Semi-Monotone Sets and Triangulation of Tame Monotone Families

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Joint work with
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Motivation: Approximation of **tame** sets by compact sets.

Tame = definable in an o-minimal structure over $\mathbf{R}$.

All sets and families below are tame.

A family of compact sets $\{S_\delta,\ \delta > 0\}$ is **monotone** if $S_\delta \subset S_\eta$ for $\delta > \eta$. We say that S_δ **approximates** $S = \cup_{\delta>0} S_\delta$.

A monotone family S_δ can be defined as $\{f \geq \delta\}$ where f is an upper semi-continuous function. Then $S = \{f > 0\}$.

Theorem. (A.G., Vorobjov, 2009). Let S_δ be a monotone family approximating S. For each δ, let $S_{\delta,\epsilon} \searrow S_\delta$ as $\epsilon \searrow 0$, so that $S_{\delta,\epsilon}$ is a compact neighborhood of S_η for $\eta > \delta$.
Then, for $0 \le \epsilon_0 \ll \delta_0 \ll \ldots \ll \epsilon_k \ll \delta_k \ll 1$,

$$T_k = S_{\delta_0,\epsilon_0} \cup \ldots \cup S_{\delta_k,\epsilon_k}$$

satisfies $\pi_i(T_k) \twoheadrightarrow \pi_i(S)$ for $i \le k$.

Conjecture. $\pi_i(T_k) \cong \pi_i(S)$ for $i < k$.
If $k \ge \dim S$, then T_k is homotopy equivalent to S.

Proved when $S_\delta = \{f \ge \delta\}$ is **separable**: There is a triangulation of K such that, for any open simplex Λ, the closures of the sets $\{f = \delta\} \cap \Lambda$ and $\{f = \eta\} \cap \Lambda$ are disjoint for $0 < \eta \ll \delta$.

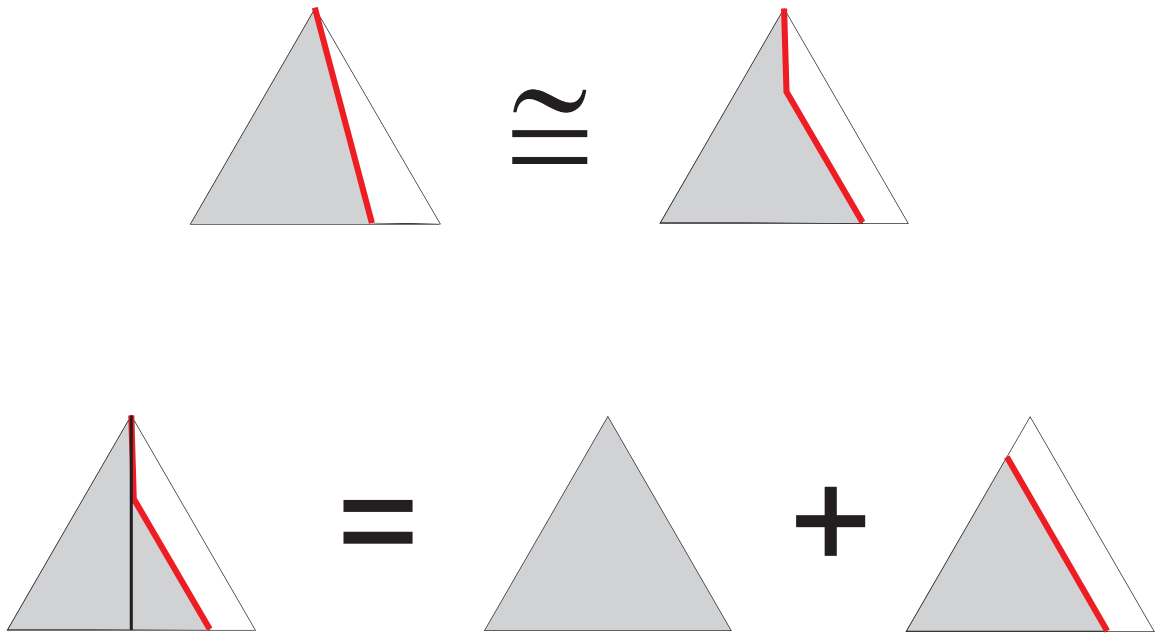
Triangulation of Monotone Families

Conjecture. Given a monotone family S_δ in a compact $K \subset \mathbf{R}^n$, there is an (ordered) triangulation of K such that, for each open k-simplex Λ, $\Lambda \cap S_\delta$ is **equivalent** to one of explicitly defined **standard families** in the standard k-simplex Δ.

