Geometry and symmetry in multi-physics models for magnetized plasmas

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Kinetic and fluid models in continuum dynamics

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• They arise from a *microscopic (kinetic) approach* by an averaging process

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KINETIC THEORY ON PHASE SPACE

Vlasov: $\partial_t f + v \cdot \partial_x f - \nabla\Phi \cdot \partial_v f = 0$

Micro-scales govern global dynamics

Moments $\longrightarrow$ $\int v^n f \, dv$

FLUID MODELS

Euler: $\partial_t u + (u \cdot \nabla) u = -\nabla p, \quad \nabla \cdot u = 0$

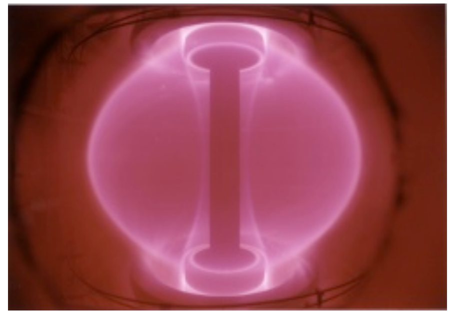
Neglects small scales

Geometric processes between kinetic and macroscopic theories

Accelerator beam optics

statistical moments

Integrable closures



physics of Vlasov plasmas

Collisionless kinetic equations

Fluid plasma models

Kinetic moments

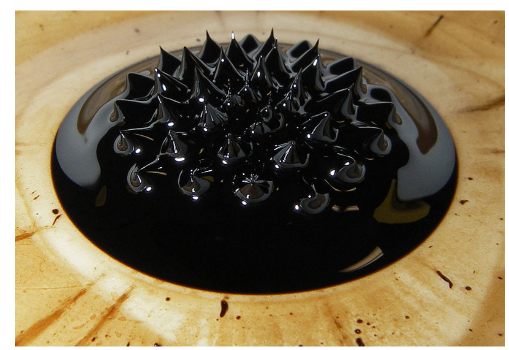
Integrable closures

Physics of quark-gluon plasmas

Kinetic theory for Yang-Mills interactions

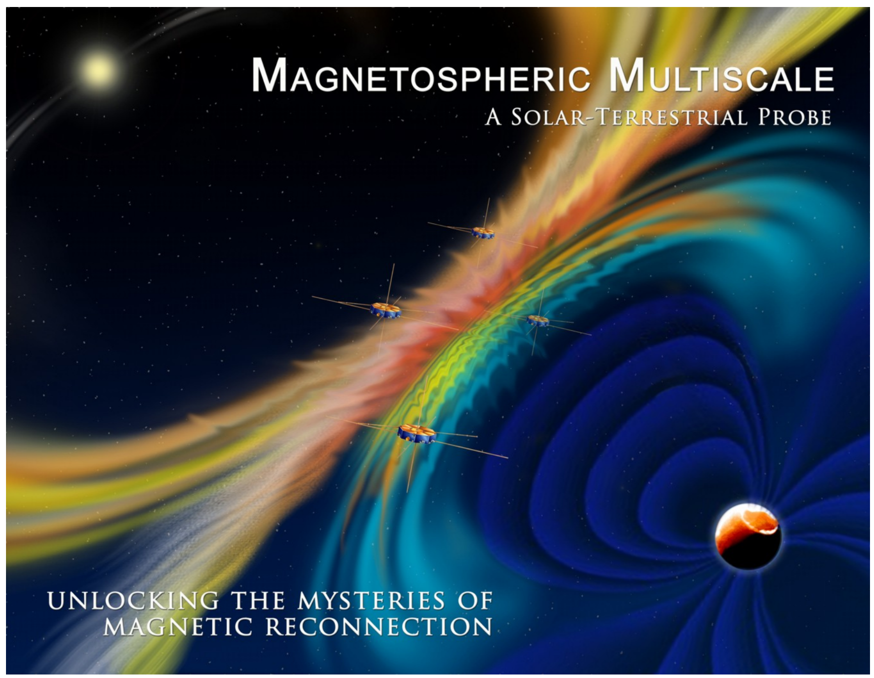
Moments

Fluid models for soft matter



Hybrid kinetic-fluid models for plasma physics

- Plasma simulations are mostly based on fluid (MHD) models
- These are invalidated by the presence of energetic particles
- Then, small-scale processes may control large-scale phenomenology



Energetic Solar wind interacts with Earth's magnetosphere

- *Microscopic effects* need to be considered along with fluid macro-scales
- **Hybrid philosophy: a fluid interacts with a hot particle gas**

Multi-physics approach!

