Approximation of definable sets by compact families, and upper bounds on homotopy and homology

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joint work with Andrei Gabrielov (Purdue)

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Construction which produces a *homotopy equivalent compact* definable set $T(S)$ via certain approximation scheme.

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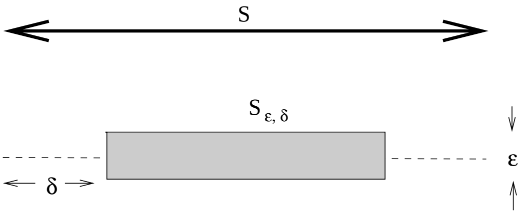
Example

$S := \bigcup_j \bigcap_i \{ \mathbf{x} \in \mathbb{R}^n | \ f_{ij}(\mathbf{x}) = 0, \ g_{ij}(\mathbf{x}) > 0 \}$ where f_{ij}, g_{ij} are continuous definable functions

$S_\delta := \bigcup_j \bigcap_i \{ f_{ij}(\mathbf{x}) = 0, \ g_{ij}(\mathbf{x}) \geq \delta \}$

$S_{\varepsilon,\delta} := \bigcup_j \bigcap_i \{ |f_{ij}(\mathbf{x})| \leq \varepsilon, \ g_{ij}(\mathbf{x}) \geq \delta \}$

$0 < \varepsilon \ll \delta \ll 1$



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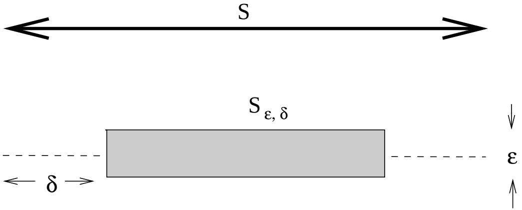

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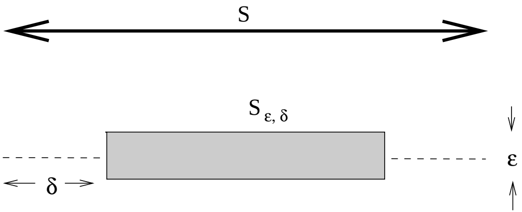

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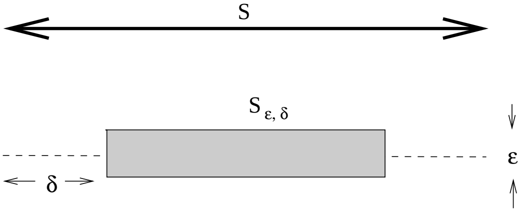

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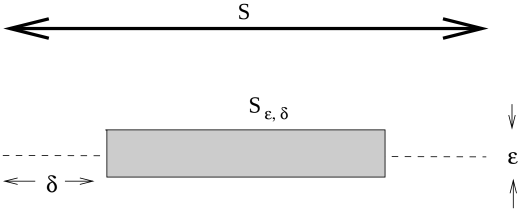
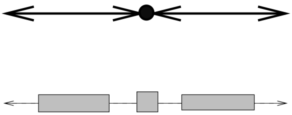




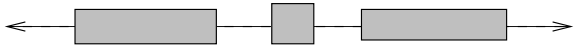
For $0 < \varepsilon_0 \ll \delta_0 \ll \cdots \ll \varepsilon_m \ll \delta_m$ define

$T_m(S) := S_{\varepsilon_0,\delta_0} \cup \cdots \cup S_{\varepsilon_m,\delta_m}$

Theorem (A. Gabrielov, NV)

$T_m(S)$ is m-equivalent to S.
If $m \geq \dim S$ then $T_m(S)$ is homotopy equivalent to S

i.e., there is a map $\varphi : \ T_m(S) \to S$ such that the induced $\varphi_{\# j} : \ \pi_j(T_m(S)) \to \pi_j(S)$ is an isomorphism for $1 \leq j \leq m-1$ and an epimorphism for $j = m$. Same for homology.



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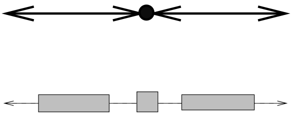

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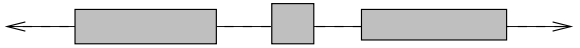

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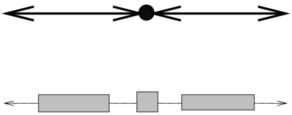

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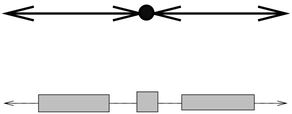

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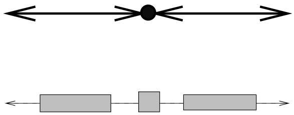