Proved for $n \leq 3$.

Equivalent means that
(a) There exist a standard family $\{V_\delta\}$ in Δ and a face-preserving PL-homeomorphism $h : \overline{\Lambda} \to \overline{\Delta}$ such that, for every $\delta > 0$, there is $\eta > 0$ such that $V_\delta \subset h(S_\eta)$ and $h(S_\delta) \subset V_\eta$;
(b) For small $\delta > 0$, there exist face-preserving PL-homeomorphisms $h_\delta : \overline{\Lambda} \to \overline{\Delta}$ such that $h_\delta(S_\delta) = V_\delta$.

Theorem. Each standard family is equivalent to a family that can be partitioned into separable families.



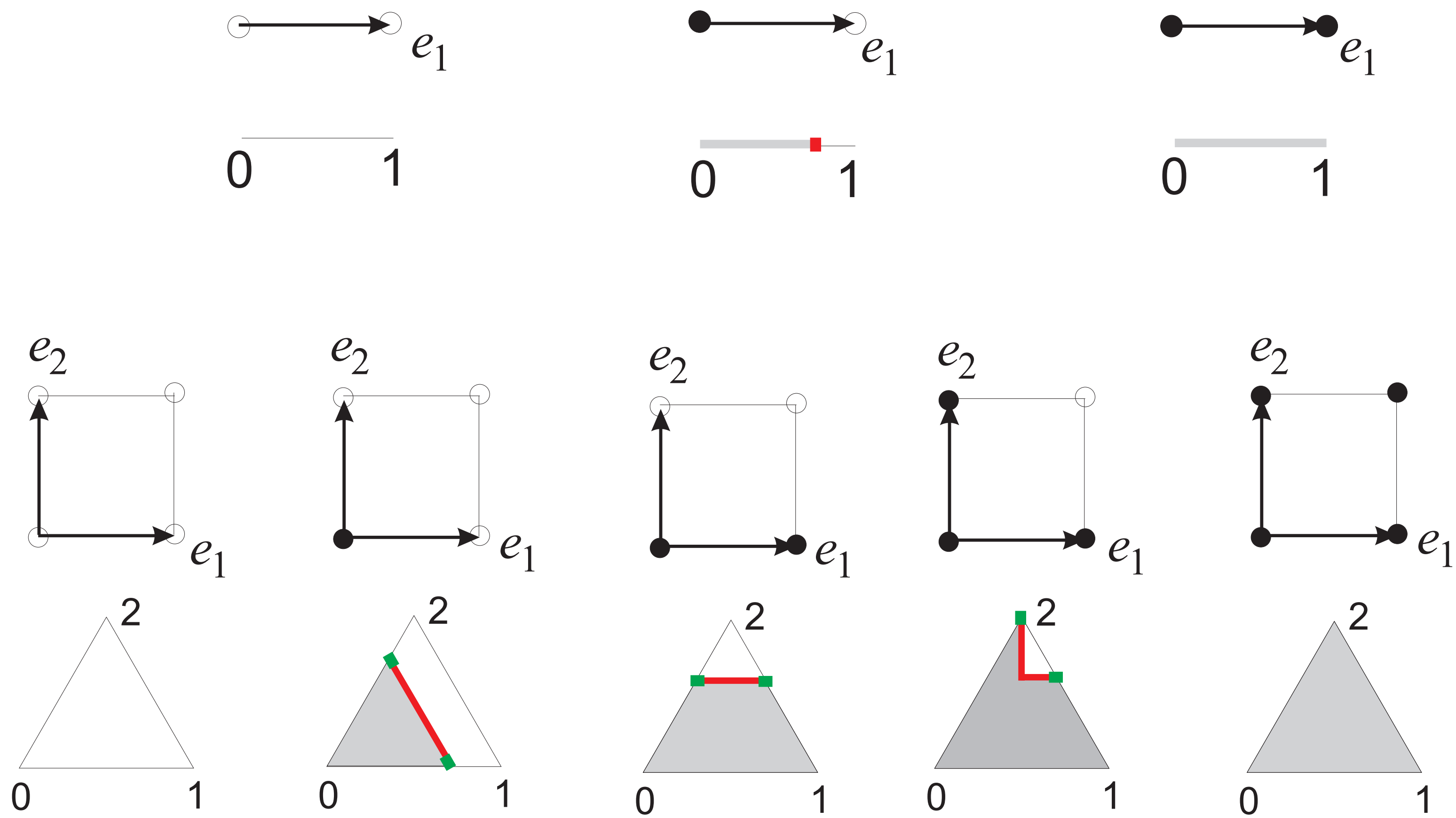
Example. A non-separable 2D family, and an equivalent family that can be partitioned into two separable families.