$\longrightarrow$ MHD fluid models need to be coupled to kinetic-like equations

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→ MHD fluid models need to be coupled to kinetic-like equations

- Several coupling options are available, which need special care
- These usually arise by *inserting assumptions in the equations of motion*, cf [Park et al. (1992); Kim et al. (1994); Todo et al. (1995)]

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. . . *we shall use geometry!*

Fluid and kinetic models for plasmas

Plasma models

- **Particle trajectories** on phase space (Liouville):
  traces particles $(\mathbf{x}(t), \mathbf{p}(t)) \longrightarrow$ *solves all details.*

- **Kinetic approach** (Vlasov, Boltzmann):
  probability distribution $f(\mathbf{x}, \mathbf{p}, t) \longrightarrow$ *retains most details.*

- **Fluid approach** (MHD, Hall-MHD):
  local averages (momentum $\mathbf{m}(\mathbf{x}, t)$, density $\rho(\mathbf{x}, t)$) $\longrightarrow$ *forget details*.

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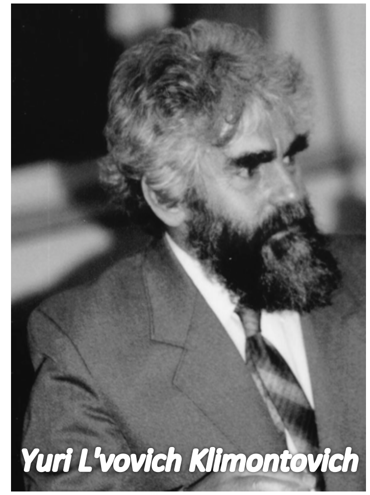
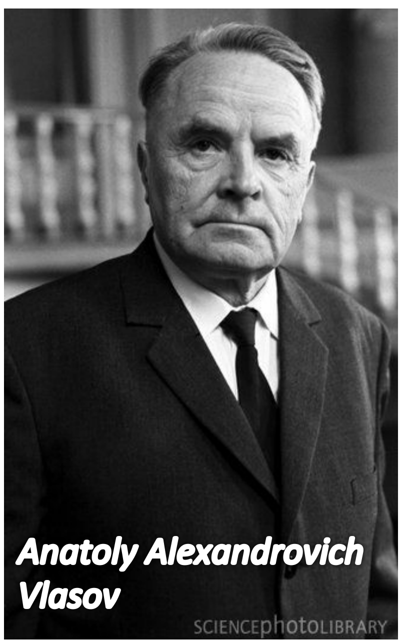
- *Collisional*: no energy conservation $\longrightarrow$ *Boltzmann* (H-theorem)

- *Collisionless*: energy is conserved $\longrightarrow$ **Vlasov** (mean field model)

$$\partial_t f + \{f, H\} = 0$$

For more info, look at these guys' work…





Kinetic approaches are expensive!

Better forget details? Fluid approaches are very convenient!

Magnetohydrodynamics (MHD)

- Fluid plasma model in which the magnetic field $\mathbf{B}$ is 'frozen in':

$$\partial_t(\mathbf{B}\cdot \mathrm{d}\mathbf{S}) + \pounds_{\boldsymbol{u}}(\mathbf{B}\cdot \mathrm{d}\mathbf{S}) = 0\,, \quad \text{or, equivalently,} \quad \partial_t\mathbf{B} + \nabla\times(\mathbf{B}\times\boldsymbol{u}) = 0$$

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- Fluid equation is

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla \mathrm{p} - \frac{1}{\mu_0 \rho}\mathrm{B}\times\nabla\times\mathrm{B}$$

