REDUCED VARIATIONAL PRINCIPLES FOR FREE-BOUNDARY CONTINUA

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Based on two papers: Holm, Marsden, TR, 1986, Montreal Lecture Notes, Gay-Balmaz, Marsden, TR 2012, one of the last papers Jerry wrote that was finished after he passed away

PLAN OF THE PRESENTATION

- *Continuum mechanical setup*
- *The heavy top*
- *Affine semidirect product Lagrangian reduction*
- *Body and spatial equations for the heavy top*
- *Fixed boundary barotropic fluids*
- *Elasticity*
- *Free boundary fluids*
- *Eringen's equations for micropolar liquid crystals*

Continuum mechanical setup

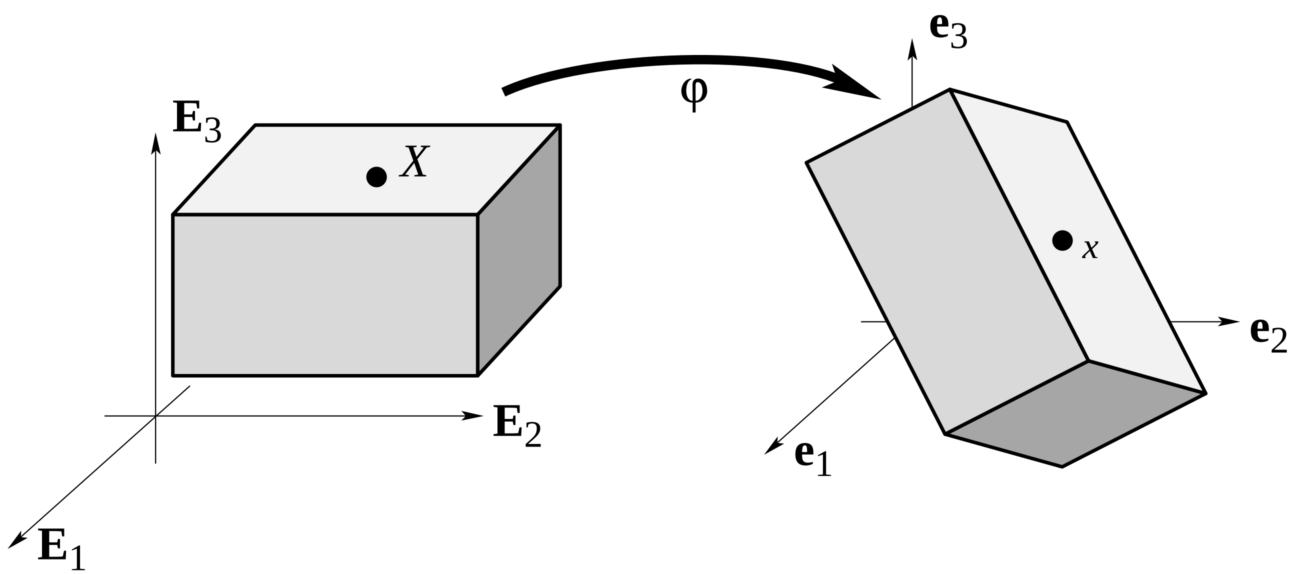
Reference configuration: $(\mathcal{B}, \mathbf{G})$ oriented Riemannian manifold
Usually $\mathcal{B} \subset \mathbb{R}^3 = \{\mathbf{X} = (X^1, X^2, X^3)\}$; $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ orthonormal

Spatial configuration: $(\mathcal{S}, \mathbf{g})$ oriented Riemannian manifold
Usually $\mathcal{S} = \mathbb{R}^3 = \{\mathbf{x} = (x^1, x^2, x^3)\}$; $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ orthonormal

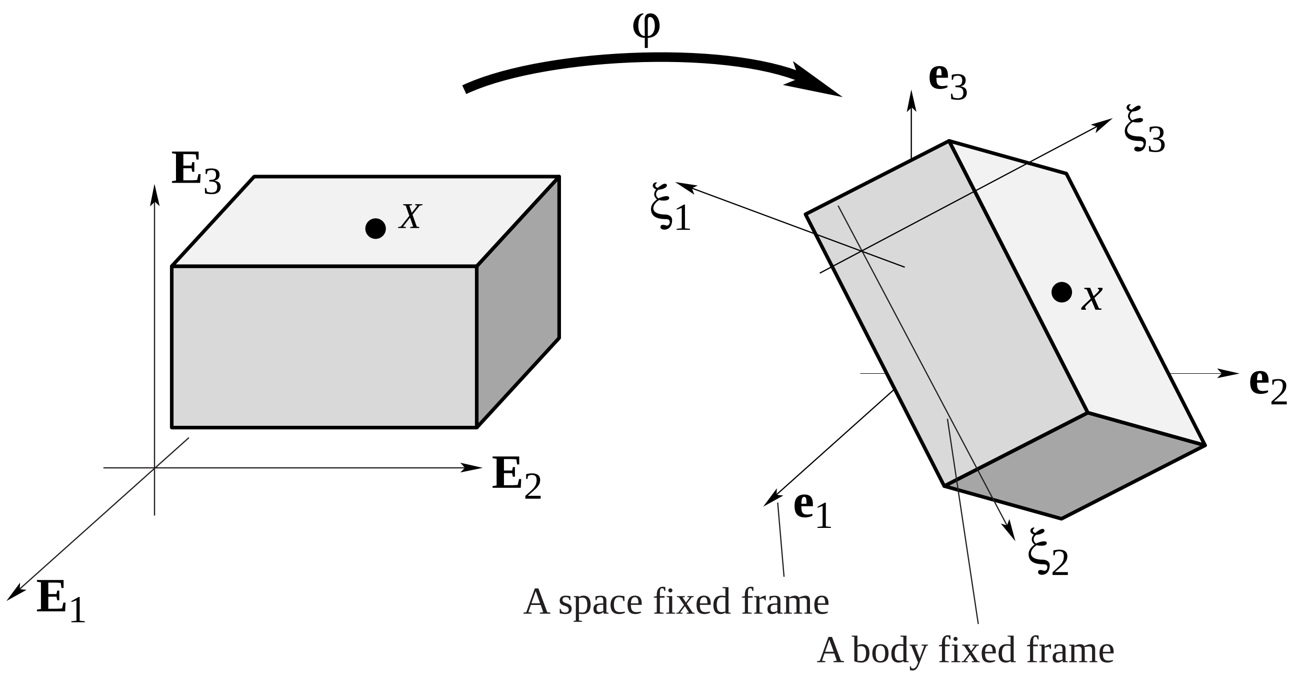
Configuration: orientation preserving embedding $\varphi : \mathcal{B} \to \mathcal{S}$, so the ***configuration space*** is $\mathrm{Emb}_+(\mathcal{B}, \mathcal{S})$

Motion: $\varphi_t(\mathbf{X}) = \mathbf{x}(\mathbf{X}, t)$ time dependent family of configurations

Time dependent basis anchored in the body moving together with it: $\boldsymbol{\mathcal{E}}_i := \varphi_t(\mathbf{E}_i)$, $i = 1, 2, 3$. ***Body*** or ***convected coordinates***: coordinates relative to $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$.



φ
E3
X
E2
E1
e3
x
e2
e1



φ
E3
X
E2
E1
e3
ξ3
ξ1
x
e2
e1
ξ2
A space fixed frame
A body fixed frame

The ***material*** or ***Lagrangian velocity*** is defined by

$$\mathbf{V}(\mathbf{X},t) := \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \frac{\partial}{\partial t}\varphi_t(\mathbf{X}).$$

The ***spatial*** or ***Eulerian velocity*** is defined by

$$\mathbf{v}(\mathbf{x},t) := \mathbf{V}(\mathbf{X},t) \Longleftrightarrow \mathbf{v}_t \circ \varphi_t = \mathbf{V}_t.$$

The ***body*** or ***convective velocity*** is defined by

$$\mathcal{V}(\mathbf{X},t) := -\frac{\partial \mathbf{X}(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial t}\varphi_t^{-1}(\mathbf{x}) \Longleftrightarrow \mathcal{V}_t = T\varphi_t^{-1} \circ \mathbf{V}_t = \varphi_t^* \mathbf{v}_t$$

The ***particle relabeling group*** Diff$(\mathcal{B})$ acts on the *right* on Emb$_+(\mathcal{B},\mathcal{S})$. The ***material frame indifference group*** Diff$(\mathcal{S})$ acts on the *left* on Emb$_+(\mathcal{B},\mathcal{S})$.

In continuum mechanics it is important to keep all options open and always have three descriptions available. They serve different purposes and the interactions between them gives interesting physical insight.

HEAVY TOP

Material or ***Lagrangian description:*** $\mathcal{B} \subset \mathbb{R}^3$ compact region with non-empty interior, $\mathbf{G}_{ij} = \delta_{ij}$. Relative to an orthonormal basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ we get ***material/Lagrangian coordinates*** X^1, X^2, X^3

Spatial or ***Eulerian description:*** $\mathcal{S} = \mathbb{R}^3$, $\mathbf{g}_{ij} = \delta_{ij}$. Relative to an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ we get ***spatial/Eulerian coordinates*** x^1, x^2, x^3

Body: Time dependent orthonormal basis anchored in the body moving together with it: $\boldsymbol{\mathcal{E}}_i := A(t)\mathbf{E}_i$, $i = 1, 2, 3$. Relative to the orthonormal basis $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$ we get the ***body/convected coordinates*** $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$

Components of a vector $\mathbf{U}$ in the basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are the same as the components of the vector $A(t)\mathbf{U}$ in the basis $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$.
Note that the body coordinates of $\mathbf{x}(\mathbf{X}, t) = A(t)\mathbf{X}$ are X^1, X^2, X^3.