Monotone Boolean Functions

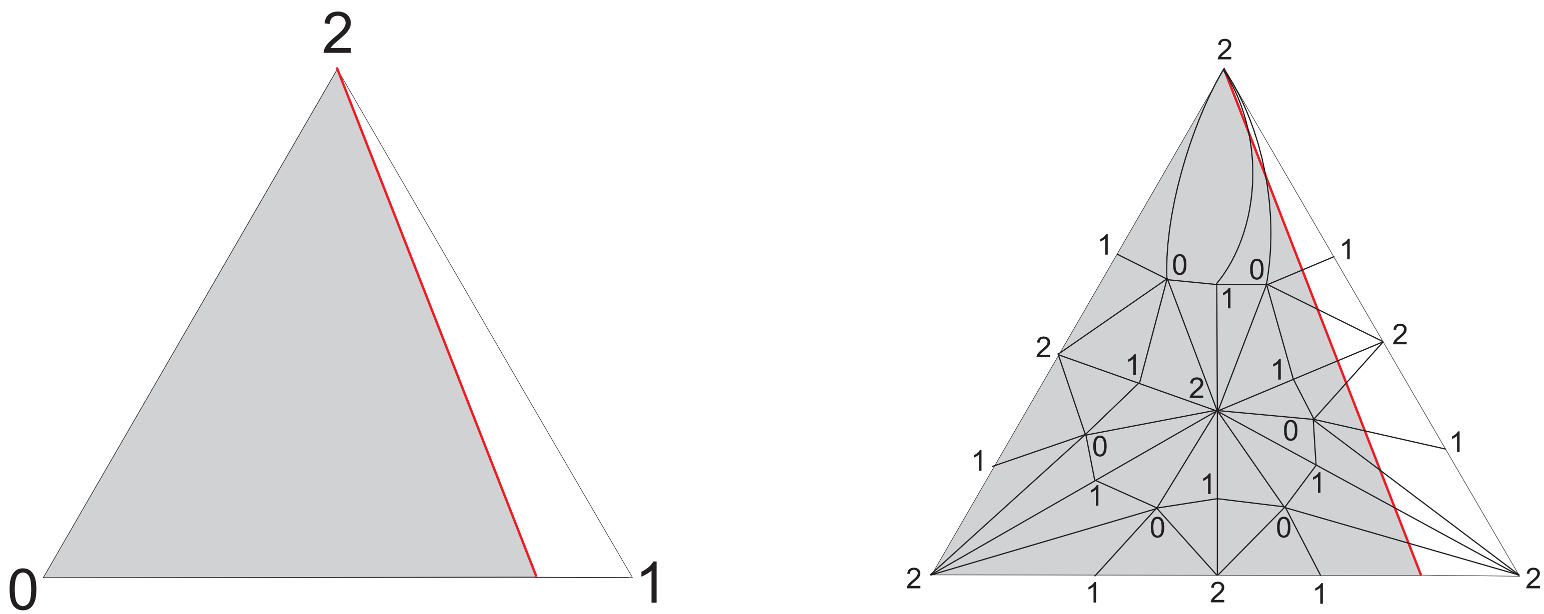
A Boolean function $\psi : \{0,1\}^n \to \{0,1\}$ is **monotone** (decreasing) if replacing 0 by 1 at any position of its argument either preserves its value or changes it from 1 to 0.

Function ψ is **lex-monotone** if it is monotone with respect to the lexicographic order of its arguments, assuming $x_1 \prec \ldots \prec x_n$.

