Well quasi-ordering Aronszajn lines.

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Definition

$(\mathcal{K}, \preceq)$ is well-quasi-ordered (wqo, in short) if for any sequence A_n $(n \in \omega)$ of elements of $\mathcal{K}$ there are $n < m$ so that $A_n \preceq A_m$.

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Theorem (Laver 1971)

The class of countable linear orderings is wqo.

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Theorem (Baumgartner 1981)

Assuming PFA. Every two separable linear orders of size $\aleph_1$ are equivalent.

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3. Here chain refers to the coordinate-wise partial order on C^2.

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A further important observation is that if C is Countryman and C^* denotes its reverse, then no uncountable linear order can embed into both C and C^* .

It was known for some time that, assuming MA_{ω_1}, the class of Countryman lines have a two-element basis.

Well-Quasi-Ordering Aronszajn Lines.

Theorem (Moore 2006)

Assuming PFA. The uncountable linear orderings have a five element basis consisting of $X, \omega_1, \omega_1^*, C$ and C^* whenever X is a set of reals of cardinality $\aleph_1$ and C is any Countryman line.

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Theorem (Moore 2008)

Assuming PFA. There is a universal Aronszajn line η_C*. Moreover,* η_C *can be described as the subset of the lexicographical power* $(\zeta_C)^\omega$ *consisting of those elements which are eventually zero where* ζ_C *is the direct sum* $C^* \oplus 1 \oplus C$.

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Theorem (M.-R. 2011)

The class of Aronszajn lines is wqo by embeddability.

Fragmented A-lines and their ranks.

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Definition

Let $\mathcal{A}_0$ be the class of Countryman lines. For each $\alpha < \omega_2$, let $\mathcal{A}_\alpha$ be the class of all elements of the form

$$\sum_{x \in I} A_x$$

so that $I \preceq C$ or $I \preceq C^*$ and $\forall x \in I \;\; A_x \in \mathcal{A}_\xi$ for some $\xi < \alpha$.

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There is a natural rank on the fragmented Aronszajn lines given by $rank(A) = \{\alpha : A \in \mathcal{A}_\alpha\}$.

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(i) C and C^* play the role of ω and ω^*, respectively.

(ii) η_C play the role of the rationals.

(iii) and being fragmented is analogous to being scattered in this context.

Details of the proof

Lemma (Main Lemma)

(MA_{ω_1}) For every ordinal $\alpha < \omega_2$ there exist two incomparable Aronszajn lines D^+_α, and D^-_α of rank α such that :

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2. $D^-_\alpha \preceq C^* \times D^+_\alpha$, $D^+_\alpha \preceq C \times D^-_\alpha$ *and*

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2. $D^-_\alpha \preceq C^* \times D^+_\alpha$, $D^+_\alpha \preceq C \times D^-_\alpha$ *and*
3. *For every* $A \in \mathcal{A}_\alpha$ *either* $A \equiv D^+_\alpha$ *or* $A \equiv D^-_\alpha$ *or else both* $A \preceq D^+_\alpha$ *and* $A \preceq D^-_\alpha$ *holds.*

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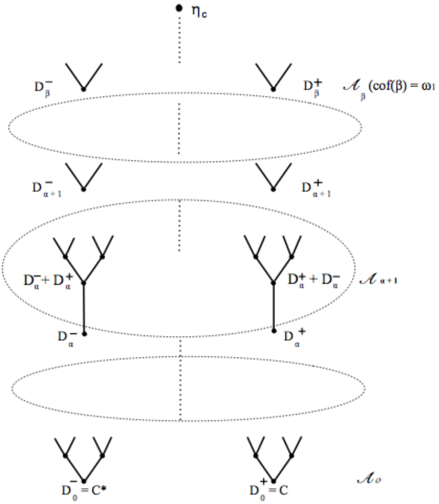

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This implies that in some sense the class $\mathcal{A}$ is too big to have a meaningful classification theorem.

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Definition

A tree T is *coherent* if it can be represented as a downward closed subtree of $\omega^{<\omega_1}$ with the property that for every pair of nodes $t, s \in T$ $\{\xi \in dom(t) \cap dom(s) : t(\xi) \neq s(\xi)\}$ is a finite set.

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Assuming $MA_{\aleph_1}$. The class $\mathcal{C}$ of coherent Aronszajn trees is cofinal, coinitial and linearly ordered.

Moreover, assuming *PFA*, any coherent Aronszajn tree is comparable with any Aronszajn tree.

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Corollary (M.-R., Todorcevic 2011)

Assuming PFA. The class of Aronszajn trees is universal for linear orders of cardinality at most $\aleph_2$