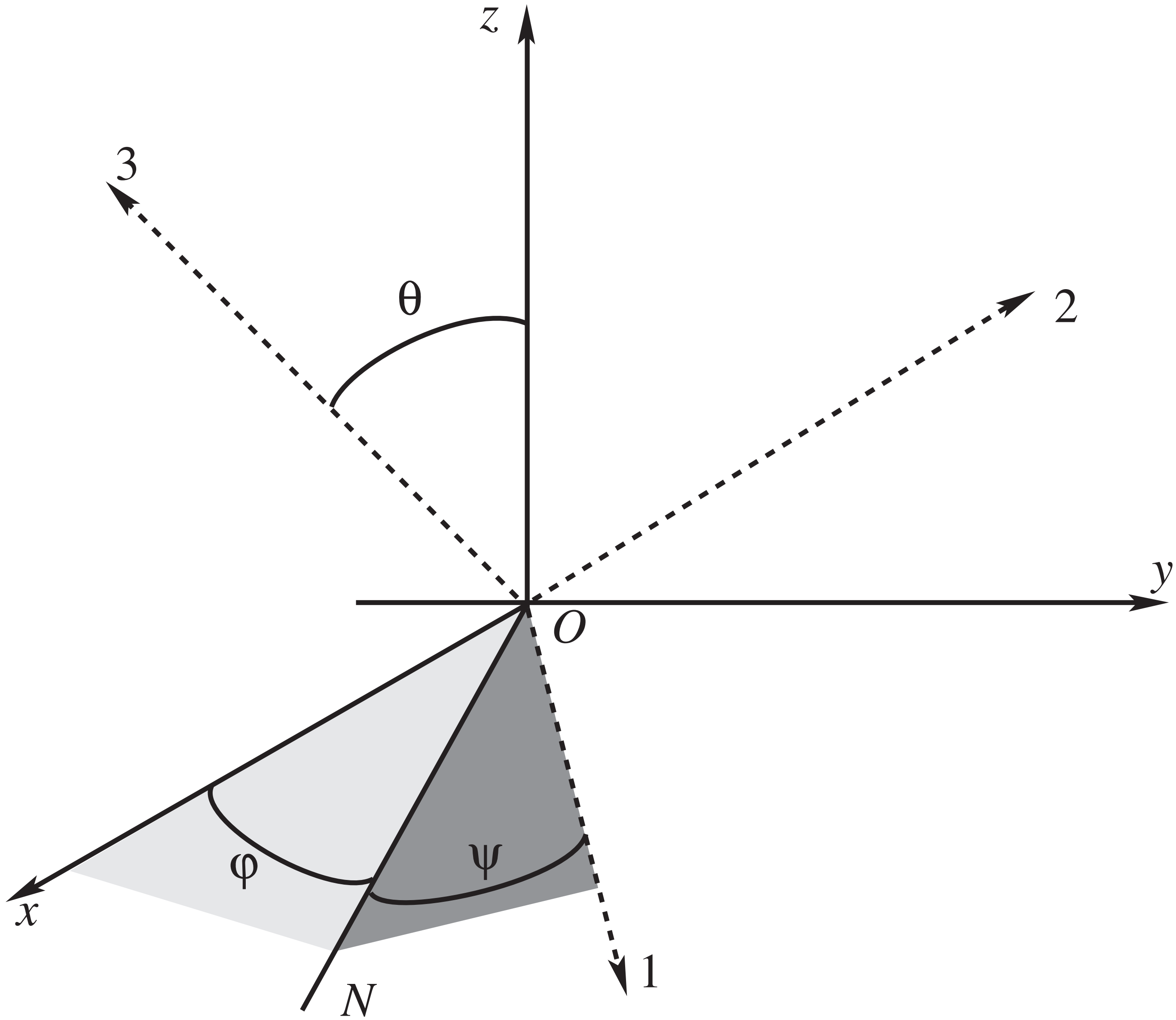
Euler angles

Passage from orthonormal spatial basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to orthonormal basis in the body $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$ by three consecutive counterclockwise rotations (specific order): first rotate around the axis $\mathbf{e}_3$ by the angle φ and denote the resulting position of $\mathbf{e}_1$ by ON (line of nodes), then rotate about ON by the angle θ and denote the resulting position of $\mathbf{e}_3$ by $\boldsymbol{\mathcal{E}}_3$, finally rotate about $\boldsymbol{\mathcal{E}}_3$ by the angle ψ.

By construction: $0 \leq \varphi, \psi < 2\pi$, $0 \leq \theta < \pi$. Get a bijection between $\{(\varphi, \psi, \theta)\}$ and SO(3). It is *not* a chart: its differential vanishes at $\varphi = \psi = \theta = 0$. But for $0 < \varphi, \psi < 2\pi$, $0 < \theta < \pi$ the ***Euler angles*** (φ, ψ, θ) do form a chart.

The resulting linear map has matrix relative to the bases $\boldsymbol{\mathcal{E}}_1, \boldsymbol{\mathcal{E}}_2, \boldsymbol{\mathcal{E}}_3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ equal to

$$A = \begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}$$


z
3
θ
2
y
O
φ
ψ
x
1
N

For the rigid body moving about a fixed point, the motions are rotations: $\mathbf{x}(\mathbf{X}, t) := A(t)\mathbf{X}$, where $A(t) \in SO(3)$.

The ***material*** or ***Lagrangian velocity***

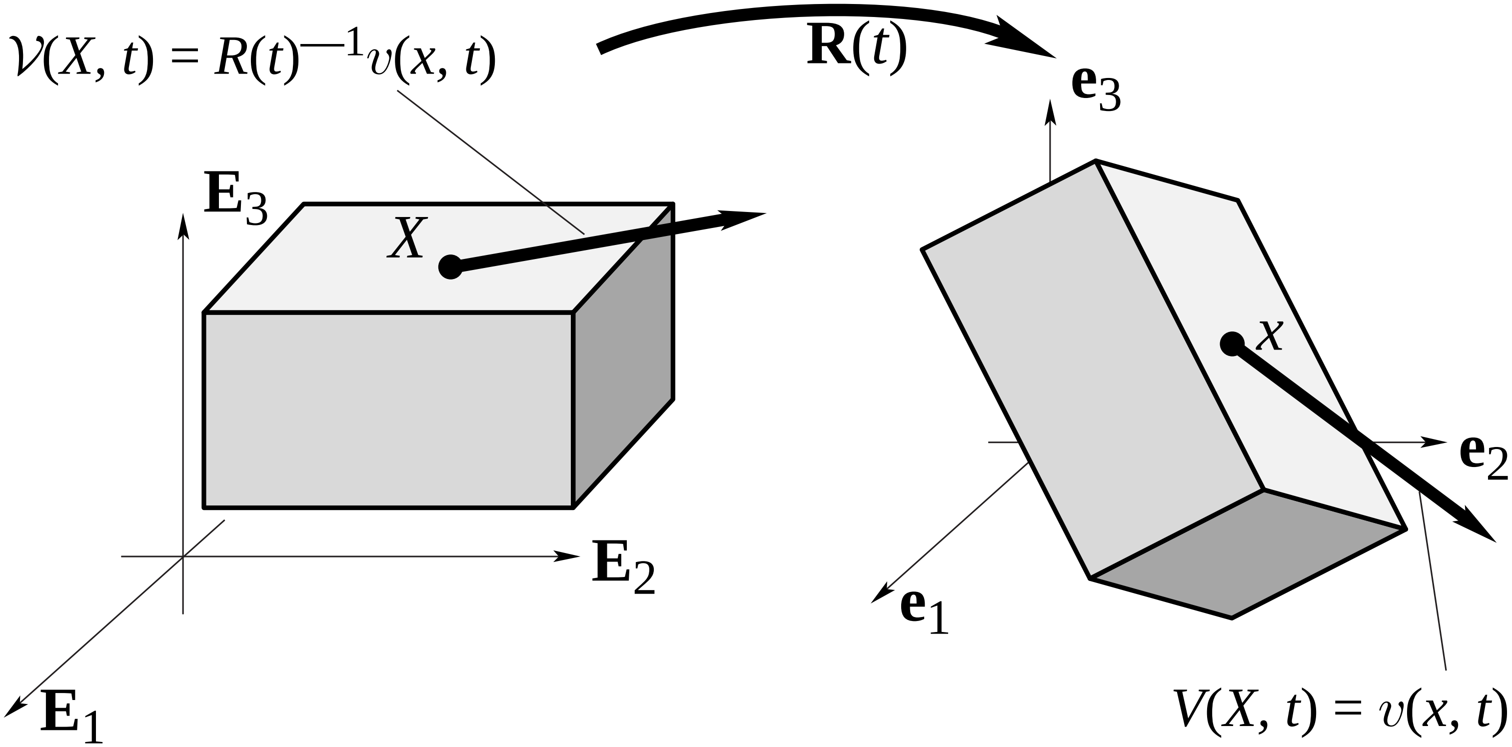
$$\mathbf{V}(\mathbf{X}, t) := \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} = \dot{A}(t)\mathbf{X}.$$

The ***spatial*** or ***Eulerian velocity***

$$\mathbf{v}(\mathbf{x}, t) := \mathbf{V}(\mathbf{X}, t) = \dot{A}(t)\mathbf{X} = \dot{A}(t)A(t)^{-1}\mathbf{x}.$$

The ***body*** or ***convective velocity***

$$\begin{aligned}\mathcal{V}(\mathbf{X}, t) : &= -\frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial t} = A(t)^{-1}\dot{A}(t)A(t)^{-1}\mathbf{x} \\ &= A(t)^{-1}\mathbf{V}(\mathbf{X}, t) = A(t)^{-1}\mathbf{v}(\mathbf{x}, t).\end{aligned}$$



Material velocity $\mathbf{V}$, spatial velocity $\mathbf{v}$, and body velocity $\mathcal{V}$.

