Differential Geometry of Singular Spaces and Reduction of Symmetries
Lecture 4

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- (H, ϖ) is a symplectic distribution on P.

- For each $f \in C^\infty(P)$, the distributional Hamiltonian vector field of f is the unique vector field Y_f in H such that, for every $u \in P$ and $w \in H_u$,
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- We say that (P, H, ϖ, h) is a distributional Hamiltonian system.

Almost Poisson structure

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- The ring $C^\infty(P)$ endowed with an almost Poisson bracket is called an almost Poisson algebra.
- We can put equations of motion in the almost Poisson form:

$$\frac{d}{dt} f(c(t)) = \{h, f\}(c(t))$$

for every $f \in C^\infty(P)$ and each integral curve of Y_h.

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- $C^\infty(R)$ is isomorphic to $C^\infty(R)^G$. Hence, it inherits an almost Poisson bracket such that

$$\rho^*\{\bar{f}_1, \bar{f}_2\} = \{\rho^*\bar{f}_1, \rho^*\bar{f}_2\}$$

for all $\bar{f}_1, \bar{f}_2 \in C^\infty(R)$.

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- The reduced system $(R, \{.,.\}, \bar{h})$ is an almost Poisson system.
- R is stratified by orbits of the family of all vector fields on R. Each stratum is an almost Poisson manifold.
- Each stratum inherits the structure of a constrained Hamiltonian system.

Pontryagin bundle

- The Pontryagin bundle of a manifold Q is the direct sum $TQ \oplus T^*Q$ of the tangent and cotangent bundle of Q. It is naturally isomorphic to the fibre product $P = TQ \times_Q T^*Q$. Let $\tau : TQ \longrightarrow Q$ and $\vartheta : T^*Q \longrightarrow Q$ be the tangent and the cotangent bundle projections, respectively, and

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$$\pi : P = TQ \times_Q T^*Q \to Q : (u,p) \mapsto \pi(u,p) = (\tau(u), \vartheta(p)).$$
- The Pontryagin bundle carries a symmetric form $\langle\langle \cdot, \cdot \rangle\rangle$ defined as follows. For each (u_1, p_1) and (u_2, p_2) in the same fibre of π,
$$\langle\langle (u_1,p_1),(u_2,p_2) \rangle\rangle = \langle p_1 \mid u_2 \rangle + \langle p_2 \mid u_1 \rangle.$$
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- Moreover, the space $\Gamma(P)$ of smooth sections of the Pontryagin bundle carries a bilinear skew-symmetric bracket, called the Courant bracket,

$$[(X, \alpha), (Y, \beta)] = ([X, Y], \$_X \beta - \$_Y \alpha + \frac{1}{2} d\,(\alpha(Y) - \beta(X))).$$

Symmetries of a Dirac structure

- A Dirac structure on Q is a subbundle D of $TQ \times_Q T^*Q$, which is maximal isotropic with respect to the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$. Thus, $\mathrm{rank}\, D = \dim Q$. We denote by $\iota : D \to P$ the inclusion map and by $\delta = \pi \circ \iota : D \to Q$ the projection of D onto Q.

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- Let
$$\Phi : G \times Q \to Q : (g, x) \mapsto \Phi_g(x) = gx$$
be an action of a connected Lie group G on the manifold Q. It induces an action
$$T\Phi : G \times TQ \to TQ : (g, u) \mapsto T\Phi_g(u)$$
of G on the tangent bundle TQ of Q. The push-forward of a vector field X on Q by Φ_g is given by
$$(\Phi_g)_* X = T\Phi_g \circ X \circ \Phi_{g^{-1}},$$
where, we treat X as a section of the tangent bundle projection $\tau : TQ \to Q$. A vector field X is G-invariant if $(\Phi_g)_* X = X$ for each $g \in G$.