where ρ is the transported mass density and p denotes pressure

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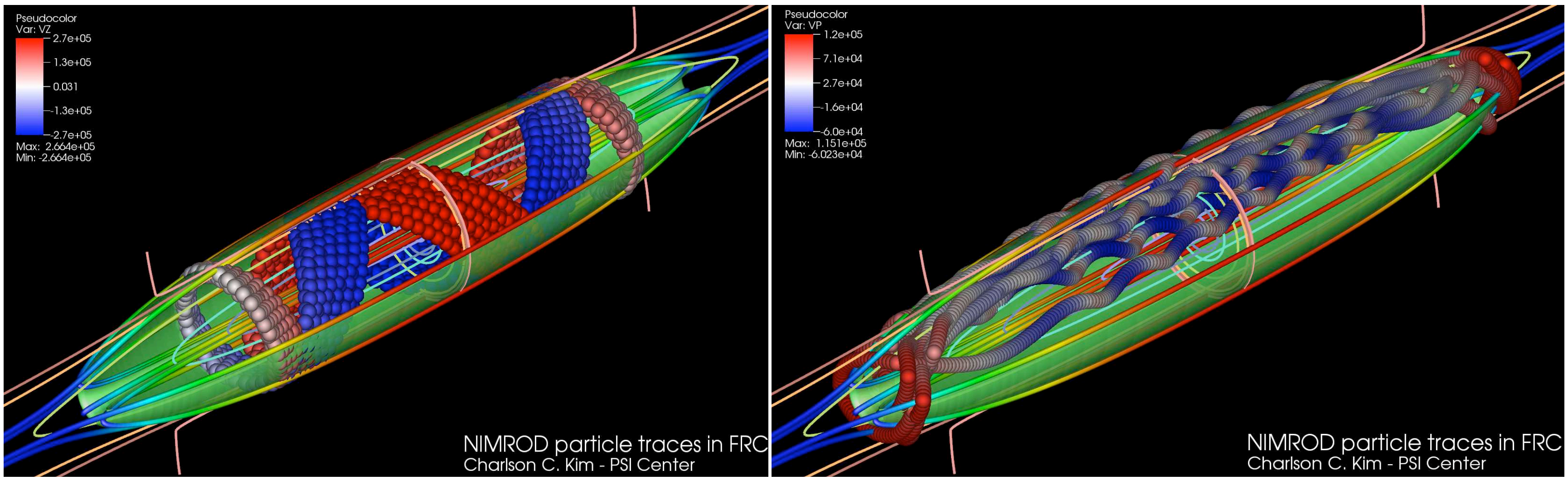
- Most plasma studies are based on this Hamiltonian (Lie-Poisson) model!

Still, energetic particles require kinetic theory!



Field Reversed Configuration experiments (FRCs) for nuclear fusion require kinetic descriptions as ordinary fluid approximations do not apply. No particular phenomenon is observed for low energy particles (right), while certain patterns emerge at high energies (left). In particular, hot particles confine to the outboard region (higher magnetic gradients) and never cross the origin.

Kinetic theory & electromagnetism: Maxwell-Vlasov

- Vlasov kinetic equation for $f(\mathbf{x}, \mathbf{p}, t)\ldots$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{x}} + q\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

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- . . . coupled to Maxwell's equations

$$\epsilon_0\mu_0\,\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{B} - \mu_0\,\frac{q}{m}\int \mathbf{p}\,f\,\mathrm{d}^3\mathbf{p}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\,\nabla\times\mathbf{E}$$

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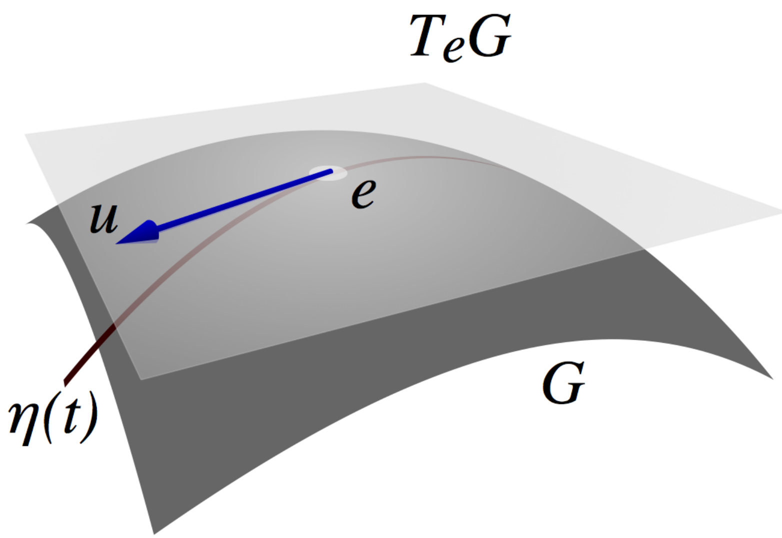
$$\epsilon_0\nabla\cdot\mathbf{E}=q\int f\,\mathrm{d}^3\mathbf{p}\,,\qquad \nabla\cdot\mathbf{B}=0$$

- Again, this is a Lie-Poisson system!