Kinetic energy

ρ_0 density in the reference configuration. The kinetic energy at time t in material, spatial, and convective representation:

$$\begin{aligned} K(t) &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X})\|\mathbf{V}(\mathbf{X},t)\|^2 d^3\mathbf{X} && \textit{material} \\ &= \frac{1}{2}\int_{A(t)\mathcal{B}} \rho_0(A(t)^{-1}\mathbf{x})\|\mathbf{v}(\mathbf{x},t)\|^2 d^3\mathbf{x} && \textit{spatial} \\ &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X})\|\mathcal{V}(\mathbf{X},t)\|^2 d^3\mathbf{X}. && \textit{body} \end{aligned}$$

Define $\quad \widehat{\boldsymbol{\omega}}(t) := \dot{A}(t)A(t)^{-1}, \qquad \widehat{\boldsymbol{\Omega}}(t) := A(t)^{-1}\dot{A}(t), \quad$ then

$$\mathbf{v}(\mathbf{x},t) = \boldsymbol{\omega}(t) \times \mathbf{x}, \qquad \mathcal{V}(\mathbf{X},t) = \boldsymbol{\Omega}(t) \times \mathbf{X},$$

$\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are ***spatial*** and ***body angular velocities***; $\boldsymbol{\omega}(t) = A(t)\boldsymbol{\Omega}(t)$.
$\widehat{\ }$: $(\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\,,])$ is the Lie algebra isomorphism $\hat{\mathbf{u}}\mathbf{v} = \mathbf{u} \times \mathbf{v}$.

So $\quad K(t) = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{X})\|\boldsymbol{\Omega}(t) \times \mathbf{X}\|^2 d^3\mathbf{X} =: \frac{1}{2}\langle\langle \boldsymbol{\Omega}(t), \boldsymbol{\Omega}(t) \rangle\rangle$

which is the quadratic form of the bilinear symmetric map on $\mathbb{R}^3$

$$\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle := \int_{\mathcal{B}} \rho_0(\mathbf{X})(\mathbf{a} \times \mathbf{X}) \cdot (\mathbf{b} \times \mathbf{X}) d^3\mathbf{X} = \mathbb{I}\mathbf{a} \cdot \mathbf{b},$$

where $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the symmetric isomorphism (relative to the dot product) whose components are $\mathbb{I}_{ij} := \mathbb{I}\mathbf{E}_j \cdot \mathbf{E}_i = \langle\langle \mathbf{E}_j, \mathbf{E}_i \rangle\rangle$, i.e.,

$$\mathbb{I}_{ij} = -\int_{\mathcal{B}} \rho_0(\mathbf{X}) X^i X^j d^3\mathbf{X} \quad \text{if} \quad i \neq j$$

$$\mathbb{I}_{ii} = \int_{\mathcal{B}} \rho_0(\mathbf{X}) \left(\|\mathbf{X}\|^2 - (X^i)^2 \right) d^3\mathbf{X}.$$

So $\mathbb{I}$ is the ***moment of inertia tensor***. ***Principal axis body frame***: basis in which $\mathbb{I}$ is diagonal; diagonal elements I_1, I_2, I_3 of $\mathbb{I}$ are the ***principal moments of inertia*** of the top. From now on, choose $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ to be a principal axis body frame.

$\langle\langle \boldsymbol{\Omega}, \cdot \rangle\rangle \in (\mathbb{R}^3)^*$ is identified with the ***angular momentum in the body frame*** $\boldsymbol{\Pi} := \mathbb{I}\boldsymbol{\Omega} \in \mathbb{R}^3$, so

$$K(\boldsymbol{\Pi}) = \frac{1}{2}\boldsymbol{\Pi} \cdot \mathbb{I}^{-1}\boldsymbol{\Pi} \quad \text{or} \quad K(\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega}$$

where $\mathbb{I} = \operatorname{diag}(I_1, I_2, I_3)$.

This is the expression of the kinetic energy in the body representation, either as a function of $\boldsymbol{\Omega}$ or $\boldsymbol{\Pi}$.

So the kinetic energy $K : T\,\mathrm{SO}(3) \to \mathbb{R}$ in material representation

$$K(A, \dot{A}) = \frac{1}{2}\left(\mathbb{I}A^{-1}\dot{A}\right) \cdot \left(A^{-1}\dot{A}\right)$$

is *left invariant* (action is $B \cdot (A, \dot{A}) := (BA, B\dot{A})$). It is the kinetic energy of the left invariant Riemannian metric on $SO(3)$ obtained by left translating the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Define the ***spatial moment of inertia tensor*** $\mathbb{I}_S(t) := A(t)\mathbb{I}A(t)^{-1}$. Since $\mathbf{\Omega} = A^{-1}\boldsymbol{\omega}$ it follows that the ***spatial angular momentum*** is $\boldsymbol{\pi} := \mathbb{I}_S\boldsymbol{\omega} = A\mathbf{\Pi}$, so

$$K(\boldsymbol{\omega}, \mathbb{I}_S) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}_S\boldsymbol{\omega}$$

This is the expression of the kinetic energy in the spatial representation.

Note that a major complication has arisen: the new dynamic variable $\mathbb{I}_S$ has been introduced.

Potential energy

The potential energy U is determined by the height of the center of mass over the horizontal plane in the spatial representation.

- ℓ length of segment from fixed point to center of mass
- $\boldsymbol{\chi}$ unit vector from origin on this segment
- $M = \int_{\mathcal{B}} \rho_0(\mathbf{X}) d^3\mathbf{X}$ total mass of the body
- g magnitude of gravitational acceleration
- $\boldsymbol{\Gamma}(t) := Mg\ell A(t)^{-1}\mathbf{e}_3$, spatial Oz unit vector viewed in body description
- $\boldsymbol{\lambda}(t) := Mg\ell A(t)\boldsymbol{\chi}$, unit vector on the line connecting the origin with the center of mass viewed in the spatial description

$$\begin{aligned} U &= Mg\ell\mathbf{e}_3 \cdot A(t)\boldsymbol{\chi} && \textit{material} \\ &= \mathbf{e}_3 \cdot \boldsymbol{\lambda} && \textit{spatial} \\ &= \boldsymbol{\Gamma} \cdot \boldsymbol{\chi} && \textit{body} \end{aligned}$$

New complications appear: **There are new variables, depending on the representation; λ in the spatial and Γ in the body representation**

AFFINE SEMIDIRECT PRODUCT LAGRANGIAN REDUCTION

Give the general theory on the *right* because it is useful for fluids. For heavy top, everything is on the *left*, so there are sign changes.

$\rho : G \to \mathrm{Aut}(V)$ *right* Lie group representation. Form $S = G \circledS V$

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + \rho_{g_2}(v_1)).$$

The Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$ of S has bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \left.\frac{d}{dt}\right|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

If $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ we have

$$\mathrm{ad}^*_{(\xi,v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are given by

$$a\xi := \frac{d}{dt}\Big|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v\rangle_V ,$$

$\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

$c \in \mathcal{F}(G, V^*)$ a right one-cocycle: $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$, $\forall f, g \in V^*$. So $c(e) = 0$ and $c(g^{-1}) = -\rho^*_g(c(g))$. Form the affine *right* representation

$$\theta_g(a) = \rho^*_{g^{-1}}(a) + c(g).$$

Note that

$$\frac{d}{dt}\Big|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi).$$

and

$$\langle a\xi + \mathbf{d}c(\xi), v\rangle_V = \langle \mathbf{d}c^{\mathsf{T}}(v) - v \diamond a, \xi \rangle_{\mathfrak{g}},$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ is defined by $\mathbf{d}c(\xi) := T_e c(\xi)$, and $\mathbf{d}c^{\mathsf{T}} : V \to \mathfrak{g}^*$ is defined by

$$\langle \mathbf{d}c^{\mathsf{T}}(v), \xi \rangle_{\mathfrak{g}} := \langle \mathbf{d}c(\xi), v \rangle_V .$$

- $L : TG \times V^* \to \mathbb{R}$ right G-invariant under the affine action $(v_h, a) \mapsto (T_hR_g(v_h), \theta_g(a)) = (T_hR_g(v_h), \rho^*_{g^{-1}}(a) + c(g))$.

- So, if $a_0 \in V^*$, define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of $G^c_{a_0}$ on G, where $G^c_{a_0} := \{g \in G \mid \theta_g(a_0) = a_0\}$.

- Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(T_gR_{g^{-1}}(v_g), \theta_{g^{-1}}(a_0)) = L(v_g, a_0).$$

- For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following affine differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c(\xi(t)),$$

with initial condition $a(0) = a_0$. The solution can be written as $a(t) = \theta_{g(t)^{-1}}(a_0)$.

i With a_0 held fixed, Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.

iii The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv The affine Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\,\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^{\mathsf{T}}\left(\frac{\delta l}{\delta a}\right).$$

Heavy top in body representation

QUESTION: The parameters are $Mg\ell\mathbf{e}_3, Mg\ell\boldsymbol{\chi} \in \mathbb{R}^3$, $\mathbb{I} \in \mathrm{Sym}_2$. This is the representation space $V^* = \mathbb{R}^3 \times \mathrm{Sym}_2 \times \mathbb{R}^3$. There is no cocycle, so $c = 0$ in theorem (no internal variables).

$$L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = \frac{1}{2}\left(\mathbb{I}A^{-1}\dot{A}\right) \cdot \left(A^{-1}\dot{A}\right) - Mg\ell\mathbf{e}_3 \cdot A\boldsymbol{\chi}$$

Left SO(3)-***representation***: $B \cdot (\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) := (B\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi})$. Note

$$L(BA, B\dot{A}, B\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi})$$

So, general theory says that we have Euler-Poincaré equations and associated variational principles for the **body Lagrangian**

$$L_B(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \mathbb{I}, \boldsymbol{\chi}) := L(I, A^{-1}\dot{A}, A^{-1}\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} - \boldsymbol{\Gamma} \cdot \boldsymbol{\chi}$$