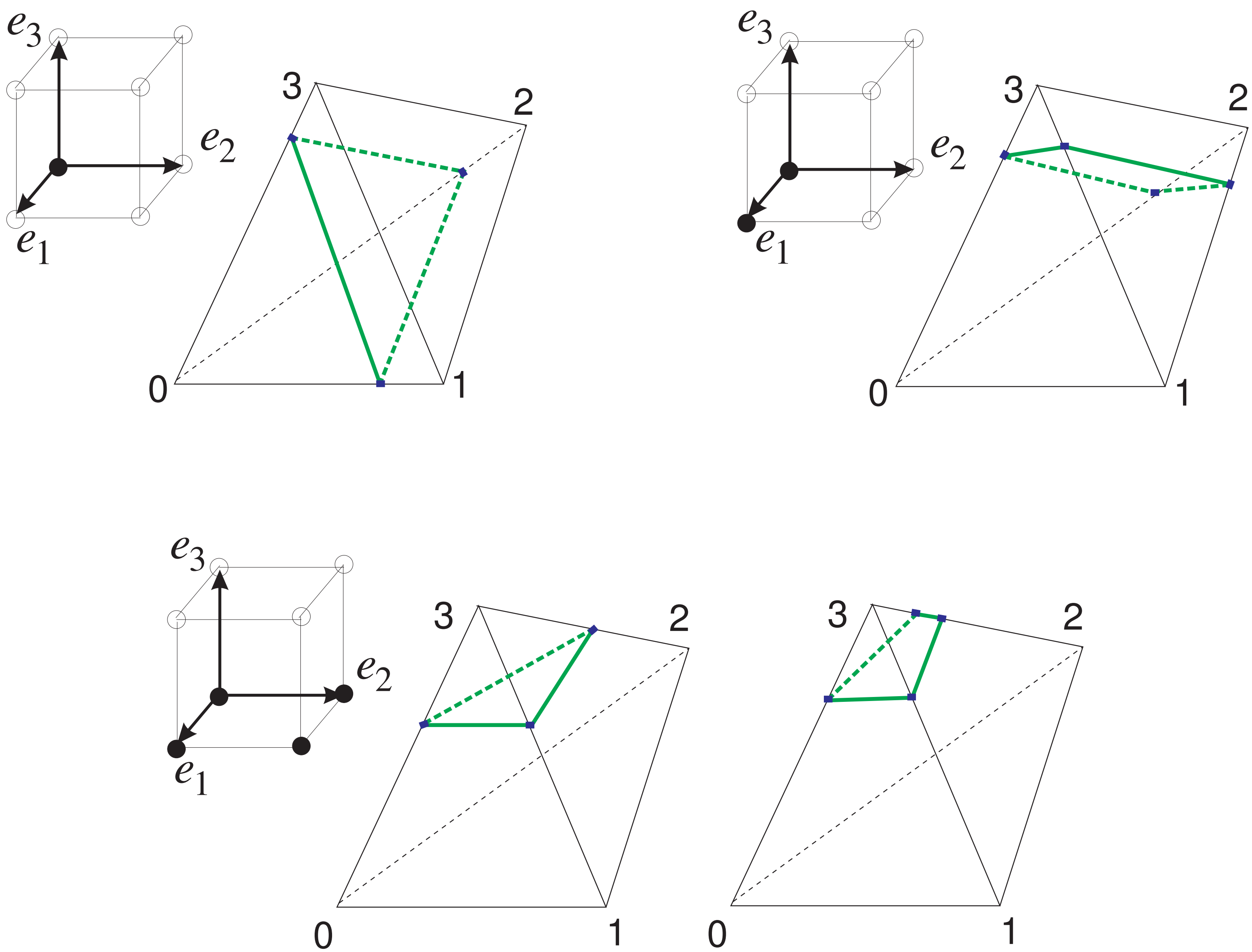
Each standard family $\{V_\delta\}$ in the standard n-simplex Δ is assigned a lex-monotone Boolean function $\psi(x_1, \ldots, x_n)$ so that
$\psi|_{x_j=0}$ is assigned to $\overline{V_\delta}|_{\Delta_j}$ for $j \neq 0$,
$\psi|_{x_1=1}$ is assigned to $\overline{V_\delta}|_{\Delta_0}$.
Here Δ_j is the facet of Δ opposite its vertex j.



