On Gowers' classification program

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Back to Fields Colloquium, October 22, 2012

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Hilbert spaces and Banach spaces: terminology

A Hilbert space is a vector space with an inner product which turns it into a complete space.
The Hilbert space H is the unique (up to linear isometrical isomorphism) separable infinite dimensional Hilbert space, e.g. $H = \ell_2$ or L_2.
A Banach space is a normed complete space.

Question

How different may a general separable Banach space X be from the Hilbert space ℓ_2?

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In general one may be interested either in the isometric structure of a Banach space X, or in its isomorphic structure of X. In the second case, one may replace the initial norm $\|.\|$ by an equivalent one $|||.|||$, that is for which the identity map is an isomorphism, or

$$\forall x, c\|x\| \leq |||x||| \leq C\|x\|.$$

preserving the topology, as well as operator convergence. So we shall also use the definition:

A Banach space $(X, \|.\|)$ is hilbertian if it is isomorphic to a Hilbert space, or equivalently, if there is an equivalent norm $|||.|||$ so that $(X, |||.|||)$ is a Hilbert space.

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Classical spaces

- the sequences spaces c_0 and ℓ_p ($||x||_p = (\sum_n |x_n|^p)^{1/p}$).
- the function spaces $L_p(\mu)$ (which contain copies of ℓ_p),
- the function spaces $C(K)$ (which contain copies of c_0).

The first non-classical space was due to B.S. Tsirelson in 1974.

Theorem (Tsirelson, 1974)

There exists a Banach space T which does not contain a copy of c_0 or $\ell_p, 1 \leq p < +\infty$.

The norm of T is defined by induction to "force" a local ℓ_1-behaviour on finite dimensional subspaces without implying a global ℓ_1-behaviour.

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Exotic spaces

This paved the way for Gowers and Gowers-Maurey's constructions of exotic spaces which solved some open questions in Banach space theory from the 1930s.

Theorem (Gowers-Maurey, 1993)

There exists a HI space GM, i.e. a space with few operators. In particular GM is not isomorphic to its hyperplanes, not even to its proper subspaces.

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Gowers' classification program

In his famous paper "An infinite Ramsey theorem and some Banach space dichotomies", Gowers proved Ramsey type dichotomies in Banach spaces, and used these to prove that the previously mentioned examples form an inevitable list of spaces.

Theorem (Gowers, 2002)

Every Banach space contains a subspace:

- *of the type of GM,*
- *of the type of G_u,*
- *of the type of T,*
- *of the type of c_0 and ℓ_p.*

Gowers observes that this list of 4 classes are defined by properties which make the list inevitable in the following sense:

a) If X belongs to a class, then all its subspaces belong again to the same class,
b) every space has a subspace in one of the classes,
c) the classes are very obviously disjoint,
d) belonging to a class gives a lot of information on the operators that may be defined on the space.

Any list of classes satisfying a)b)c)d), obtained by Ramsey type dichotomies, will be an answer to Gowers' classification program.

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Gowers' classification program

Of course each class in such a list should be defined by one or several hereditary properties, as in Gowers' initial list of 4 classes, and the list could always be refined by using some more properties, possibly dividing each classe in several subclasses.

Question (Gowers' classification program)

How to refine Gowers' inevitable list of 4 classes? How to be more precise about the properties defining the classes? How to divide some classes in several subclasses?

In particular, according to Gowers' program the last or "nicest" class should the class of spaces *isomorphic* to c_0 or ℓ_p. This is not the case in his list of 4 classes, as we shall explain later on.

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Recent developments: complexity

- The idea of defining a list of specific structures, present as a substructure of any given structure is of course not original.
- Such ideas to consider simpler substructures present in every structure may come from the feeling that the general classification of the structures themselves is out of reach (say here, the classification of separable Banach spaces up to linear isomorphisms by some identifiable invariants is much too complex).
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- The theory of complexity of equivalence relations deals with such questions of complexity of classification.

Complexity of isomorphism: definition

Definition

Let R and S be two analytic equivalence relations on Polish spaces E and F respectively. We say that E is Borel reducible to F if there exists a Borel function $f : X \to Y$ such that

$$\forall x, y \in E, xRy \Leftrightarrow f(x)Sf(y).$$

We obtain in this way a relative mesure of complexity of (analytic) equivalence relations on Polish spaces.

Most natural relations of isomorphism of structures in analysis belong to this setting.