Since $\frac{\delta L_B}{\delta \boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} = \boldsymbol{\Pi}$ and $\frac{\delta L_B}{\delta \boldsymbol{\Gamma}} = -\boldsymbol{\chi}$, we get the equations

$$\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega} + \boldsymbol{\Gamma} \times \boldsymbol{\chi}, \quad \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega}, \quad \dot{\mathbb{I}} = 0, \quad \dot{\boldsymbol{\chi}} = 0$$

Right SO(3)-***representation***: $(\mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi}) \cdot B := (\mathbf{e}_3, B^{-1}\mathbb{I}B, B^{-1}\boldsymbol{\chi})$.

$$L(AB, \dot{A}B, \mathbf{e}_3, B^{-1}\mathbb{I}B, B^{-1}\boldsymbol{\chi}) = L(A, \dot{A}, \mathbf{e}_3, \mathbb{I}, \boldsymbol{\chi})$$

So, general theory says that we have Euler-Poincaré equations and associated variational principles for the **spatial Lagrangian**

$$L_S(\boldsymbol{\omega}, \mathbf{e}_3, \mathbb{I}_S, \boldsymbol{\lambda}) := L(I, \dot{A}A^{-1}, \mathbf{e}_3, A\mathbb{I}A^{-1}, A\boldsymbol{\chi}) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}_S\boldsymbol{\omega} - \mathbf{e}_3 \cdot \boldsymbol{\lambda}$$

Since $\frac{\delta L_S}{\delta \boldsymbol{\omega}} = \mathbb{I}_S\boldsymbol{\omega} = \boldsymbol{\pi}$, $\frac{\delta L_S}{\delta \boldsymbol{\lambda}} = -\mathbf{e}_3$, $\frac{\delta L_S}{\delta \mathbb{I}_S} = \boldsymbol{\omega} \otimes \boldsymbol{\omega}$, we get

$$\dot{\boldsymbol{\pi}} = \mathbf{e}_3 \times \boldsymbol{\lambda}, \quad \dot{\mathbf{e}}_3 = 0, \quad \dot{\mathbb{I}}_S = [\mathbb{I}_S, \widehat{\omega}]\,, \quad \dot{\boldsymbol{\lambda}} = \boldsymbol{\omega} \times \boldsymbol{\lambda}$$

Remark: In body representation, we have equations on $\mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$. Four dimensional generic orbits; Casimirs are $\boldsymbol{\Pi} \cdot \boldsymbol{\Gamma}$, $\|\boldsymbol{\Gamma}\|^2$.

In spatial representation, equations are on the dual of the semidirect product $\mathfrak{so}(3) \circledS (\mathrm{Sym}^2 \times \mathbb{R}^3)$. This is 12 dimensional. It has 6 Casimirs: the three invariants of $\mathbb{I}_S$, $\|\boldsymbol{\lambda}\|^2$, $(\mathbb{I}_S\boldsymbol{\lambda}) \cdot \boldsymbol{\lambda}$, $\|\mathbb{I}_S\boldsymbol{\lambda}\|^2$. The coadjoint orbit is symplectomorphic to $(T^*\mathrm{SO}(3), can)$. One more integral: $\boldsymbol{\pi} \cdot \mathbf{e}_3$. Reduce and get to 4 dimensions $(TS^2, magnetic)$.

Special case: Free or Euler top

$\ell = 0$, so $L(A, \dot{A}) = K(A, \dot{A}) = \frac{1}{2}\left(\mathbb{I}A^{-1}\dot{A}\right)\cdot\left(A^{-1}\dot{A}\right)$, i.e., we study geodesic motion on SO(3) for the left invariant metric whose value at I is $\langle\langle \mathbf{a}, \mathbf{b}\rangle\rangle = \mathbb{I}\mathbf{a}\cdot\mathbf{b}$. The equations in body representation decouple:

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi}\times\mathbf{\Omega}, \quad \mathbf{\Gamma} = 0,$$

Geodesic equations on $\mathrm{SO}(3)\times\mathbb{R}^3 \cong T\,\mathrm{SO}(3)$ (left trivialized) are

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi}\times\mathbf{\Omega}, \quad \dot{A} = A\widehat{\mathbf{\Omega}}$$

The left action induces a momentum map $\mathbf{J}_L : T\,\mathrm{SO}(3) \to \mathbb{R}^3$ which is conserved. Recall $\mathbf{J}_L(\alpha_A) = T_I^*R_A(\alpha_A)$ which after the identifications becomes $\mathbf{J}_L(A, \mathbf{\Pi}) = A\mathbf{\Pi}$. Direct verification:

$$\dot{\boldsymbol{\pi}} = \dot{A}\mathbf{\Pi} + A\dot{\mathbf{\Pi}} = A\widehat{\mathbf{\Omega}}\mathbf{\Pi} + A(\mathbf{\Pi}\times\mathbf{\Omega}) = A(\mathbf{\Omega}\times\mathbf{\Pi} + \mathbf{\Pi}\times\mathbf{\Omega}) = 0$$

We shall see a similar phenomenon for fluids.

Momentum map of SU(2)-action on $\mathbb{C}^2$, the Cayley-Klein parameters, the Kustaanheimo-Stiefel coordinates, and the family of Hopf fibrations on concentric three-spheres in $\mathbb{C}^2$ are ***the same map****.*

FIXED BOUNDARY BAROTROPIC FLUIDS

Group = Diff($\mathcal{D}$), $V^* = |\Omega^n(\mathcal{D})| \times S_2(\mathcal{D})$, Riemannian metrics $G = g$ on $\mathcal{B} = \mathcal{S}$

$$L_{(\bar{\varrho},g)}(V_\eta) = \frac{1}{2}\int_{\mathcal{D}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X) - \int_{\mathcal{D}} E(\bar{\varrho}(X), g(\eta(X)), T_X\eta)\bar{\varrho}(X), \qquad \textit{material}$$

$$\ell_{spat}(\mathbf{v}, \bar{\rho}, g) = \frac{1}{2}\int_{\mathcal{D}} g(x)(\mathbf{v}(x), \mathbf{v}(x))\bar{\rho}(x) - \int_{\mathcal{D}} e(\rho(x))\bar{\rho}(x), \qquad \textit{spatial}$$

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{D}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{D}} \mathcal{E}(\bar{\varrho}, C)\bar{\varrho}, \qquad \textit{convective}$$

- $\bar{\varrho}(X) =: \varrho(X)\mu(g)(X) := (\eta^*\bar{\rho})(X), \qquad \bar{\rho}(x) := \rho(x)\mu(g)(x)$ mass density
- $C := \eta^* g$ Cauchy-Green tensor
- $E(\bar{\varrho}(X), g(\eta(X)), T_X\eta) := e\left(\frac{\bar{\varrho}(X)}{\mu(\eta^* g)(X)}\right), \quad \mathcal{E}(\bar{\varrho}(X), C(X)) := e\left(\frac{\bar{\varrho}(X)}{\mu(C)(X)}\right)$ internal energy density

L is *right*-invariant under the action of $\varphi \in \text{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto (V_\eta \circ \varphi, \varphi^* \bar{\varrho}, g)$$

and the reduction map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathbf{v}, \bar{\rho}, g) := (V_\eta \circ \eta^{-1}, \eta_* \bar{\varrho}, g)$$

induces the spatial Lagrangian $\ell_{spat}(\mathbf{v}, \rho, g)$ because

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E(\varphi^* \bar{\varrho}, g \circ \eta \circ \varphi, T\eta \circ T\varphi) = E(\bar{\varrho}, g \circ \eta, T\eta) \circ \varphi$$

when $(\eta, \bar{\varrho}) \mapsto (\eta \circ \varphi, \varphi^* \bar{\varrho})$. g is not acted on by $\text{Diff}(\mathcal{D})$.
L is *left*-invariant under the action of $\psi \in \text{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto (T\psi \circ V_\eta, \bar{\varrho}, \psi_* g).$$

and the reduction map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathcal{V}, \bar{\varrho}, C) := (T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g),$$

induces the convective Lagrangian $\ell_{conv}(\mathcal{V}, \bar{\varrho}, C)$ because

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E(\bar{\varrho}, \psi_* g \circ (\psi \circ \eta), T\psi \circ T\eta) = E(\bar{\varrho}, g \circ \eta, T\eta)$$

when $(\eta, g) \mapsto (\psi \circ \eta, \psi_* g)$. $\bar{\varrho}$ is not acted on by $\text{Diff}(\mathcal{D})$.

General semidirect product reduction gives spatial equations

$$\begin{cases} \partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\frac{1}{\rho} \operatorname{grad}_g p, \quad p := \rho^2 \frac{\partial e}{\partial \rho} \\ \partial_t \rho + \operatorname{div}_g(\rho \mathbf{v}) = 0, \quad \mathbf{v} \| \partial \mathcal{D}, \end{cases}$$

and convective equations

$$\begin{cases} \bar{\varrho}\left(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}\right) = 2 \operatorname{Div}_C \left(\frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} \right) \\ \partial_t C - \pounds_{\mathcal{V}} C = 0, \quad \mathcal{V} \| \partial \mathcal{B}, \end{cases}$$

right hand side is related to the spatial pressure p by the formula

$$2 \frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} = -(p \circ \eta) \mu(C) C^\sharp, \quad \text{so} \quad 2 \operatorname{Div}_C \left(\frac{\partial \mathcal{E}}{\partial C} \bar{\varrho} \right) = -\operatorname{grad}_C (p \circ \eta) \mu(C),$$

$C^\sharp \in S^2(\mathcal{D})$ is the cometric, grad_C is the gradient relative to C.