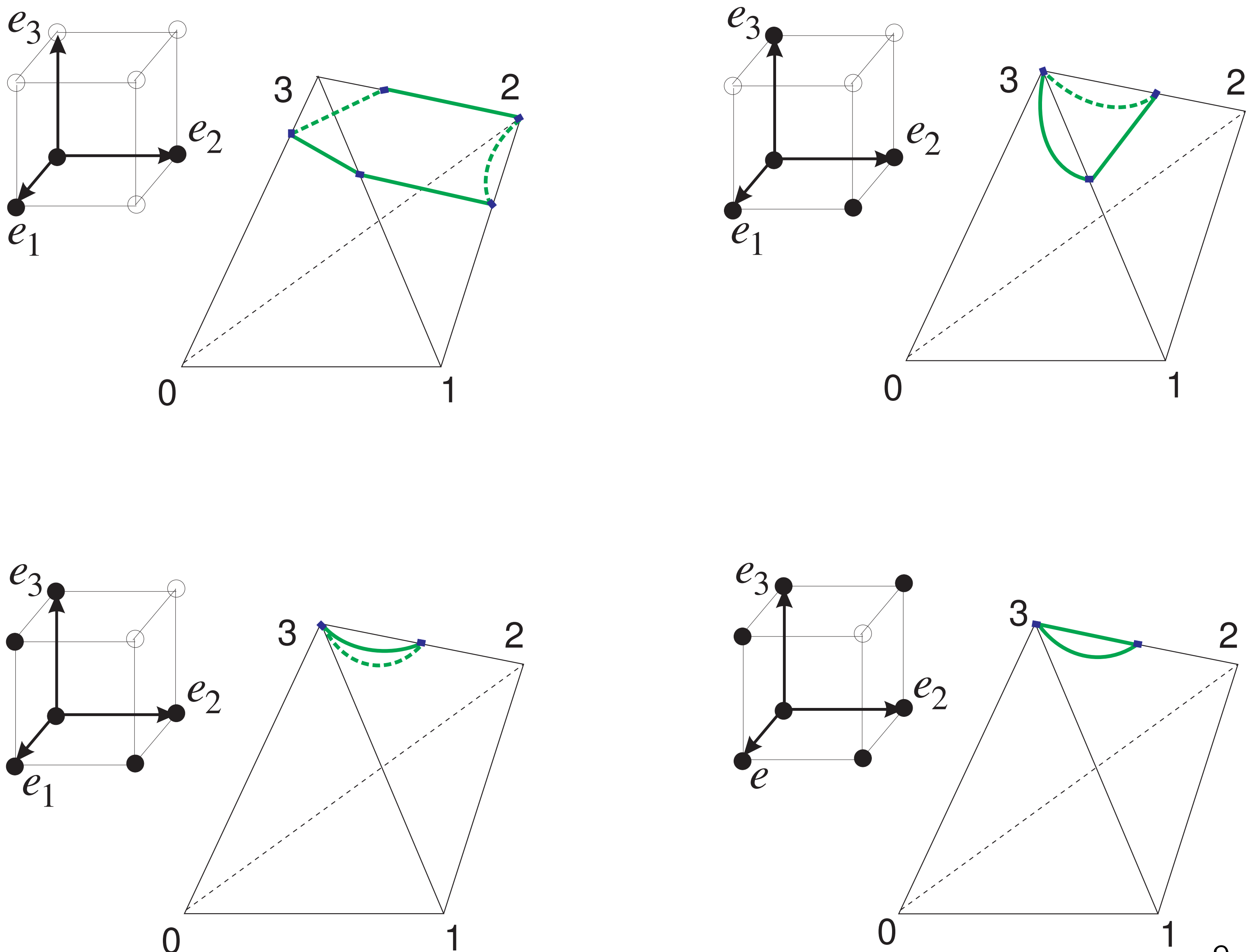
Standard 1D and 2D families



Partition (iterated barycentric subdivision) of a non-standard family into standard families



Standard 3D families (proper, separable)



Standard 3D families (non-separable)

Regular Cells

Definition. A bounded set $X \subset \mathbf{R}^m$ is a **regular** n**-cell** if $(X, \overline{X})$ is homeomorphic to $(B, \overline{B})$ where $B = (0,1)^n$.
X is PL-**regular** if $(X, \overline{X})$ is PL-homeomorphic to $(B, \overline{B})$.

Conjecture. Given a tame monotone family S_δ in a compact K, there exists a PL-**regular cell decomposition** of K such that, for each open n-cell C,
$C \cap S_\delta$ is a family of PL-regular n-cells,
$C \cap \partial S_\delta$ is a family of PL-regular $(n-1)$-cells in ∂C.

Need a decent supply of regular cells to prove this Conjecture.