- Similarly, we have an induced action

$$T^*\Phi : G \times T^*Q \to T^*Q : (g, p) \mapsto T^*\Phi_g(p),$$

where

$$\langle T^*\Phi_g(p) \mid u\rangle = \langle p \mid T\Phi_{g^{-1}}(u)\rangle$$

for every pair $(u, p) \in P = TQ \times_Q T^*Q$. This definition implies that the action of G on P preserves the evaluation. In other words,

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- If α is a 1-form on Q, considered as a section of the cotangent bundle $\vartheta : T^*Q \to Q$, then

$$(\Phi_g)_*\alpha = T^*\Phi_g \circ \alpha \circ \Phi_{g^{-1}}$$

is a section of ϑ that we shall also call the push-forward of α by Φ_g. A form α is G-invariant if $(\Phi_g)_*\alpha = \alpha$ for every $g \in G$. For every 1-form α on Q,

$$(\Phi_g)_*\alpha = (\Phi^*_{g^{-1}})\alpha.$$

- The product of $T\Phi$ and $T^*\Phi$ gives rise to an action

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- For a section $\sigma = (X, \alpha)$ of $\pi : P \to Q$, we denote by $(\Phi_g)_*\sigma$ the section of π given by

$$(\Phi_g)_*\sigma = \Psi_g \circ \sigma \circ \Phi_{g^{-1}} = ((\Phi_g)_*X, (\Phi_g)_*\alpha) = ((\Phi_g)_*X, \Phi^*_{g^{-1}}\alpha).$$

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- A section σ of π is G-invariant if $(\Phi_g)_*\sigma = \sigma$ for each $g \in G$.
- We consider here a Dirac structure $D \subset P$ that is invariant under the action of G on P.

Free and proper action

- If the action Φ of G on Q is free and proper, the action Ψ of G on the Pontryagin bundle is also free and proper.
- Therefore, Q is a left principal fibre bundle with structure group G, the base manifold Q/G and the projection map $\rho_Q : Q \longrightarrow Q/G$.
- Similarly, P is a left principal G-bundle with base manifold P/G and the projection map $\rho_P : P \longrightarrow P/G$.
- Since the Pontryagin bundle projection $\pi : P \longrightarrow Q$ intertwines the action of G in P and Q; that is, for each $g \in G$, $\pi \circ \Psi_g = \Phi_g \circ \pi$, it follows that there exists a map $\overline{\pi} : P/G \longrightarrow Q/G$ such that the following diagram

$$\begin{array}{ccccc} & & \rho_P & & \\ & P & \longrightarrow & P/G & \\ \pi & \downarrow & & \downarrow & \overline{\pi} \\ & Q & \longrightarrow & Q/G & \\ & & \rho_Q & & \end{array}$$

commutes.

- Moreover, the action Ψ on P is linear on fibres of the projection π.
- Therefore, $\overline{\pi} : P/G \to Q/G$ is a vector bundle.
- If $\sigma = (X, \alpha) : Q \to P$ is a G-invariant section of π, there exists a section $\overline{\sigma} = (\overline{X}, \overline{\alpha}) : Q/G \to P/G$ of $\overline{\pi}$ such that the following diagram

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Principal connection

- A connection on the principal bundle Q is a G-invariant distribution $\mathrm{hor}\, TQ$, that is complementary to the vertical distribution $\mathrm{ver}\, TQ = \ker T\pi$. This implies that we have a direct sum decomposition
$$TQ = \mathrm{ver}\, TQ \oplus \mathrm{hor}\, TQ.$$
Every vector $u \in T_q Q$ can be decomposed into the vertical part $\mathrm{ver}\, u$ and the horizontal part $\mathrm{hor}\, u$, that is $u = \mathrm{ver}\, u + \mathrm{hor}\, u$. Similarly, every covector $p \in T_q^* Q$ can be decomposed into the vertical part $\mathrm{ver}\, p$ and the horizontal part $\mathrm{hor}\, p$ such that
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$$\langle p \mid u \rangle = \langle \mathrm{ver}\, p \mid \mathrm{ver}\, u \rangle + \langle \mathrm{hor}\, p \mid \mathrm{hor}\, u \rangle.$$
- The decompositions described here lead to a decomposition of the Pontryagin bundle $P = \mathrm{ver}\, P \oplus \mathrm{hor}\, P$, where the vertical Pontryagin bundle $\mathrm{ver}\, P$ and the horizontal Pontryagin bundle $\mathrm{hor}\, P$ are given by
$$\mathrm{ver}\, P = \mathrm{ver}\, TQ \oplus \mathrm{ver}\, T^*Q \quad \text{and} \quad \mathrm{hor}\, P = \mathrm{hor}\, TQ \oplus \mathrm{hor}\, T^*Q.$$

- We get a decomposition of the bilinear form on P into its vertical and horizontal components

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- The orbit spaces $(\mathrm{ver}\, P)/G$ and $(\mathrm{hor}\, P)/G$ are vector bundles over P/G. We call $(\mathrm{ver}\, P)/G$ the reduced vertical Pontryagin bundle and $(\mathrm{hor}\, P)/G$ the reduced horizontal Pontryagin bundle.