Special case: ideal homogeneous incompressible fluid. Group is $\operatorname{Diff}_\mu(g) := \{\eta \in \operatorname{Diff}(\mathcal{D}) \mid \eta^* \mu(g) = \mu(g)\}$, $V^* = S_2(\mathcal{D})$.
Lagrangian in spatial and convective rep. (suppose $H^1(\mathcal{D}, \mathbb{R}) = 0$):

$$\ell_{spat}(\mathbf{v}, g) = \frac{1}{2}\int_{\mathcal{D}} g(x)(\mathbf{v}(x), \mathbf{v}(x))\mu(g)(x)$$

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{D}} C(X)(\mathcal{V}(X), \mathcal{V}(X))\mu(g)(X)$$

In spatial representation: if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \mathfrak{X}_{div,\|}(\mathcal{D}) \Longrightarrow$

$$\partial_t \mathbf{v} + \nabla_{\mathbf{v}}\mathbf{v} = -\operatorname{grad} p \qquad \textit{Euler equations}$$

if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \mathbf{d}\Omega^1_{\delta,\|}(\mathcal{D}) := \{\mathbf{d}\mathbf{v}^{\flat_g} \mid \mathbf{v} \in \mathfrak{X}_{div,\|}(\mathcal{D})\} = \Omega^2_{ex}(\mathcal{D}) \Longrightarrow$

$$\partial_t \omega + \pounds_{\mathbf{v}}\omega = 0, \text{ where } \omega := \mathbf{d}\mathbf{v}^{\flat_g} \qquad \textit{vorticity advection}$$

In convective representation: if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \Omega^1_{\delta,\|}(\mathcal{D}) \Longrightarrow$

$$\partial_t \mathbb{P}\left(\mathcal{V}^{\flat_C}\right) = 0 \quad \text{and} \quad \partial_t C - \pounds_{\mathcal{V}} C = 0.$$

$\mathbb{P} : \Omega^1(\mathcal{D}) \to \Omega^1_{\delta,\|}(\mathcal{D})$ orthogonal Hodge projector for the metric g

if $\mathfrak{X}_{div,\|}(\mathcal{D})^* = \Omega^2_{ex}(\mathcal{D}) \Longrightarrow$

$$\partial_t \Omega = 0 \quad \text{and} \quad \partial_t C - \pounds_{\mathcal{V}} C = 0.$$

where $\Omega := \mathbf{d}\mathcal{V}^{\flat_C}$ is the convective vorticity.

ELASTICITY

Configuration space Emb$(\mathcal{B}, \mathcal{S})$. Material Lagrangian:

$$L(V_\eta, \bar{\varrho}, g, G) = \frac{1}{2}\int_{\mathcal{B}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X) - \int_{\mathcal{B}} W(g(\eta(X)), T_X\eta, G(X))\bar{\varrho}(X).$$

Material frame indifference: the *material stored energy function* W is invariant under the transformations

$$(\eta, g) \mapsto (\psi \circ \eta, \psi_* g), \quad \psi \in \mathrm{Diff}(\mathcal{S}), \qquad i.e.,$$

$$W\left(\psi_* g(\psi(\eta(X))), T_{\eta(X)}\psi \circ T_X\eta, G(X)\right) = W\left(g(\eta(X)), T_X\eta, G(X)\right).$$

$\forall \eta \in$ Emb$(\mathcal{B}, \mathcal{S})$, $\forall \psi : \eta(\mathcal{B}) \to \mathcal{B}$

So can define the *convective stored energy* $\mathcal{W}$ by

$$\mathcal{W}(C(X), G(X)) := W\left(\eta^* g(X), \mathbf{I}, G(X)\right) = W\left(g(\eta(X)), T_X\eta, G(X)\right).$$

$C := \eta^* g$ **Cauchy-Green tensor**

Material and convective tensors

Cauchy-Stress tensor $\boldsymbol{\sigma} \in S^2(\eta(\mathcal{B}))$ is given in terms of the stored energy function by the *Doyle-Ericksen formula*

$$\boldsymbol{\sigma} = 2\rho \frac{\partial W}{\partial g},$$

where $\bar{\rho} = \rho\mu(g)$. Obtained by the axioms for constitutive theory.

Pull back $\boldsymbol{\sigma}$ to $\mathcal{B}$; get we **convected stress tensor** $\boldsymbol{\Sigma} := \eta^* \boldsymbol{\sigma} \in S^2(\mathcal{B})$, related to the convective stored energy function by

$$\boldsymbol{\Sigma} = 2\mathcal{R} \frac{\partial \mathcal{W}}{\partial C},$$

where $\mathcal{R}$ is the *convected mass density* defined by the equality $\bar{\varrho} = \mathcal{R}\mu(C)$, i.e., $\mathcal{R} \circ \eta = \rho$.

First Piola-Kirchhoff tensor is the two-point tensor over η

$$\mathbf{P}(\alpha_X, \beta_x) := J_\eta(X)\boldsymbol{\sigma}(\eta(X))(T^*\eta^{-1}(\alpha_X), \beta_x), \quad x = \eta(X),$$

where $\alpha_X \in T^*\mathcal{B}$, $\beta_x \in T^*\mathcal{S}$, J_η is the Jacobian of η relative to the metrics g and G, i.e., $\eta^*\mu(g) = J_\eta \mu(G)$.

$$\begin{aligned}\mathbf{P}(\alpha_X, \beta_x)\mu(G) &= \boldsymbol{\sigma}(\eta(X))\left(T^*\eta^{-1}(\alpha_X), \beta_x\right)\mu(C) \\ &= \mathbf{\Sigma}(X)\left(\alpha_X, T^*\eta(\beta_x)\right)\mu(C)\end{aligned}$$

and the Doyle-Ericksen relation reads

$$\mathbf{P} = 2\varrho\left(\frac{\partial W}{\partial T\eta}\right)^{\sharp_g},$$

$\sharp_g$ is index raising operator associated to the Riemannian metric g.

Boundary conditions

Pure displacement boundary conditions: $\eta \in \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ is prescribed on the boundary $\partial\mathcal{B}$:

$$\eta|_{\partial\mathcal{B}} = \tilde{\eta} \quad \text{given.}$$

Traction boundary conditions:, traction $\mathbf{P}\cdot\mathbf{N}_C$ given on $\partial\mathcal{B}$:

$$\mathbf{P}\cdot\mathbf{N}_C|_{\partial\mathcal{B}} = \tilde{\boldsymbol{\tau}} \quad \text{given,}$$

$\mathbf{N}_C$ normal to $\partial\mathcal{B}$ relative to the Cauchy-Green tensor C.

Below we will treat only the case $\tilde{\boldsymbol{\tau}} = 0$; the case $\tilde{\boldsymbol{\tau}} \neq 0$ requires the addition of another term in the Lagrangian.

Can consider *mixed boundary conditions* by imposing the first condition on $\partial_d \mathcal{B}$ and the second condition on $\partial_{\boldsymbol{\tau}} \mathcal{B}$, where

$$\overline{\partial_d \mathcal{B} \cup \partial_{\boldsymbol{\tau}} \mathcal{B}} = \partial \mathcal{B}, \quad \partial_d \mathcal{B} \cap \partial_{\boldsymbol{\tau}} \mathcal{B} = \varnothing.$$

We call $\eta|_{\partial_d \mathcal{B}} = \tilde{\eta}$ the *essential boundary condition* and build it directly into the configuration space, defined to be

$$\mathcal{C} := \{\eta \in \mathrm{Emb}(\mathcal{B}, \mathcal{S}) \mid \eta|_{\partial_d \mathcal{B}} = \tilde{\eta}\}.$$

Convective representation

Euler-Poincaré theory does not apply; do by hand with EP as guide.

Convective quantities: $C := \eta^* g$ Cauchy-Green tensor,

$(\mathcal{V}, \bar{\varrho}, C, G) := \left(T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g, G\right) \in \mathfrak{X}(\mathcal{B}) \times |\Omega^n(\mathcal{B})| \times S_2(\mathcal{B}) \times S_2(\mathcal{B})$,

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C, G) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{B}} \mathcal{W}(C, G)\bar{\varrho}.$$

Compute variations

$$\delta\mathcal{V} = \dot{\zeta} - [\mathcal{V}, \zeta], \qquad \delta C = \pounds_\zeta C,$$

ζ curve of vector fields vanishing at the endpoints.

Reduced variational principle:

$$\delta \int_{t_0}^{t_1} \ell_{conv}(\mathcal{V}, \bar{\varrho}, C, G)dt = 0,$$

for these constrained variations yields

Convective equations of motion:

$$\bar{\varrho}\,(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = 2\,\mathrm{Div}_C\left(\frac{\partial \mathcal{W}}{\partial C}\bar{\varrho}\right), \qquad \partial_t C - \pounds_{\mathcal{V}} C = 0,$$

∇ is the Levi-Civita connection of the Riemannian metric C.

Boundary conditions:

$$\mathcal{V}|_{\partial_d \mathcal{B}} = 0, \quad \boldsymbol{\Sigma} \cdot \mathbf{N}_C^{\flat_C}|_{\partial_{\boldsymbol{\tau}} \mathcal{B}} = 0.$$

The first equation can equivalently be written as

$$\mathcal{R}\,(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = \mathrm{Div}_C(\boldsymbol{\Sigma}),$$

where $\boldsymbol{\Sigma}$ is the convected stress tensor.

So, elasticity has always a convective representation.

What is the spatial representation?

Spatial representation

Need invariance under the *right* action of Diff($\mathcal{B}$):

$$(V_\eta, \varrho, g, G) \mapsto (V_\eta \circ \varphi, \varphi^* \varrho, g, \varphi^* G), \quad \varphi \in \text{Diff}(\mathcal{B})$$

Kinetic energy is right-invariant. So sufficient condition is

$$W\Big(g(\eta(\varphi(X))), T_X(\eta\circ\varphi), \varphi^* G(X)\Big) = \Big(W(g(\eta(\_)), T_\_\eta, G(\_))\circ\varphi\Big)(X),$$

for all $\varphi \in \text{Diff}(\mathcal{B})$. This is equivalent to

$$\mathcal{W}(\varphi^* C, \varphi^* G) = \mathcal{W}(C, G) \circ \varphi, \qquad \forall \varphi \in \text{Diff}(\mathcal{B})$$

This is *material covariance* which is equivalent to isotropy.