Remark. A cylindrical n-cell is called regular in "Tame topology and o-minimal structures" by L. van den Dries if its upper and lower bounds are monotone in each of the variables, and its projection to $\mathbb{R}^{n-1}$ is a regular (in the same sense) $(n-1)$-cell. Such a cell is **not** necessarily topologically regular.

Example Let $X = \{x > 0,\ y > 0,\ x + y < 1,\ 0 < z < x^2 + y^2\}$, and
$Y = \{(x, y, z, t) : 0 < t < 1,\ (x/t, y/t, z) \in X\}$.
Then Y is regular in the sense of van den Dries.
However, for $1/2 < c < 1$, $\partial Y \cap \{z = c\}$ is a cone over two disjoint segments, so ∂Y is not a manifold, hence Y is not topologically regular.

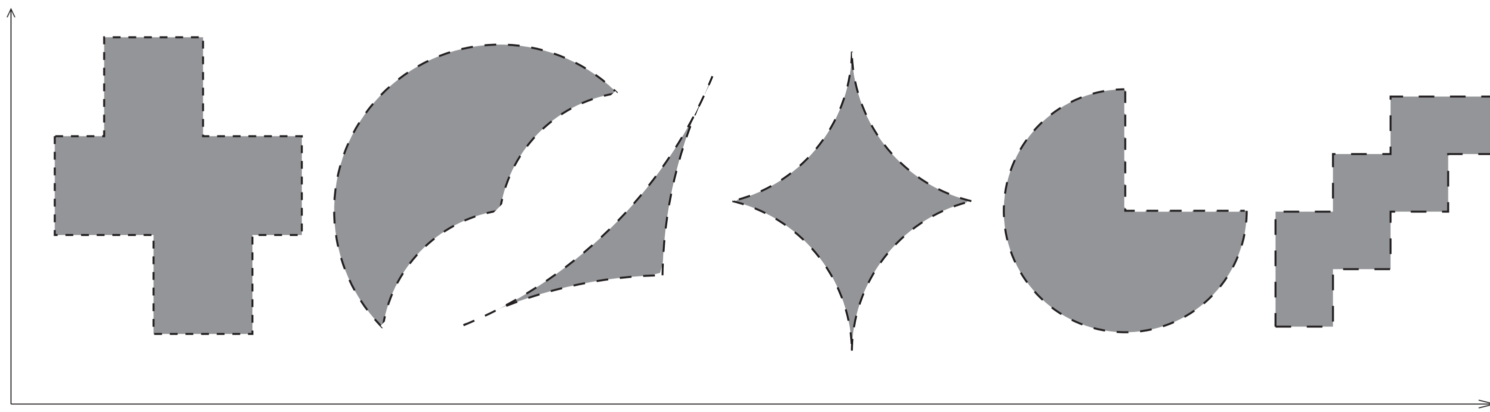
Semi-Monotone Sets

A **coordinate cone** is an intersection of the sets $\{x_j \, ? \, 0\}$ where $? \in \{<, =, >\}$.

An open bounded set $X \subset \mathbf{R}^n$ is **semi-monotone** if its intersection with any translation of any coordinate cone is either empty or connected.

Theorem. (Basu, A.G., Vorobjov, 2010) A tame semi-monotone set $X \subset \mathbf{R}^n$ is a PL-regular n-cell.

Remark. Theorem can be proved for semi-algebraic sets over any **real closed field**.



Examples of semi-monotone (above) and not semi-monotone (below) sets in $\mathbf{R}^2$

Proof of Theorem: Induction on the dimension n. Use local conical structure of tame sets. A cone over a regular $(n-1)$-cell is a regular n-cell.