Geometric mechanics for fluid and kinetic models

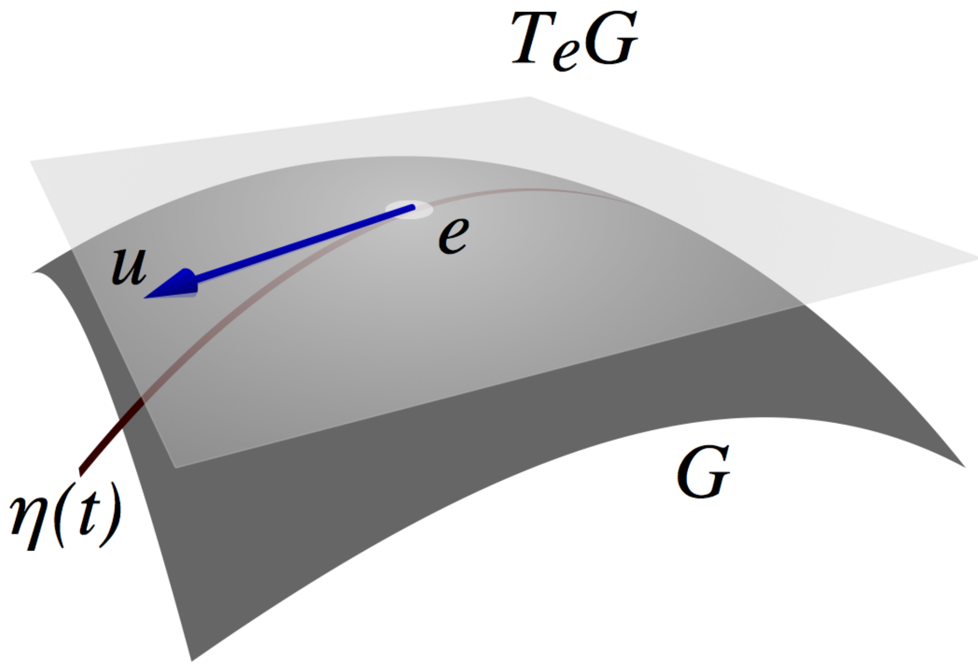
Geometric fluid dynamics



Lagrangian and Eulerian variables are related by the relabeling symmetry, which produces an *intrinsic geometric description* [Arnold (1966)] capturing essential features such as *circulation laws* and dynamical invariants.

Ex. Incompressible ideal fluids move along geodesics on $G = \mathrm{Diff}_{\mathrm{vol}}(M)$

Geometric approach possesses variational and Hamiltonian formulations!