Isotropy

Material symmetry for W at $X_0 \in \mathcal{B}$:

$$\mathcal{W}(\boldsymbol{\lambda}^* C(X_0), G(X_0)) = \mathcal{W}(C(X_0), G(X_0))$$

for all isomoetries $\boldsymbol{\lambda} : T_{X_0}\mathcal{B} \to T_{X_0}\mathcal{B}$ relative to $G(X_0)$.

Material symmetry group $\mathfrak{S}_{X_0}$ at X_0: all such isometries

Isotropy at X_0: $\mathfrak{S}_{X_0} \supseteq \mathrm{SO}(T_{X_0}\mathcal{B} = \mathrm{SO}(3)$

Isotropy means isotropy at every $X_0 \in \mathcal{B}$.

Material covariance is equivalent to isotropy.

So, assume from now on that the material is isotropic. ♦

Spatial quantities: $c := \eta_* G \in \mathcal{S}_2(D_\Sigma)$ Finger deformation tensor

$$\mathbf{u} := \dot{\eta} \circ \eta^{-1} \in \mathfrak{X}(D_\Sigma), \quad \bar{\rho} := \eta_* \bar{\varrho} \in |\Omega^n(D_\Sigma)|,$$

$\Sigma = \eta(\partial \mathcal{B})$ boundary of *current configuration* $D_\Sigma := \eta(\mathcal{B}) \subset \mathcal{S}$,

$$w_\Sigma(c, g) := \mathcal{W}(\eta^* g, \eta^* c) \circ \eta^{-1}$$

spatial stored energy function. w_Σ, $\mathcal{W}$, and W are related by

$$(w_\Sigma(c, g) \circ \eta)(X) = \mathcal{W}(\eta^* g(X), \eta^* c(X)) = W\left(g(\eta(X)), T_X \eta, \eta^* c(X)\right).$$

Doyle-Ericksen formula for the Cauchy stress tensor can be written:

$$\boldsymbol{\sigma} = 2\rho \frac{\partial w_\Sigma}{\partial g} \in S^2(D_\Sigma)$$

Reduced spatial Lagrangian

$$\ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) = \frac{1}{2} \int_{D_\Sigma} g(\mathbf{v}, \mathbf{v}) \bar{\rho} - \int_{D_\Sigma} w_\Sigma\,(c, g)\, \bar{\rho},$$

variables defined on current configuration D_Σ and Σ is a variable.

Compute the variations of all variables:

$$\delta \mathbf{v} = \dot{\xi} + [\mathbf{v}, \xi] \in \mathfrak{X}(D_\Sigma)$$

ξ is an arbitrary curve in $\mathfrak{X}(D_\Sigma)$ vanishing at the endpoints. Since the only part of ξ contributing to the motion of the boundary Σ is its normal part, we define for any $x \in \Sigma$,

$$\delta\Sigma(x) := g(x)\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \eta_\varepsilon(\eta^{-1}(x)), \mathbf{n}_g(x) \right),$$

$\mathbf{n}_g$ is the outward-pointing unit normal vector field relative to g. Therefore,

$$\delta\Sigma = g(\xi, \mathbf{n}).$$

The other variations are

$$\delta\bar{\rho} = -\boldsymbol{\pounds}_\xi \bar{\rho}, \qquad \delta c = -\boldsymbol{\pounds}_\xi c.$$

Reduced spatial variational principle

$$\delta \int_{t_0}^{t_1} \ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g, c)dt = 0,$$

relative to the above constrained variations.

Spatial equations of motion:

$$
\begin{aligned}
\rho\left(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}\right) &= \mathsf{Div}_g\left(\boldsymbol{\sigma}\right), \\
\partial_t c + \boldsymbol{\pounds}_{\mathbf{v}} c &= 0, \\
\partial_t \bar{\rho} + \boldsymbol{\pounds}_{\mathbf{v}} \bar{\rho} &= 0, \\
\partial_t \Sigma &= g(\mathbf{v}, \mathbf{n}_g)
\end{aligned}
$$

with boundary conditions

$$
\mathbf{v}|_{\Sigma_d} = 0, \quad \boldsymbol{\sigma} \cdot \mathbf{n}_g|_{T\Sigma_\tau} = 0
$$

FREE BOUNDARY FLUIDS

Configuration space $\mathrm{Emb}(\mathcal{B}, \mathcal{S})$. Material Lagrangian:

$$L_{(\bar{\varrho},g)}(V_\eta) = \frac{1}{2}\int_{\mathcal{B}} g(V_\eta, V_\eta)\bar{\varrho} - \int_{\mathcal{B}} E(\bar{\varrho}(X), g(\eta(X)), T_X\eta)\bar{\varrho} - \tau\int_{\partial\mathcal{B}} \gamma(\eta^* g),$$

E internal energy density related to the spatial energy e as before, τ a constant. Third term proportional to area of current configuration and represents the potential energy associated with surface tension; $\gamma(\eta^* g)$ boundary volume form of Riemannian volume form for $\eta^* g$.

Convective representation: L left $\mathrm{Diff}(\mathcal{S})$-invariant, so produces

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{B}} \mathcal{E}(\bar{\varrho}, C)\bar{\varrho} - \tau\int_{\partial\mathcal{B}} \gamma(C).$$

Convective equations of motion

$$\begin{cases} \bar{\varrho}\left(\partial_t\mathcal{V} + \nabla_{\mathcal{V}}\mathcal{V}\right) = 2\,\mathrm{Div}_C\left(\dfrac{\partial\mathcal{E}}{\partial C}\bar{\varrho}\right) \\ \partial_t C - \pounds_{\mathcal{V}} C = 0, \end{cases}$$

(BC) $p \circ \eta|_{\partial\mathcal{B}} = \tau\kappa_C$, κ_C mean curvature of $\partial\mathcal{B}$ relative to C and p is the spatial pressure. In terms of the pressure p, the right hand side of the motion equation reads $-\,\mathrm{grad}_C(p \circ \eta)\mu(C)$.

Spatial representation: L right Diff($\mathcal{B}$)-invariant: $(V_\eta, \bar{\varrho}, g) \mapsto (V_\eta \circ \varphi, \varphi^* \bar{\varrho}, g)$, $\forall \varphi \in$ Diff($\mathcal{B}$). This leads to the spatial Lagrangian:

$$\ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g) = \frac{1}{2} \int_{D_\Sigma} g(\mathbf{v}, \mathbf{v}) \bar{\rho} - \int_{D_\Sigma} e(\rho) \bar{\rho} - \tau \int_\Sigma \gamma(g)$$

and the spatial equations of motion

$$\begin{cases} \rho \left(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} \right) = - \operatorname{grad}_g p \\ \partial_t \bar{\rho} + \pounds_{\mathbf{v}} \bar{\rho} = 0 \end{cases} \quad \text{on} \quad \Sigma$$

with the boundary condition and boundary movement

$$p|_\Sigma = \tau \kappa_g, \quad \partial_t \Sigma = g(\mathbf{v}, \mathbf{n}_g).$$

COMPARISON BETWEEN THE POTENTIAL ENERGIES

FLUID: $$\int_{\mathcal{D}} E(\bar{\varrho}(X), g(\eta(X)), T_X\eta)\bar{\varrho}(X)$$

$\bar{\varrho}(X) =: \varrho(X)\mu(g)(X) := (\eta^*\bar{\rho})(X)$, $\bar{\rho}(x) := \rho(x)\mu(g)(x)$ mass density

$$E(\bar{\varrho}(X), g(\eta(X)), T_X\eta) := e\left(\frac{\bar{\varrho}(X)}{\mu(\eta^*g)(X)}\right) = e(\rho)(x),$$

$$\mathcal{E}(\bar{\varrho}(X), C(X)) := e\left(\frac{\bar{\varrho}(X)}{\mu(C)(X)}\right) = e(\rho)(x)$$

internal energy density, $C := \eta^*g$ Cauchy-Green tensor

SOLID: $$\int_{\mathcal{B}} W(g(\eta(X)), T_X\eta, G(X))\bar{\varrho}(X)$$

$$\mathcal{W}(C(X), G(X)) := W\left(\eta^*g(X), \mathbf{I}, G(X)\right) = W\left(g(\eta(X)), T_X\eta, G(X)\right).$$

convective stored energy $\mathcal{W}$. Material covariance

$$\mathcal{W}(\varphi^*C, \varphi^*G) = \mathcal{W}(C, G) \circ \varphi, \qquad \forall \varphi \in \mathrm{Diff}(\mathcal{B})$$

is equivalent to isotropy.

NOTE: Potential energy for elastica does not recover potential energy for fluids.

GENERAL CONTINUA

$$L(V_\eta, \bar{\varrho}, g, G) = \frac{1}{2}\int_{\mathcal{B}} g(\eta(X))(V_\eta(X), V_\eta(X))\bar{\varrho}(X) \\ - \int_{\mathcal{B}} U(g(\eta(X)), T_X\eta, G(X), \bar{\varrho}(X)) - \tau \int_{\partial\mathcal{B}} \gamma(\eta^* g),$$

U density on $\mathcal{B}$. This form is more general than the Lagrangian for free boundary fluids and for elastic materials considered before.