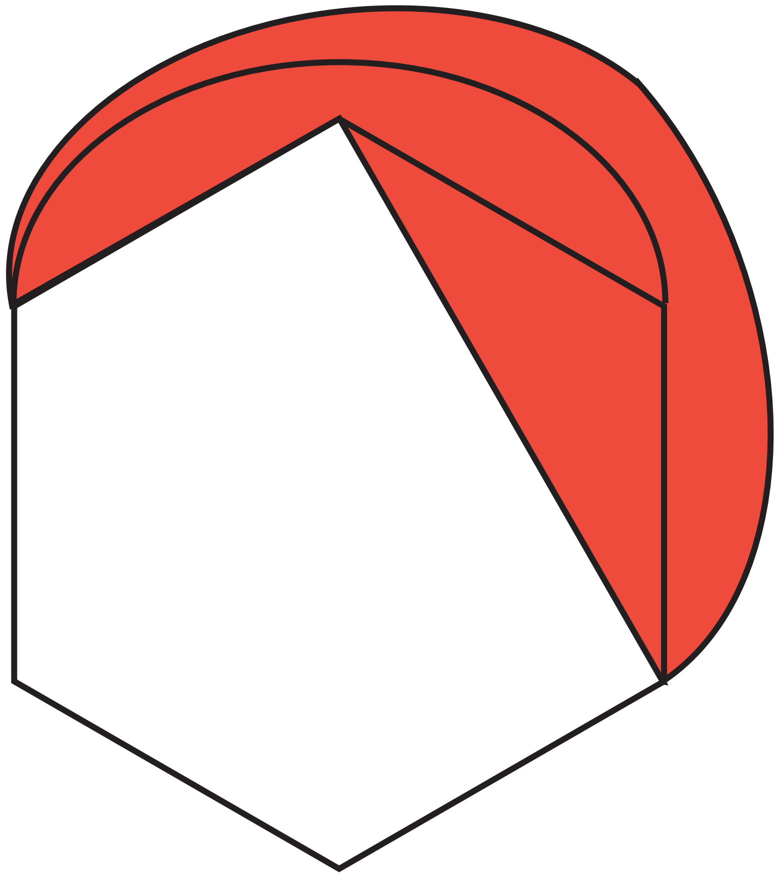
To glue things together, we need to cut a semi-monotone regular cell by generic coordinate hyperplanes and prove that the pieces are again regular cells.

Generalized Schönflies Theorem. If S^{m-1} is a locally flat PL-sphere embedded in S^m, then it cuts S^m into two PL-cubes.

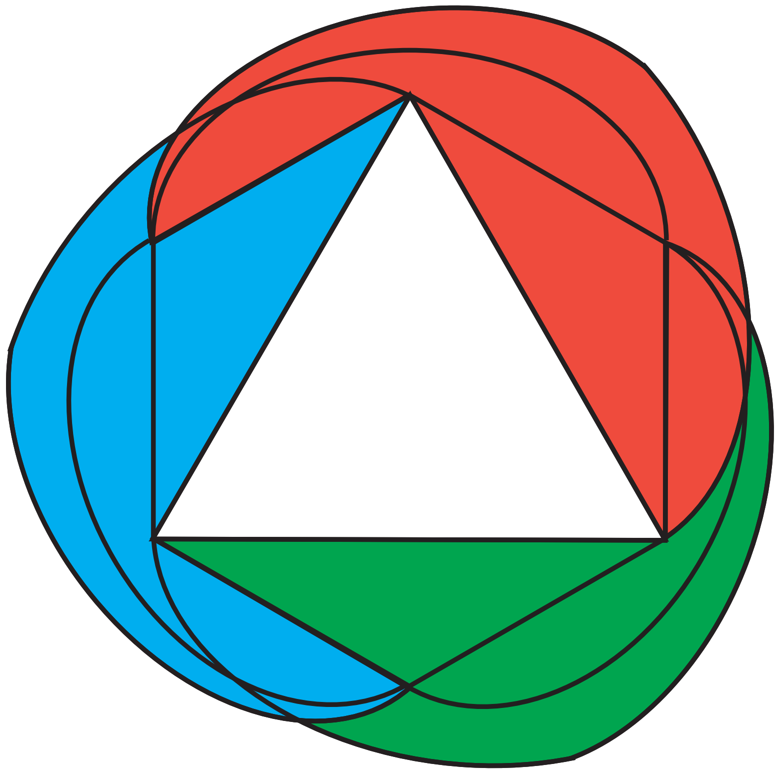
True for $m \neq 4$, unknown for $m = 4$. We need it for $m = n$, $n-1$.

