ - sheaf of perfectoid K-alg. with

(3) X affinoid $\Rightarrow H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

(4) $\exists$ étale site X_{et}, $X_{et} \cong X^\flat_{et}$

(If X point, then (4) $\Rightarrow$ isom. of Galois grps)

The map $X \longmapsto X^\flat$ (X affinoid associated to R).

If $|\cdot| : R \to \Gamma \cup \{0\}$, then $f \longmapsto |f f^\#|$ defines a function on $R^\flat$.

Need generalization of Faltings almost purity thm.

Thm: R perfectoid K-alg. S/R finite étale, then S perfectoid K-alg. S°/R° $\underbrace{\text{almost}}_{\text{technical sense}}$ finite étale.

(S°, R° spl. fiber)

Thm: $|(\mathbb{P}^n_{K^\flat})^{ad}| = \varprojlim_{\Phi} |(\mathbb{P}^n_K)^{ad}|$, $\varphi(x_0, \dots, x_n) = (x_0^p, \dots, x_n^p)$

$(\mathbb{P}^n_{K^\flat})^{ad}_{et} \cong \varprojlim_{\Phi} (\mathbb{P}^n_K)^{ad}_{et} \rightsquigarrow \pi : \mathbb{P}^n_{K^\flat} \to \mathbb{P}^n_K$ defined on top. spaces and étale topoi of adic spaces.

Weight-monodromy conjecture: $X/\mathbb{Q}_p$ proper smooth variety $l \neq p$.

$V = H^i_{et}(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l)$

Fix $\Phi \in Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, geometric Frob.

Structure of V: • weight $V = \bigoplus_{j=0}^{2i} V_j$, where all eigenvalues of Φ on V_j are Weil #'s of wt j.

• monodromy ; action of inertia $N V_j \to V_{j-2}$

Conj (Deligne 1970) $\forall j \geq 0$, $N V_{i+j} \xrightarrow{\sim} V_{i-j}$

Thm (Deligne): Analog over $\mathbb{F}_p((t))$ is true. ((p,p) characteristic case is true)

Thm: Assume X geometrically connected, complete intersection in a proj. smooth toric variety (eg $\mathbb{P}^n$). Then conj is true.

Pf: Extend scalars to $K = \mathbb{Q}_p(p^{1/p^\infty})$. Let $X \subseteq \mathbb{P}^n$ be a hypersurface.

$$\begin{array}{ccc} \mathbb{P}^n_{K^\circ} & \xrightarrow{\pi} & \mathbb{P}^n_K \\ & & \cup \\ \pi^{-1}(X) & \longrightarrow & X \end{array} \rightsquigarrow \text{map } H^i(X) \to H^i(\pi^{-1}(X))$$

If $\pi^{-1}(X)$ is alg., would conclude using Deligne

($\pi^{-1}(X)$ fractal, $H^i(\pi^{-1}(X))$ inf. dim'l, $\pi^{-1}(X)$ will generally meet a line at ∞ many places)

Thm: If X complete intersection, $X \subseteq \tilde{X} \subset \mathbb{P}^n$ be a small open subset (in p-adic top)

$\exists$ alg. variety $Y \subset \pi^{-1}(\tilde{X})$ with $\dim Y = \dim X$

If $\tilde{X}$ small, then $H^i(X) = H^i(\tilde{X}) \longrightarrow H^i(\pi^{-1}(\tilde{X})) \longrightarrow H^i(Y)$

apply Deligne