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- The reduced horizontal Pontryagin bundle is isomorphic to the Pontryagin bundle of the orbit space Q/G.

- We get a decomposition of the bilinear form on P into its vertical and horizontal components

$$\langle\langle\cdot,\cdot\rangle\rangle = \mathrm{ver}\,\langle\langle\cdot,\cdot\rangle\rangle + \mathrm{hor}\,\langle\langle\cdot,\cdot\rangle\rangle.$$

- The bracket on the space of sections of P need not decompose into horizontal and vertical parts because the bracket of a horizontal section of P with the vertical section of P need not vanish.
- The orbit spaces $(\mathrm{ver}\,P)/G$ and $(\mathrm{hor}\,P)/G$ are vector bundles over P/G. We call $(\mathrm{ver}\,P)/G$ the reduced vertical Pontryagin bundle and $(\mathrm{hor}\,P)/G$ the reduced horizontal Pontryagin bundle.
- The reduced vertical Pontryagin bundle $(\mathrm{ver}\,P)/G$ is isomorphic to the direct sum of $Q[\mathfrak{g}] \oplus Q[\mathfrak{g}^*]$ of the adjoint and co-adjoint bundles of Q.
- The reduced horizontal Pontryagin bundle is isomorphic to the Pontryagin bundle of the orbit space Q/G.
- Note that the adjoint bundle of a principal fibre bundle Q is $Q[\mathfrak{g}] = (Q \times \mathfrak{g})/G$. Similarly, the co-adjoint bundle of Q is $Q[\mathfrak{g}^*] = (Q \times \mathfrak{g}^*)/G$.

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- The bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ on P is G-invariant and it gives rise to a bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_{P/G}$ on the reduced Pontryagin bundle such that

$$\langle\langle p_1, p_2 \rangle\rangle = \langle\langle \rho_P(p_1), \rho_P(p_2) \rangle\rangle_{P/G}$$

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  for every p_1, p_2 in the same fibre of P.
- The Courant bracket evaluated on G-invariant sections of $P \longrightarrow Q$ gives a G-invariant section of $P \longrightarrow G$. Hence, there is a bracket $[\cdot, \cdot]_{P/G}$ on the space $\Gamma(P/G)$ of sections of $\overline{\pi} : P/G \longrightarrow Q/G$ such that if σ_1 and σ_2 are G-invariant sections of $P \longrightarrow G$, then

$$[\overline{\sigma}_1, \overline{\sigma}_2]_{P/G} = \overline{[\sigma_1, \sigma_2]}.$$

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- The space D/G of G-orbits in D is a subbundle of P/G, which is maximally isotropic with respect to the bilinear form on P/G induced by the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_{P/G}$.

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- The space D/G of G-orbits in D is a subbundle of P/G, which is maximally isotropic with respect to the bilinear form on P/G induced by the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_{P/G}$.
- If the Dirac structure D is closed in the sense that for each pair σ_1 and σ_2 of G-invariant sections of $D \to Q$, the bracket $[\sigma_1, \sigma_2]$ has values in D, then $[\overline{\sigma}_1, \overline{\sigma}_2]_{P/G}$ has values in D/G for every pair of sections $\overline{\sigma}_1, \overline{\sigma}_2$ of $(D/G) \to Q/G$.

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- For each compact subgroup H of G and each connected component L of
$$Q_H = \{q \in Q \mid G_q = Q\},$$
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Theorem

D is uniquely determined by the collection of all structures D_L, as L varies over connected components of Q_H and H varies over compact subgroups of G.

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- The action of $G_L = N_L/H$ on L is free and proper.
- Since the Dirac structure D is G-invariant, it follows that D_L is G_L-invariant.
- Hence, we need to analyze the structure of D_L and apply regular reduction.