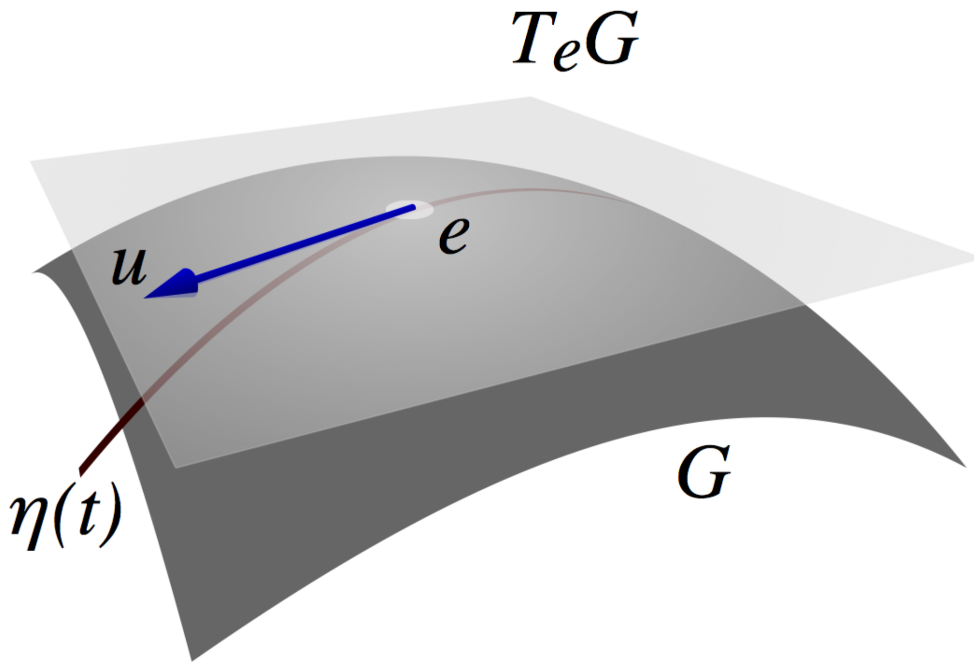


Lagrangian fluid dynamics of $\boldsymbol{\eta}(\mathbf{a},t)$ on the Lie group G possesses the

canonical Poisson bracket: $$\{F,G\} = \int \left(\frac{\delta F}{\delta \boldsymbol{\eta}} \cdot \frac{\delta G}{\delta \boldsymbol{\psi}} - \frac{\delta F}{\delta \boldsymbol{\psi}} \cdot \frac{\delta G}{\delta \boldsymbol{\eta}} \right) \mathrm{d}^3\mathbf{a}\,,$$

Eulerian dynamics on the (dual) tangent space at identity possesses the

Lie-Poisson bracket : $$\{F,G\}(\boldsymbol{\sigma}) = \left\langle \boldsymbol{\sigma}, \left[\frac{\delta F}{\delta \boldsymbol{\sigma}}, \frac{\delta G}{\delta \boldsymbol{\sigma}} \right] \right\rangle$$



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Fluids: *(η,ψ) are Lagrangian coordinates, while $\boldsymbol{\sigma}$ = fluid momentum $\mathbf{m}$.*
Vlasov: *(η,ψ) are Lagrangian coordinates, while σ = distribution function f.*

Symmetry is everywhere in mechanics

- Rotational symmetry for vectors (*rigid body motion*):

$$[\mathbf{g}, \mathbf{k}] = \mathbf{g} \times \mathbf{k} \ \longrightarrow \ \{F, G\} = \boldsymbol{\mu} \cdot \frac{\mathrm{d}F}{\mathrm{d}\boldsymbol{\mu}} \times \frac{\mathrm{d}G}{\mathrm{d}\boldsymbol{\mu}}$$

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- Relabeling symmetry for velocities (*Euler fluid dynamics*):

$$[\mathbf{v}, \mathbf{u}] = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} \;\rightarrow\; \{F, G\} = \int \boldsymbol{\mu}(\mathbf{x}) \cdot \left[\frac{\delta F}{\delta \boldsymbol{\mu}}, \frac{\delta G}{\delta \boldsymbol{\mu}}\right] \mathrm{d}^3\mathbf{x}$$

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- Canonical (symplectic) symmetry for matrices (*beam optics*):

$$[A, B] = AB - BA \;\longrightarrow\; \{F, G\} = \mathrm{Tr}\left(X^T \left[\frac{\mathrm{d}F}{\mathrm{d}X}, \frac{\mathrm{d}G}{\mathrm{d}X}\right]\right)$$

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- Canonical symmetry for phase-space functions (**Vlasov equation**):

$$[h,k] = \frac{\partial h}{\partial\mathbf{x}}\cdot\frac{\partial k}{\partial\mathbf{p}} - \frac{\partial h}{\partial\mathbf{p}}\cdot\frac{\partial k}{\partial\mathbf{x}} \;\rightarrow\; \{F,G\} = \int f(\mathbf{x},\mathbf{p})\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$$

Intermezzo: geometry of Vlasov kinetic theory

- Let $\zeta_t : (S, w) \hookrightarrow (\mathcal{P}, \omega)$ be an embedding from a volume manifold (S, w) to the symplectic manifold $(\mathcal{P}, \omega)$. Vlasov has the following soln

$$f(z,t) = \int_S w\, \delta(z - \zeta(s,t))$$

- In more generality, the following Lie groups act on $\mathrm{Emb}(S, \mathcal{P})$:
  - *Canonical transformations on* $\mathcal{P}$: $\psi \cdot \zeta = \psi \circ \zeta$ (left action)
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- The actions of Diff_{Ham} and Diff_{Vol} on Emb$(S, \mathcal{P})$ produce the (dual pair of) **momentum maps** [Marsden&Weinstein(1983), Holm&CT(2009)]