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Complexity of isomorphism: definition

For example, the Banach-Stone theorem

$$K_1 \text{ homeomorphic to } K_2 \Leftrightarrow C(K_1) \text{ isometric to } C(K_2)$$

means in this setting that

homeomorphism of compact metric spaces

is Borel reducible to (i.e. not more complex than)

isometry of separable Banach spaces

Complexity of linear isomorphism

Theorem (Ferenczi - Louveau - Rosendal, 2006)

The complexity of linear isomorphism between separable Banach spaces is E_{max}*, the maximum complexity among all analytic equivalence relations.* *The same holds for*

- *linear isomorphic beembedding, complemented linear isomorphic biembedding, Lipschitz isomorphism of separable Banach spaces,*
- *uniform homeomorphism of complete metric spaces,*
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This may give support to the idea that classifying Banach spaces in general is out of reach and that one should concentrate on a "simpler" classification such as Gowers'.

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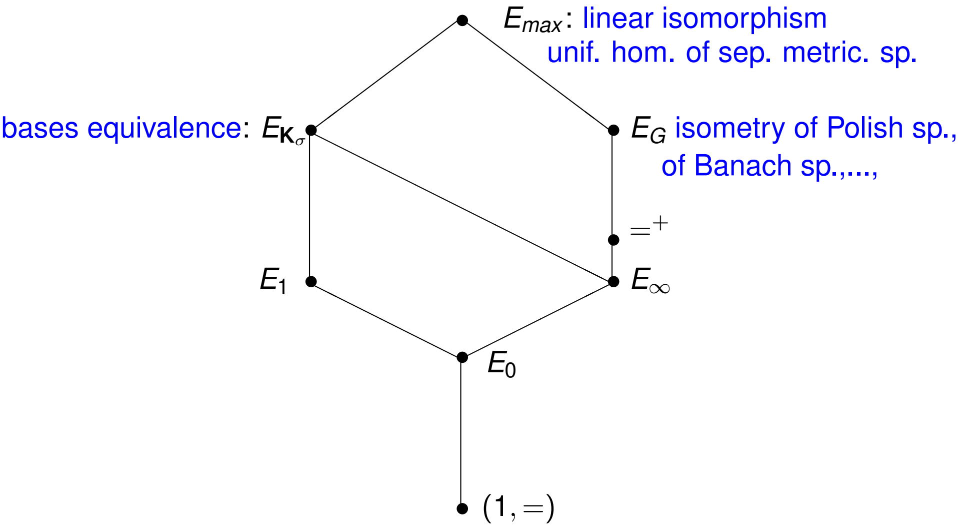
Complexities of some equivalence relations



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Gowers' list of 4 classes

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- *minimal, like c_0, ℓ_p, but also others: S, T^*...*

A space X is minimal if every subspace of X has a further subspace isomorphic to X. Such spaces may be thought of as spaces which can not be "simplified" by passing to a subspace, "self-similar" spaces, "fractal" spaces,...

Question

What is the correct dichotomy for minimality? And how may we distinguish between c_0, ℓ_p and other minimals?

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Tsirelson's space

Let us take a look at the first "non-classical" space, Tsirelson's space T.
Note that the shift S on T is an isomorphism, so T is isomorphic to its hyperplanes. However

Fact

We have $\lim_n \|S^n\| = +\infty$.

This shows that T is different from c_0 or ℓ_p, where S is isometric. Even more

Fact

The space T is not uniformly isomorphic to its tail subspaces (i.e. there is no K such that T is K-isomorphic to $[e_i, i \geq n]$ for all n, where $(e_i)_i$ is the natural basis of T).

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It is not possible to embed T uniformly into its tail subspaces (i.e. there is no K such that T is K-isomorphic to a subspace of every tail subspace).

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Let us call this property of a Banach space property (t).

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Property (t)

Definition

A space X with a basis has property (t) if no subspace of X embeds uniformly into the tail subspaces of X.

Note that

- property (t) is hereditary,
- c_0 or ℓ_p do not satisfy (t), therefore, property (t) spaces do not contain copies of c_0 or ℓ_p
- minimal spaces do not satisfy (t): therefore, property (t) spaces do not contain minimal subspaces.

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Is it not true that every Banach space contains a minimal subspace or a subspace with property (t).

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Topological characterization of property (t)

So to obtain a dichotomy with minimality, we are looking for a more general property than (t).

If X has a Schauder basis, let us consider $b(X)$, the set of subspaces generated of sequences of vectors with rational coordinates and successive supports on the basis.

This is easily seen as a Polish space.

On the other hand, classical results tell us that subspaces in $b(X)$ capture enough of the general structure of the set of subspaces of X.

So $b(X)$ will be the Polish space of (approximately all) subspaces of X.

Topological characterization of property (t)

Proposition (F. Godefroy 2011)

X has property (t) if and only if for any $K \in \mathbb{N}$, for any $Y \in b(X)$, the set

$$\mathrm{Emb}_K(Y) = \{Z \in b(X) : Z \text{ contains a K} - \text{isomorphic copy of } Y\}$$

is nowhere dense.

Corollary

If X has property (t) then the set

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is meager (i.e. has topological measure 0) for all $Y \in b(X)$.