For $n \leq 5$, we circumvent Generalized Schönflies Theorem with

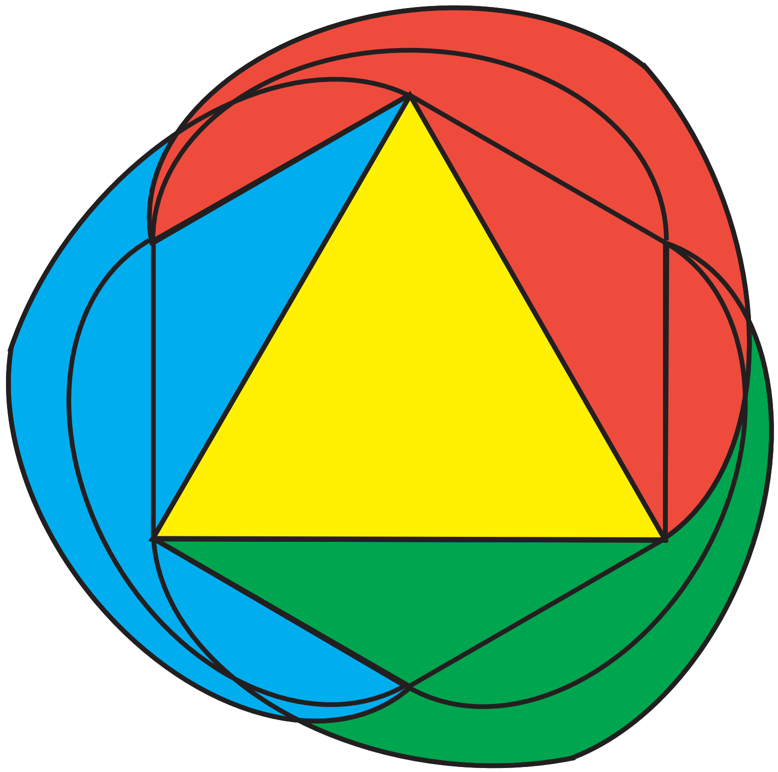
Proposition. Any acyclic simplicial complex with ≤ 5 vertices has a vertex with the acyclic link.



Acyclic 2D complex with 6 vertices, each having non-acyclic link



Acyclic 2D complex with 6 vertices, each having non-acyclic link



Acyclic 2D complex with 6 vertices, each having non-acyclic link

Regular Boolean Functions

A Boolean function $\psi : \{0,1\}^n \to \{0,1\}$ is **regular** if, for any sequence of quantifiers $\exists_j$ and $\forall_k$ applied to ψ, the result **does not depend** on the order of quantifiers.
Here $\exists_j(\psi) = \psi|_{x_j=0} \vee \psi|_{x_j=1}, \quad \forall_k(\psi) = \psi|_{x_k=0} \wedge \psi|_{x_k=1}$.

Theorem. Let us subtract from the cube $(-1,1)^n$ the union of closed octants corresponding to $\{\psi = 1\}$ for a Boolean function ψ.
The result is a regular cell iff ψ is regular.

Theorem. (Basu, A.G., Vorobjov, 2010) A tame open bounded set is semi-monotone iff, for each $x \notin X$, the set of octants with the vertex at x that do not intersect X corresponds to a non-zero regular Boolean function.

A bounded upper semi-continuous function f defined on a semi-monotone set $U \subset \mathbf{R}^n$ is **submonotone** if, for any t, the set $\{f < t\}$ is either empty or semi-monotone.
A function f is **supermonotone** if $-f$ is submonotone.

Theorem. (Basu, A.G., Vorobjov, 2010). An open and bounded set $X \subset \mathbf{R}^{n+1}$ is semi-monotone iff $X = \{f(x) < t < g(x)\}$ for some functions f and g on a semi-monotone set $U \subset \mathbf{R}^n$, where $f(x) < g(x)$ for all $x \in U$, f is submonotone and g is supermonotone.

A bounded continuous function f defined on a semi-monotone set X is **monotone** if it is sub- and supermonotone, and either strictly monotone or constant in each variable.

A map $\mathbf{f} : X \to \mathbf{R}^k$ is **monotone** if each f_j is a monotone function on X and, for any n functions selected from x_i and f_j, each of them is monotone (either strictly increasing, or strictly decreasing, or constant) on the level curves of the other $n-1$ functions.

In both definitions, independence of the type of monotonicity on the choice of constants should be assumed.

This is true if all f_j are monotone and smooth, and all differentials dx_i, df_j are in general position at each point of X.

Theorem. Let $\mathbf{f} : X \to \mathbf{R}^k$ be a monotone map, $X \subset \mathbf{R}^n$. Let $Y = \{x \in X,\ y = \mathbf{f}(x)\} \subset \mathbf{R}^{n+k}$ be the graph of $\mathbf{f}$. Then, for every n-dimensional coordinate subspace L of $\mathbf{R}^{n+k}$ such that projection Z of Y to L is open, Z is a semi-monotone set, and Y is a graph of a monotone map $Z \to \mathbf{R}^k$.