Material frame indifference: for all $\eta \in \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ and all diffeomorphisms $\psi : \eta(\mathcal{B}) \to \mathcal{B}$ we have

$$U\left(\psi_* g(\psi(\eta(X))), T_{\eta(X)}\psi \circ T_X\eta, G(X), \bar{\varrho}(X)\right) \\ = U\left(g(\eta(X)), T_X\eta, G(X), \bar{\varrho}(X)\right).$$

Therefore, we can define the **convective energy density** $\mathcal{U}$ by

$$\mathcal{U}(C(X), G(X), \bar{\varrho}(X)) := U\left(\eta^* g(X), \mathbf{I}, G(X), \bar{\varrho}(X)\right) \\ = U\left(g(\eta(X)), T_X\eta, G(X), \bar{\varrho}(X)\right).$$

Convective representation

Covariance assumption on $U \Longrightarrow$ material Lagrangian depends on the Lagrangian variables only through the convective quantities

$(\mathcal{V}, \bar{\varrho}, C, G) := \left(T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g, G\right) \in \mathfrak{X}(\mathcal{B}) \times |\Omega^n(\mathcal{B})| \times S_2(\mathcal{B}) \times S_2(\mathcal{B})$.

In terms of these variables, the Lagrangian is

$$\ell_{conv}(\mathcal{V}, \bar{\varrho}, C, G) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{B}} \mathcal{U}(C, G, \bar{\varrho}) - \tau \int_{\partial\mathcal{B}} \gamma(C).$$

The convective equations of motion are derived via reduced variational principle:

$$\begin{cases} \bar{\varrho}\,(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = 2\,\mathrm{Div}_C \left(\dfrac{\partial \mathcal{U}}{\partial C}\right) \\ \partial_t C - \pounds_{\mathcal{V}} C = 0, \end{cases}$$

with boundary conditions

$$\mathcal{V}|_{\partial_d \mathcal{B}} = 0, \quad \left(2\frac{\partial \mathcal{U}}{\partial C} \cdot \mathrm{N}_C^{\flat_C} + \tau \kappa_C \gamma(C) \mathrm{N}_C^{\flat_C}\right)\Big|_{\partial_\tau \mathcal{B}} = 0,$$

∇ Levi-Civita covariant derivative, N_C normal to $\partial\mathcal{B}$, both relative to the Cauchy-Green tensor C.

Spatial representation

Conditions under which the material Lagrangian L is invariant under the *right* action of Diff($\mathcal{B}$) given by

$$(V_\eta, \bar{\varrho}, g, G) \mapsto (V_\eta \circ \varphi, \varphi^* \bar{\varrho}, g, \varphi^* G), \qquad \forall \varphi \in \mathrm{Diff}(\mathcal{B})$$

The kinetic energy and the boundary term in the potential energy are right-invariant. For right invariance of the first summand in the potential energy it suffices that

$$U\Big(g(\eta(\varphi(X))), T_X(\eta \circ \varphi), \varphi^* G(X), \varphi^* \bar{\varrho}(X)\Big)$$
$$= \varphi^* \Big(U(g(\eta(\_)), T_\_\eta, G(\_), \bar{\varrho}(\_))\Big)(X), \quad \forall \varphi \in \mathrm{Diff}(\mathcal{B}).$$

This is equivalent to

$$\mathcal{U}(\varphi^* C, \varphi^* G, \varphi^* \bar{\varrho}) = \varphi^* \mathcal{U}(C, G, \bar{\varrho}), \quad \forall \varphi \in \mathrm{Diff}(\mathcal{B})$$

by the definition of $\mathcal{U}$.

U satisfying this identity is **materially covariant**.

Isotropy

This definition is more general than the standard one given before and recovers it if $\mathcal{U}(C, G, \bar{\varrho}) = \mathcal{W}(C, G)\bar{\varrho}$ for some function $\mathcal{W}$. We extend the notions of isotropy and material covariance to continua given by the material Lagrangian above. We shall see that this will include both free boundary fluids and classical nonlinear elasticity.

Define **material symmetry** at $X \in \mathcal{B}$ to be a linear isometry $\boldsymbol{\lambda} : T_X\mathcal{B} \to T_X\mathcal{B}$ relative to $G(X)$ such that

$$\boldsymbol{\lambda}^* \left[\mathcal{U}(C(X), G(X), \bar{\varrho}(X))\right] = \mathcal{U}((\boldsymbol{\lambda}^* C)(X), G(X), (\boldsymbol{\lambda}^* \bar{\varrho})(X)).$$

The material is called **isotropic** if, for each $X \in \mathcal{B}$, all proper rotations in $T_X\mathcal{B}$ are material symmetries at X. As before, one proves: *isotropy is equivalent to material covariance*.

From now on, we shall assume material covariance. ♦

The material Lagrangian L reduces to the spatial Lagrangian

$$\ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) = \frac{1}{2}\int_{D_\Sigma} g(\mathbf{v}, \mathbf{v})\bar{\rho} - \int_{D_\Sigma} u_\Sigma\,(c, g, \bar{\rho}) - \tau \int_\Sigma \gamma(g),$$

where

$$u_\Sigma(c, g, \bar{\rho}) := \eta_*\Big(\mathcal{U}(\eta^* g, \eta^* c, \eta^* \bar{\rho})\Big),$$

η is a parametrization of D_Σ, that is, $\eta \in \mathcal{C} = \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ such that $\eta(\mathcal{B}) = D_\Sigma$, $\Sigma = \eta(\partial\mathcal{B})$, and the spatial variables are

$$\mathbf{v} := V_\eta \circ \eta^{-1} \in \mathfrak{X}(D_\Sigma), \quad \bar{\rho} := \eta_* \bar{\varrho} \in \mathcal{F}(D_\Sigma)^*, \quad c := \eta_* G \in \mathcal{S}_2(D_\Sigma).$$

By Lagrangian reduction on $T\mathcal{C}$, the spatial equations of motion are given by the stationarity condition

$$\delta \int_{t_0}^{t_1} \ell_{spat}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) dt = 0,$$

relative to the constrained variations

$$\delta\mathbf{v} = \dot{\xi} + [\mathbf{v}, \xi] \in \mathfrak{X}(D_\Sigma), \quad \delta\Sigma = g(\xi, \mathbf{n}), \quad \delta\bar{\rho} = -\pounds_\xi \bar{\rho}, \quad \delta c = -\pounds_\xi c$$

ξ is an arbitrary curve in $\mathfrak{X}(D_\Sigma)$ vanishing at the endpoints.

The reduced Euler-Lagrange equations associated to ℓ_{spat} produce the equations for nonlinear elasticity in spatial formulation

$$\begin{cases} \bar{\rho}\left(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}\right) = 2\, \mathrm{Div}_g \left(\dfrac{\partial u_\Sigma}{\partial g}\right) \\ \partial_t c + \boldsymbol{\pounds}_{\mathbf{v}} c = 0, \qquad \partial_t \bar{\rho} + \boldsymbol{\pounds}_{\mathbf{v}} \bar{\rho} = 0 \\ \partial_t \Sigma = g(\mathbf{v}, \mathbf{n}_g) \end{cases}$$

with boundary conditions

$$\mathbf{v}|_{\Sigma_d} = 0, \quad \left.\left(\tau \kappa_g \left(\mathbf{n}_g\right)^{\flat_g} + 2\mathbf{i}_{\mathbf{n}_g} \left(\frac{\partial u_\Sigma}{\partial g}\right)^{\flat_g}\right)\right|_{T\Sigma_\tau} = 0.$$

The second boundary condition says that the total sum of forces exerted on the free boundary Σ_τ is zero: it is the sum of the surface tension force and of the internal traction force.

Kelvin-Noether Theorems

$\mathbf{v}$ solution of the system above, γ loop in $\mathcal{B}$ and $\gamma_t := \gamma \circ \eta_t$. Then

$$\frac{d}{dt}\oint_{\gamma_t} \mathbf{v}^{\flat_g} = 2\oint_{\gamma_t}\left(\frac{1}{\bar{\rho}}\operatorname{Div}_g\left(\frac{\partial u_\Sigma}{\partial g}\right)\right)^{\flat_g}.$$

Solution $\mathcal{V}$ of the convective equations, loop γ in $\mathcal{B}$. Then

$$\frac{d}{dt}\oint_{\gamma} \mathcal{V}^{\flat_C} = 2\oint_{\gamma}\frac{1}{\bar{\varrho}}\operatorname{Div}_C\left(\frac{\partial \mathcal{U}}{\partial C}\right)^{\flat_C}.$$

Classical elasticity: $U(g, T\eta, G, \bar{\varrho}) = W(g, T\eta, G)\bar{\varrho}$. Kelvin-Noether:

$$\frac{d}{dt}\oint_{\gamma_t} \mathbf{v}^{\flat_g} = \oint_{\gamma_t}\frac{1}{\rho}\operatorname{Div}_g(\boldsymbol{\sigma})^{\flat_g}.$$

Fluid dynamics: $U(g, T\eta, G, \bar{\varrho}) = E(g, T\eta, \bar{\varrho})\bar{\varrho} = e\left(\frac{\bar{\varrho}}{\mu(\eta^* g)}\right)\bar{\varrho}$ Fluids are isotropic materials and hence they have a spatial representation.