$\mathbf{J}_L : \text{Emb}(S, \mathcal{P}) \to \mathfrak{X}^*_{\text{Ham}}(\mathcal{P})\,;$ $\quad\quad$ $\mathbf{J}_R : \text{Emb}(S, \mathcal{P}) \to \mathfrak{X}^*_{\text{Vol}}(S)$

$\zeta(s) \mapsto \int_S w\,\delta(z - \zeta(s,t))\,;$ $\quad\quad$ $\zeta(s) \mapsto \zeta^*\omega = \text{d}Q^i(s) \wedge \text{d}P_i(s)$

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- Moments $\int \mathbf{p}^n\, f\, \text{d}^3\mathbf{p}$ are also momentum maps [Gibbons, Holm&CT(2008)]

Let's apply geometric mechanics to formulate our hybrid models!

A geometric hybrid model: assumptions

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$$H = \frac{1}{2}\int \frac{|\mathbf{m}|^2}{\rho}\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mathrm{m}_h}\int f\,|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \int \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int |\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x}\,,$$

A geometric hybrid model: equations

- This process returns the same fluid equation as in the literature while inserting new transport term and circulation force in the kinetic equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla \mathsf{p} - \frac{1}{\mathsf{m}_h\,\rho}\,\nabla \cdot \int \mathbf{p}\mathbf{p} f\, \mathrm{d}^3\mathbf{p} - \frac{1}{\mu_0 \rho}\mathbf{B} \times \nabla \times \mathbf{B}$$

$$\frac{\partial f}{\partial t} + \left(\boldsymbol{u} + \frac{\mathbf{p}}{m_h}\right) \cdot \frac{\partial f}{\partial \mathbf{x}} - (\mathbf{p} \cdot \nabla \boldsymbol{u}) \cdot \frac{\partial f}{\partial \mathbf{p}} + a_h\,\mathbf{p} \times (\mathbf{B} - \nabla \times \boldsymbol{u}) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

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$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u})=0\,,\qquad \frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\mathbf{B})\ ,$$

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- Unlike previous models, the fluid interaction terms do NOT vanish in the absence of magnetic fields

A geometric hybrid model: equations

- This process returns the same fluid equation as in the literature while inserting new transport term and circulation force in the kinetic equation

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\frac{1}{\rho}\nabla \mathsf{p}-\frac{1}{\mathsf{m}_h\,\rho}\,\nabla\cdot\int \mathbf{p}\mathbf{p}f\,\mathrm{d}^3\mathbf{p}-\frac{1}{\mu_0\rho}\mathbf{B}\times\nabla\times\mathbf{B}$$

$$\frac{\partial f}{\partial t}+\left(\boldsymbol{u}+\frac{\mathbf{p}}{m_h}\right)\cdot\frac{\partial f}{\partial \mathbf{x}}-(\mathbf{p}\cdot\nabla\boldsymbol{u})\cdot\frac{\partial f}{\partial \mathbf{p}}+a_h\,\mathbf{p}\times(\mathbf{B}-\nabla\times\boldsymbol{u})\cdot\frac{\partial f}{\partial \mathbf{p}}=0$$

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- Unlike previous models, the fluid interaction terms do NOT vanish in the absence of magnetic fields

- Circulation force terms emerge since hot particle trajectories are now computed in the cold fluid frame.

- We get magnetic and cross helicity invariants:

$$\mathcal{H} = \int \mathbf{A} \cdot \mathbf{B} \, \mathsf{d}^3\mathbf{x} \,, \qquad \Lambda = \int \left(\boldsymbol{u} - m_h \frac{\mathbf{K}}{\rho} \right) \cdot \mathbf{B} \, \mathsf{d}^3\mathbf{x}$$

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- Circulation laws (see also *Euler-Poincaré approach* [Holm&Tronci(2011)])