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Theorem (3rd dichotomy, Ferenczi-Rosendal 2009)

Any Banach space contains a subspace X such that either

- *X is minimal (i.e. embeds into all its subspaces), or*
- *no Y embeds in more than a meager set of subspaces of X.*

A space X with this last property will be said to be tight.

Property (t) is just a special kind of tightness. Other kinds exist.

The dichotomy for minimality: tightness

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- *X is minimal (i.e. embeds into all its subspaces), or*
- *no Y embeds in more than a meager set of subspaces of X.*

A space X with this last property will be said to be tight.

Property (t) is just a special kind of tightness. Other kinds exist.

Refining Gowers' list: 4 dichotomies

Unconditional basis	$\ast$ 1st dichotomy $\ast$	Hered. indecomp.
$\Uparrow$	(Gowers 96)	$\Downarrow$
Tight by support	$\ast$ 2nd dichotomy $\ast$	Quasi minimal
$\Downarrow$	(Gowers 02)	$\Uparrow$
Tight by range	$\ast$ 4rd dichotomy $\ast$	Seq. minimal
$\Downarrow$	(F.R. 09)	$\Uparrow$
Tight	$\ast$ 3rd dichotomy $\ast$	Minimal
	(F.R. 09)	

Refining Gowers' list: 4+2=6 dichotomies

st. as. $\ell_p, 1 \leq p < +\infty$	$\ast$ Tcaciuc $\ast$	Unif. inhomogeneous
$\Downarrow$	(Tcaciuc 07)	$\Uparrow$
Unconditional basis	$\ast$ 1st dichotomy $\ast$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$\ast$ 2nd dichotomy $\ast$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$\ast$ 4th dichotomy $\ast$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$\ast$ 3rd dichotomy $\ast$	Minimal
$\Uparrow$		$\Downarrow$
Property (t)	$\ast$ 5th dichotomy $\ast$	Loc. minimal
	(F.R. 09)	

Refining Gowers' list: 4+2=6 dichotomies

st. as. $\ell_p, 1 \leq p < +\infty$	$*$ Tcaciuc $*$	Unif. inhomogeneous
$\Downarrow$		$\Uparrow$
Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomie $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomie $*$	Minimal
$\Uparrow$		$\Downarrow$
Property (t)	$*$ 5th dichotomie $*$	Loc. minimal

Combining the 6 dichotomies one should obtain $2^6 = 64$ classes of Banach spaces, but because of the different relations between the properties, one obtains 19 classes.

Refining Gowers' list: 4+2=6 dichotomies

st. as. $\ell_p, 1 \leq p < +\infty$	$*$ Tcaciuc $*$	Unif. inhomogeneous
$\Downarrow$		$\Uparrow$
Unconditional basis	$*$ 1st dichotomy $*$	Hered. indecomp.
$\Uparrow$		$\Downarrow$
Tight by support	$*$ 2nd dichotomie $*$	Quasi minimal
$\Downarrow$		$\Uparrow$
Tight by range	$*$ 4th dichotomy $*$	Seq. minimal
$\Downarrow$		$\Uparrow$
Tight	$*$ 3rd dichotomie $*$	Minimal
$\Uparrow$		$\Downarrow$
Property (t)	$*$ 5th dichotomie $*$	Loc. minimal

More precisely, one obtains 6 classes from the first 4 dichotomies, and 19 sub-classes by also using the two others.

Refining Gowers' list: a list of 6 classes

Theorem (Ferenczi, Rosendal, 2009)

Every Banach space of infinite dimension contains a subspace of one of the following 6 types:

Type	Properties	Examples
(1)	HI, tight by range	Gowers, 95 (F.R.)
(2)	HI, tight, seq. minimal	Gowers-Maurey', 11 (F. Schlumprecht)
(3)	tight by support	Gowers, 94
(4)	unc. basis, quasi min., tight by range	Argyros,Manoussakis, Pelczar,12
(5)	unc. basis, tight, seq. minimal	Tsirelson, 74
(6)	unc. basis, minimal	c_0, ℓ_p

Refining Gowers' list: a list of 19 subclasses

Type	Properties	Examples
(1)	HI, tight by range	1a: ?, 1b; G_{as}
(2)	HI, tight, seq. minimal	2a: ?, 2b: GM'
(3)	tight by support	3a:?, 3b: G^*, 3c: X_u, 3d: X_u^*
(4)	unc. basis, quasi min., tight by range	1a:?, 1b: AMP 1c:?, 1d:?
(5c)	unc. basis, seq. minimal, and - prop. (t), st. as. ℓ_p, $1 \leq p < \infty$,	T
(5abd)	- other properties	?
(6a)	minimal, unif. inhomogeneous	S
(6b)	minimal, reflexive, st. as. ℓ_∞	T^*
(6c)	isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$	c_0, ℓ_p

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