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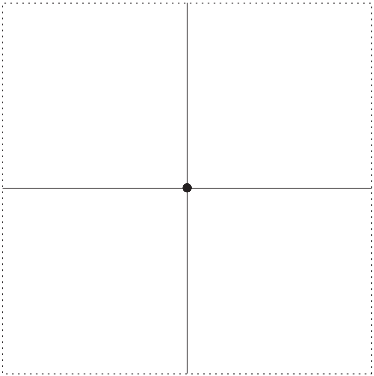
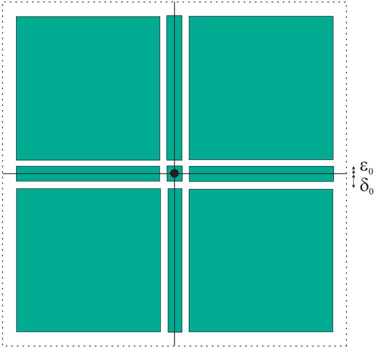
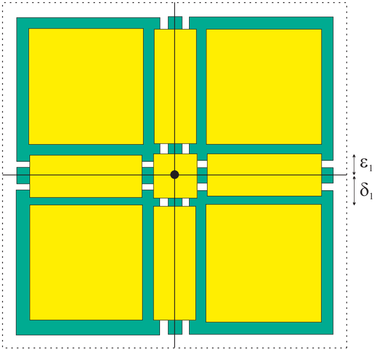


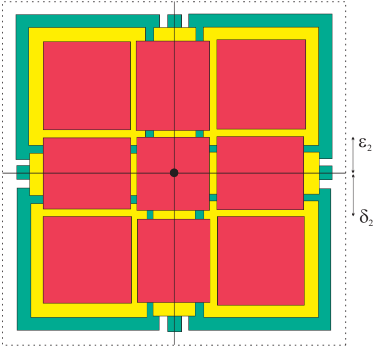
$$S = \{|x| < 1,\ |y| < 1\} \cap$$
$$(\{x > 0,\ y > 0\} \cup \ldots \cup \{x > 0,\ y = 0\} \cup \ldots \cup \{x = y = 0\})$$



Approximation $T_0 = S_{\delta_0, \epsilon_0}$



Approximation $T_1 = S_{\delta_{0},\epsilon_{0}} \cup S_{\delta_{1},\epsilon_{1}}$



Approximation $T_2 = S_{\delta_0,\epsilon_0} \cup S_{\delta_1,\epsilon_1} \cup S_{\delta_2,\epsilon_2}$

To apply in more general situations (e.g., projections of sets defined by equations and inequalities) – axiomatic definition of approximation scheme.

Let $S \subset \mathbb{R}^n$ be a union of an o-minimal monotone family of compact sets $S_\delta,\ \delta > 0$ such that $S_\delta \subset S_{\delta'}$ for $\delta > \delta'$.
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We say that S is *represented* by the families S_δ, $S_{\varepsilon,\delta}$.

Consistent with Example above.

Another examples:
Let $\rho :\ \mathbb{R}^{n+r} \to \mathbb{R}^n$ be the projection on a subspace.
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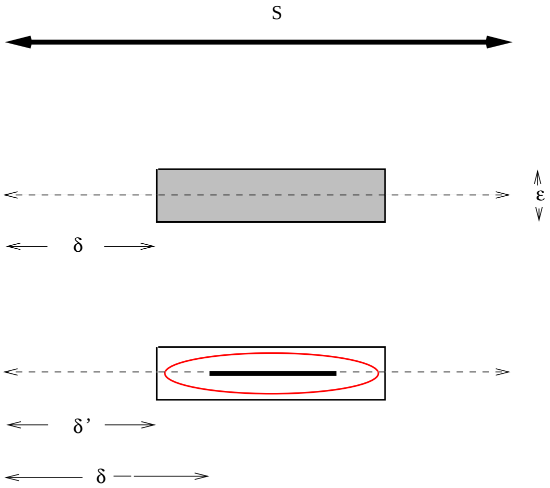
S
ε
δ
δ'
δ

Assume that S is connected.

Whenever $m > 0$ there is a natural bijection between connected components of S and $T_m(S) = S_{\varepsilon_0,\delta_0} \cup \cdots \cup S_{\varepsilon_m,\delta_m}$.

Theorem (A. Gabrielov, NV)

For every $1 \leq j \leq m$, there are epimorphisms

$$\psi_j : \pi_j(T_m(S)) \to \pi_j(S),$$

$$\varphi_j : H_j(T_m(S)) \to H_j(S),$$

in particular, $\operatorname{rank} H_j(S) \leq \operatorname{rank} H_j(T_m(S))$.

Conjecture

ψ_j and φ_j are isomorphisms for $j \leq m-1$.
If $m \geq \dim S$ then $T_m(S)$ and S are homotopy equivalent.