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \boldsymbol{u} \cdot \mathrm{d}\mathbf{x} = - \oint_{\gamma_t} \frac{1}{\rho} \left(\frac{1}{\mu_0} \mathbf{B} \times \nabla \times \mathbf{B} + m_h \nabla \cdot \mathbb{P} \right) \cdot \mathrm{d}\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \frac{\mathbf{K}}{\rho} \cdot \mathrm{d}\mathbf{x} = \oint_{\gamma_t} \frac{1}{\rho} \left(a_h \mathbf{K} \times \mathbf{B} - \nabla \cdot \mathbb{P} \right) \cdot \mathrm{d}\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \left(1 + \frac{n}{\rho} \right) \mathbf{A} \cdot \mathrm{d}\mathbf{x} = - \oint_{\gamma_t} \frac{1}{\rho} (\nabla \cdot \mathbf{K}) \, \mathbf{A} \cdot \mathrm{d}\mathbf{x} \,;$$

where $n = \int f \, \mathrm{d}^3\mathbf{p}$ is the hot particle density, while the pressure tensor

$$\mathbb{P} = \int \mathbf{p}\mathbf{p} f \, \mathrm{d}^3\mathbf{p}$$

emerges as a geometric forcing term in the cold fluid dynamics

Geometry of hybrid pressure-coupling schemes

- The momentum shift $\mathbf{M} = \rho\boldsymbol{u} + \mathbf{K}$ corresponds to an *entangling Poisson map* [Krishnaprasad&Marsen(1984); Holm(1986)]

$$\mathcal{E} : \Big(\mathfrak{X}(\mathbb{R}^3) \oplus \mathfrak{X}_{\text{can}}(\mathbb{R}^6)\Big)^* \to \Big(\mathfrak{X}(\mathbb{R}^3) \circledS \mathfrak{X}_{\text{Ham}}(\mathbb{R}^6)\Big)^*$$
$$(\rho\boldsymbol{u}, \mathbf{K}) \mapsto (\rho\boldsymbol{u} + \mathbf{K}, \mathbf{K})$$

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- Semidirect-product arises from *cotangent-lifts of* Diff$(\mathbb{R}^3)$ *acting on* Diff$(\mathbb{R}^6)$ *(subgroup action)*, whose momentum map is $\mathbf{K} = \int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p}$

- Denote two-forms by $\Omega^2(\mathbb{R}^3)$. Hybrid model is written on the Lie group

$$\underbrace{\Big(\text{Diff}(\mathbb{R}^3) \circledS \text{Diff}(\mathbb{R}^6)\Big)}_{\text{cold \& hot flows (Lagrangian variables)}} \circledS \underbrace{\Big(C^\infty(\mathbb{R}^3) \times \Omega^2(\mathbb{R}^3)\Big)}_{\text{dual to advected quantities: } (\rho, \mathbf{A})}$$

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- Other momentum map properties underlying flows on semidirect-products were presented in [Holm&CT(2008); Gay-Balmaz,Vizman&CT(2010)]

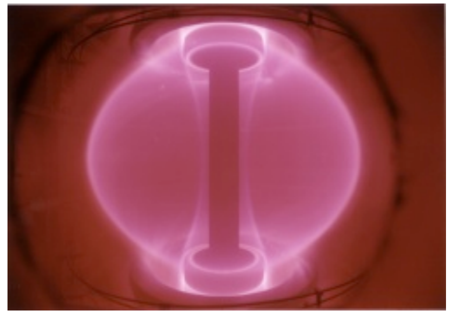
Next steps

- Nonlinear stability in toroidal geometry (fusion devices): Casimir method
- Application of hybrid models in MHD turbulence [Cowley et al. (2011)]
- New hybrid models for space plasmas. (Collisionless reconnection)
- Extend the hybrid philosophy to complex fluids and quantum plasmas

Accelerator beam optics

statistical moments

Integrable closures



physics of Vlasov plasmas

Collisionless kinetic equations

Fluid plasma models

Kinetic moments

Integrable closures

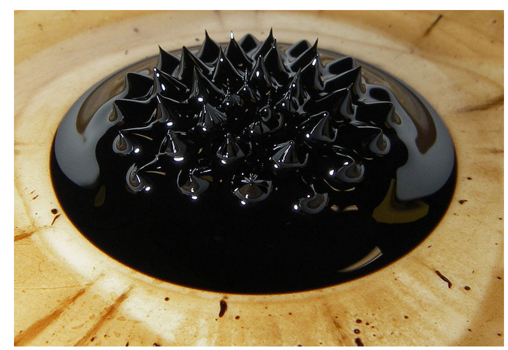
Geometric Mechanics does all at once!

Physics of quark-gluon plasmas

Kinetic theory for Yang-Mills interactions

Moments

Fluid models for soft matter



Liquid crystals

Ferrofluids + spin glasses

Ferromagnetic nanoparticles