Kelvin-Noether: $\quad \frac{d}{dt}\oint_{\gamma} \mathcal{V}^{\flat_C} = 0, \qquad \frac{d}{dt}\oint_{\gamma_t} \mathbf{v}^{\flat_g} = 0.$

ERINGEN'S EQUATIONS NON-DISSIPATIVE MICROPOLAR LIQUID CRYSTALS

$$\begin{cases} \rho\frac{D}{Dt}\mathbf{u}_l = \partial_l\frac{\partial\Psi}{\partial\rho^{-1}} - \partial_k\left(\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^a}\gamma_l^a\right), \quad \rho\sigma_l = \partial_k\left(\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^l}\right) - \varepsilon_{lmn}\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_m^a}\gamma_n^a, \\ \frac{D}{Dt}\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \qquad \frac{D}{Dt}j_{kl} + (\varepsilon_{kpr}j_{lp} + \varepsilon_{lpr}j_{kp})\boldsymbol{\nu}_r = 0, \\ \frac{D}{Dt}\gamma_l^a = \partial_l\boldsymbol{\nu}_a + \nu_{ab}\gamma_l^b - \boldsymbol{\gamma}_r^a\partial_l\mathbf{u}_r, \qquad \frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \quad \text{mat. deriv.} \end{cases}$$

Repeated indices sum. $\mathbf{u} \in \mathfrak{X}(\mathcal{D})$ Eulerian velocity, $\rho \in \mathcal{F}(\mathcal{D})$ mass density, $\boldsymbol{\nu} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$, microrotation rate, where we use the standard isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, $j_{kl} \in \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3))$ microinertia tensor (symmetric), σ_k spin inertia is defined by

$$\sigma_k := j_{kl}\frac{D}{dt}\boldsymbol{\nu}_l + \varepsilon_{klm}j_{mn}\boldsymbol{\nu}_l\boldsymbol{\nu}_n = \frac{D}{dt}(j_{kl}\boldsymbol{\nu}_l),$$

and $\gamma = (\gamma_i^{ab}) \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ wryness tensor. This variable is denoted by $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_i^a)$ when it is seen as a form with values in $\mathbb{R}^3$.

$\Psi = \Psi(\rho^{-1}, j, \gamma) : \mathbb{R} \times \mathrm{Sym}(3) \times \mathfrak{gl}(3) \to \mathbb{R}$ is the free energy.
The "axiom of objectivity" requires that

$$\Psi(\rho^{-1}, A^{-1}jA, A^{-1}\gamma A) = \Psi(\rho^{-1}, j, \gamma),$$

for all $A \in \mathrm{O}(3)$ (for nematics and nonchiral smectics), or for all $A \in \mathrm{SO}(3)$ (for cholesterics and chiral smectics).

These equations are affine Euler-Poincaré for the group

$$\left[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))\right] \circledS \left[\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \mathfrak{X}(\mathcal{D}, \mathfrak{so}(3))\right].$$

Apply general affine semidirect product EP reduction theorem.

The general theory applied to many other complex fluids: spin systems, Yang-Mills MHD (classical and superfluid), Hall MHD, multivelocity superfluids (classical and superfluid), HBVK dynamics for superfluid ^{4}He, Volovik-Dotsenko spin glasses, microfluids, Ericksen-Leslie equations, Lhuillier-Rey equations.

Cemal Eringen: February 15, 1921 - December 7, 2009. His 1999 two volume treatise on micropolar continuum mechanics is basic.

EXPLANATION:

- Diff($\mathcal{D}$) acts on $\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ via the *right* action

$$(\chi, \eta) \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3)) \times \mathrm{Diff}(\mathcal{D}) \longmapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3)).$$

Therefore, the group multiplication in $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ is

$$(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi).$$

- The bracket of $\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is

$$\mathrm{ad}_{(\mathbf{u},\nu)}(\mathbf{v}, \zeta) = (\mathrm{ad}_{\mathbf{u}} \mathbf{v}, \mathrm{ad}_\nu \zeta + \mathbf{d}\nu \cdot \mathbf{v} - \mathbf{d}\zeta \cdot \mathbf{u}),$$

where $\mathrm{ad}_{\mathbf{u}} \mathbf{v} = -[\mathbf{u}, \mathbf{v}]$, $\mathrm{ad}_\nu \zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is given by $\mathrm{ad}_\nu \zeta(x) := \mathrm{ad}_{\nu(x)} \zeta(x)$, and $\mathbf{d}\nu \cdot \mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is given by $\mathbf{d}\nu \cdot \mathbf{v}(x) := \mathbf{d}\nu(x)(\mathbf{v}(x))$.

- $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ acts on $\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \mathfrak{X}(\mathcal{D}, \mathfrak{so}(3))$ on the right. Need explicit action on the dual

$$\Big[\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \mathfrak{X}(\mathcal{D}, \mathfrak{so}(3))\Big]^* \cong \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{so}(3))$$

via L^2 inner product. The action is given in the following way:

$\lozenge$ $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3))$ acts *linearly* and on the *right* on the advected quantities $(\rho, j) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3))$, by

$$(\rho, j) \mapsto \left(J(\eta)(\rho \circ \eta), \chi^{\mathsf{T}}(j \circ \eta)\chi\right), \quad \chi^{\mathsf{T}} = \chi^{-1}.$$

$\lozenge$ $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3))$ acts on $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ by

$$\gamma \mapsto \chi^{-1}(\eta^* \gamma)\chi + \chi^{-1} T\chi.$$

This is a *right affine* action. Note that γ transforms as a connection.

- The reduced Lagrangian

$$l : \left[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathbb{R}^3)\right] \circledS \left[\mathcal{F}(\mathcal{D}) \oplus \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \oplus \Omega^1(\mathcal{D}, \mathfrak{so}(3))\right] \to \mathbb{R}$$

$$l(\mathbf{u}, \boldsymbol{\nu}, \rho, j, \gamma) = \frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \rho \left(j\boldsymbol{\nu} \cdot \boldsymbol{\nu}\right) \mu - \int_{\mathcal{D}} \rho \Psi(\rho^{-1}, j, \gamma)\mu.$$

Define the covariant γ-derivative by $\mathbf{d}^{\gamma}\nu(\mathbf{v}) := \mathbf{d}\nu(\mathbf{v}) + [\gamma(\mathbf{v}), \nu]$.

The affine Euler-Poincaré equations for l are:

$$\begin{cases} \rho\left(\frac{\partial}{\partial t}\mathbf{u}+\nabla_{\mathbf{u}}\mathbf{u}\right)=\operatorname{grad}\frac{\partial\Psi}{\partial\rho^{-1}}-\partial_k\left(\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}^a\right), \\ j\frac{D}{Dt}\boldsymbol{\nu}-(j\boldsymbol{\nu})\times\boldsymbol{\nu}=-\frac{1}{\rho}\operatorname{div}\left(\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}}\right)+\boldsymbol{\gamma}^a\times\frac{\partial\Psi}{\partial\boldsymbol{\gamma}^a}, \\ \frac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0, \qquad \frac{D}{Dt}j+[j,\nu]=0, \\ \frac{\partial}{\partial t}\gamma+\boldsymbol{\pounds}_{\mathbf{u}}\gamma+\mathbf{d}^{\gamma}\nu=0, \quad \hat{\boldsymbol{\nu}}=\nu\in\mathcal{F}(\mathcal{D},\mathfrak{so}(3)), \end{cases}$$

which are Eringen's equations after the change of variables $\gamma\mapsto-\gamma$.
$L_{(\rho_0,j_0,\gamma_0)}: T\left[\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathrm{SO}(3))\right]\to\mathbb{R}$ induced by the Lagrangian l by right translation and freezing the parameters.

A curve $(\eta,\chi)\in\operatorname{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathrm{SO}(3))$ is a solution of the Euler-Lagrange equations associated to $L_{(\rho_0,j_0,\gamma_0)}$ if and only if the curve

$$(\mathbf{u},\boldsymbol{\nu}):=(\dot\eta\circ\eta^{-1},\dot\chi\chi^{-1}\circ\eta^{-1})\in\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathrm{SO}(3))$$

is a solution of the previous equations.

The evolutions of the mass density ρ, the microinertia j, and the wryness tensor γ are given by

$$\rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}), \quad j = \left(\chi j_0 \chi^{-1}\right) \circ \eta^{-1}, \quad \gamma = \eta_* \left(\chi \gamma_0 \chi^{-1} + \chi T \chi^{-1}\right).$$

If $\gamma_0 = 0$, then the evolution of γ is given by $\gamma = \eta_* \left(\chi T \chi^{-1}\right)$. This is usually taken as a definition of γ; last Eringen equation missing.

Kelvin-Noether circulation theorem for micropolar liquid crystals

$$\frac{d}{dt} \oint_{c_t} \mathbf{u}^\flat = \oint_{c_t} \frac{\partial \Psi}{\partial j} \mathbf{d} j + \frac{\partial \Psi}{\partial \gamma} \mathbf{i}\_ \mathbf{d} \gamma - \frac{1}{\rho} \operatorname{div} \left(\rho \frac{\partial \Psi}{\partial \gamma} \right) \gamma.$$

The γ-circulation formulated in $\mathbb{R}^3$

$$\frac{d}{dt} \oint_{c_t} \boldsymbol{\gamma} = \oint_{c_t} \boldsymbol{\nu} \times \boldsymbol{\gamma}$$

A priori Lyapunov stability estimates for stationary solutions? Casimirs?

FIRST PROBLEM: Eringen defines a smectic liquid crystal by $\mathrm{Tr}(\gamma) = \gamma_1^1 + \gamma_2^2 + \gamma_3^3 = 0$. This is *not preserved by the evolution* $\gamma = \eta_* \left(\chi \gamma_0 \chi^{-1} + \chi T \chi^{-1} \right)$.

Consistent with the statement: the equation

$$\frac{\partial \gamma}{\partial t} + \pounds_{\mathbf{u}} \gamma + \mathbf{d} \boldsymbol{\nu} + \gamma \times \boldsymbol{\nu} = 0$$

does not imply that if $\mathrm{Tr}(\gamma_0) = 0$ then $\mathrm{Tr}(\gamma) = 0$ for all time.
Is Eringen's definition of smectic incorrect? Instead of the trace need an $\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$-invariant function (of γ) under the action

$$\mathbf{v} \mapsto \chi^{-1} \mathbf{v} + \chi^{-1} T \chi, \quad \mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3), \quad \chi \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3)).$$

I do not know how to choose a physically reasonable function of this type.

SECOND PROBLEM: How does Eringen's micropolar theory imply Ericksen-Leslie director theory? Ericksen [1991], Leslie [1990]. Open problem since 1997.
Solution of this problem due to Gay-Balmaz, Tronci, TR. A new equation is necessary.