Conjecture proved when the family S_δ is *separable*.
Case of equations and inequalities is separable, case of their projections – not necessarily.

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Conjecture proved when the family S_δ is *separable*.
Case of equations and inequalities is separable, case of their projections – not necessarily.

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Whenever $m > 0$ there is a natural bijection between connected components of S and $T_m(S) = S_{\varepsilon_0,\delta_0} \cup \cdots \cup S_{\varepsilon_m,\delta_m}$.

Theorem (A. Gabrielov, NV)

For every $1 \leq j \leq m$, there are epimorphisms

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Application: asymptotically tight (or close to tight) upper bounds on Betti numbers.

For semialgebraic and basic algebraic sets – a classical problem: Petrovskii, Oleinik, Milnor, Thom.
(Note: triangulations or cellular decompositions are too expensive)

Two directions for generalization: more general definable atomic functions, and more complex formulae defining sets.

More general functions.
The key ingredient in algebraic bound is Bezout's theorem.

Khovanskii: generalization of Bezout to Pfaffian functions.
Hence generalizations of Petrovskii, etc. to semi-Pfaffian sets.

One can introduce the *complexity* of a definable function axiomatically, *à la* Benedetti-Risler, and obtain Betti numbers bounds in terms of this complexity. (One of the axioms is an analogy of Bezout.)
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For definiteness, semialgebraic case.
For s distinct polynomials of degrees $\leq d$ in $\mathbb{R}^n$.

Using classical technique,

- Basu: sets defined by monotone Boolean combinations of only $\geq$-inequalities or of only $>$-inequalities $\mathrm{b}(S) \leq O(sd)^n$;
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Any of the above theorems implies

Theorem (A. Gabrielov, NV)

Let $\nu = \min\{m+1, n-m, s\}$. Then the k-th Betti number

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Proof.

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Projections

Let $\rho : \mathbb{R}^{n+r} \to \mathbb{R}^n$, and $Y = \rho(X)$ where X is a a semialgebraic set defined by a Boolean combination of atomic formulae $h * 0$ where $h \in \{>, \geq, =\}$, $\deg(h) \leq d$ and the number of distinct polynomials h is s.

Effective quantifier elimination produces Boolean combination of equations and inequalities defining Y, and implies

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Pfaffian functions do not admit quantifier elimination.

Another approach, which also produces a better bound in semialgebraic case.

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Definition

For two maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, the *fibered product*

$$X_1 \times_Y X_2 := \{(\mathbf{x}_1, \mathbf{x}_2) \in X_1 \times X_2 |\ f_1(\mathbf{x}_1) = f_2(\mathbf{x}_2)\}.$$

For $f : X \to Y$, let $W_p := \underbrace{X \times_Y \cdots \times_Y X}_{p+1 \text{ times}}$

Example

Let $(\mathbf{x}, \mathbf{y})$ be coordinates in $\mathbb{R}^{n+r}$, let $f = \rho$.
For $X \subset \mathbb{R}^{n+r}$ and $Y = \rho(X) \subset \mathbb{R}^n$, the set $W_p \subset \mathbb{R}^{n+(p+1)r}$ is defined by the same equations and inequalities as X, applied to $\mathbf{y}$ and each of $p+1$ copies of $\mathbf{x}$.

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Let $f : X \to Y$ be a continuous surjective closed o-minimal map. Then there is a spectral sequence $E^r_{p,q}$ converging to $H_*(Y)$ with

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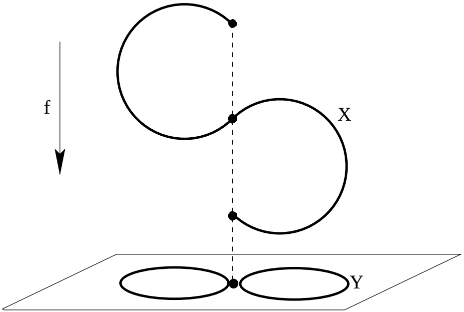

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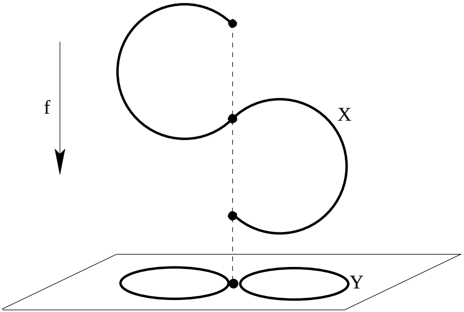

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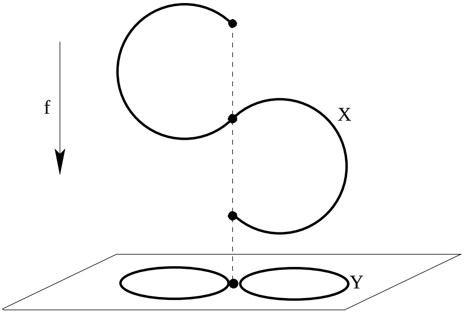

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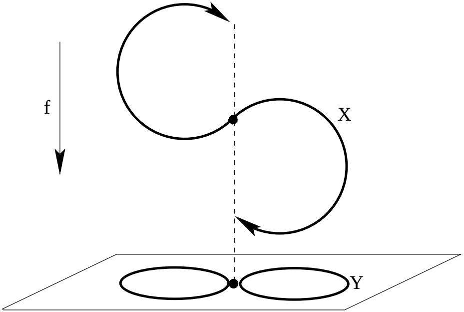

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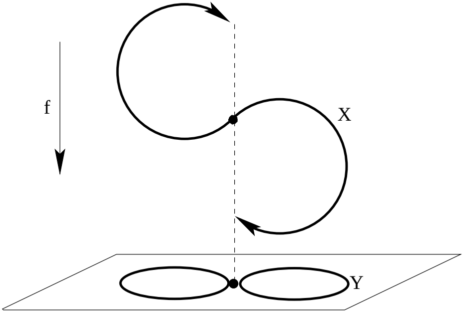

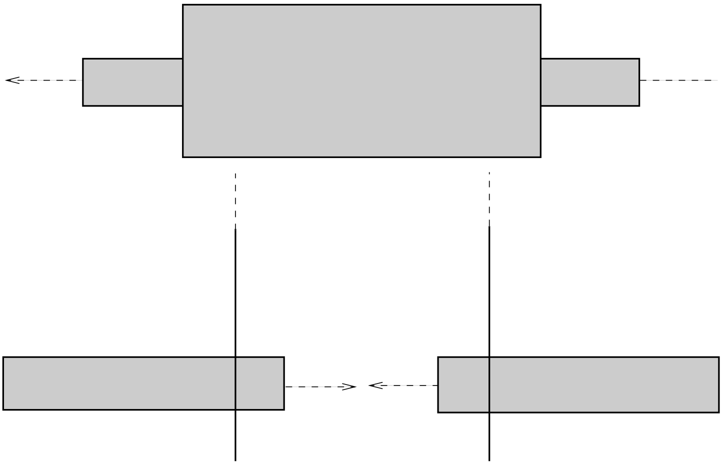

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$T_m(X)$ is compact $\Rightarrow$ the spectral sequence is applicable to ρ.

Recall that X is a semialgebraic set defined by a Boolean combination of atomic formulae $h * 0$ where $h \in \{>, \geq, =\}$, $\deg(h) \leq d$ and the number of distinct polynomials h is s.

Corollary

$$\mathrm{b}_k(Y) \leq \sum_{0 \leq i \leq k} O((i+1)(k+1)sd)^{n+(i+1)r} \leq ((k+1)sd)^{O(n+kr)}$$

Better than quantifier elimination bound

$$\mathrm{b}_k(Y) \leq (sd)^{O(n^2 r)}.$$

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Formulae with quantifiers

$Y = \rho(X)$ is equivalent to $Y = \{\mathbf{y} \in \mathbb{R}^n | \exists\, \mathbf{x} \in \mathbb{R}^r\, ((\mathbf{x}, \mathbf{y}) \in X)\}$

In general

$$Y = \{\mathbf{y} \in \mathbb{R}^n | \exists\, \mathbf{x}_1 \in \mathbb{R}^{r_1}\, \forall\, \mathbf{x}_2 \in \mathbb{R}^{r_2} \exists\, \mathbf{x}_3 \in \mathbb{R}^{r_3} \cdots \forall\, \mathbf{x}_t \in \mathbb{R}^{r_t}$$

$$((\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{y}) \in X)\},$$

where $X \subset \mathbb{R}^{r_1 + \cdots + r_t + n}$ is defined by a Boolean combination.

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Easy by Alexander's duality: $\forall = \neg\exists\neg$

General approach.

Let $\overline{X}$ be complement(X) $\Rightarrow$ $Y = \overline{\rho(\overline{X})}$

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$$((\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{y}) \in X)\},$$

where $X \subset \mathbb{R}^{r_1 + \cdots + r_t + n}$ is defined by a Boolean combination of atomic formulae $h * 0$ where $h \in \{>, \geq, =\}$, $\deg(h) \leq d$ and the number of distinct polynomials h is s.

$$\mathrm{b}_k(Y) \leq (2^{t^2} dsnr_1 \cdots r_t)^{O(2^t nr_1 \cdots r_t)}.$$

Doubly exponential in the number of quantifier alternations.

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