On bounded representations and maximal symmetry

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Fields Institute 2012,
joint work with V. Ferenczi, São Paulo

Bounded subgroups of $GL(X)$

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The strong operator topology is given by pointwise convergence on X, i.e.,

$$T_i \to T \quad \Leftrightarrow \quad \|T_i x - Tx\| \to 0 \quad \text{for all } x \in X,$$

while the weak operator topology is given by weak convergence, i.e.,

$$T_i \to T \quad \Leftrightarrow \quad \phi(T_i x) \to \phi(Tx) \quad \text{for all } x \in X \text{ and } \phi \in X^*.$$

Note that if $|||\cdot|||$ is an equivalent norm on X, i.e., such that

$$\mathrm{Id}\colon (X, \|\cdot\|) \to (X, |||\cdot|||)$$

is an isomorphism, then

$$GL(X, \|\cdot\|) = GL(X, |||\cdot|||)$$

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and the three topologies remain unaltered, although the norm of course changes.

Thus, we can talk unequivocally about $GL(X)$ and its topologies without fixing the norm.

Suppose $G \leqslant GL(X)$ is a weakly bounded subgroup, i.e., such that for any $x \in X$ and $\phi \in X^*$,

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So we can simply talk about bounded subgroups of $GL(X)$.

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In the strong operator topology, $\mathrm{Isom}(X)$ is a Polish group, i.e., a separable and complete metric topological group.

Thus, any bounded and strongly closed $G \leqslant GL(X)$ is a closed subgroup of a Polish group and hence is itself Polish in the strong operator topology.

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Here a norm $\|\cdot\|$ on X is maximal if

$$\mathrm{Isom}(X, \|\cdot\|)$$

is a maximal bounded subgroup of $GL(X)$.

In other words, if $\|\!|\cdot|\!\|$ is an equivalent norm on X such that

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Thus, if we think of $\mathrm{Isom}(X, \|\cdot\|)$ as the set of symmetries of X, then $\|\cdot\|$ is maximal if

$$(X, \|\cdot\|)$$

or rather the unit ball

$$B(X, \|\cdot\|)$$

is a maximally symmetric body.

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whence $\|\cdot\|$ is maximal.

In other words, any transitive norm $\|\cdot\|$ is maximal.

Examples

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For example, the standard norms on

- ℓ_p (Rolewicz),
- $L_p([0,1])$ (Rolewicz),
- $C(K, \mathbb{C})$ for K a compact manifold (Kalton, Wood)

are all maximal, but not on

- $C([0,1], \mathbb{R})$ (Partington).

Convex equivariant renormings

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This would be analogous to the existence of maximal compact subgroups of semisimple Lie groups.

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However, in this case, a Baire category argument shows that there is a G-invariant norm on X that is simultaneously uniformly convex and uniformly smooth (e.g., Bader, Furman, Gelander, Monod).

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Moreover, it can be shown that if $(X, \|\cdot\|)$ is locally uniformly convex and $G \leqslant GL(X)$ is compact in the strong operator topology, then the G-invariant norm

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is still locally uniformly convex.

However, e.g., L_1 does not admit a locally uniformly convex norm invariant under the original isometries.

Nevertheless, refining a result of G. Lancien (1993), we have

Theorem (Lancien)

Let X be separable reflexive and $G \leqslant GL(X)$ be a bounded subgroup. Then there is an equivalent G-invariant norm $\|\cdot\|$ on X such that both $\|\cdot\|$ and $\|\cdot\|^*$ are locally uniformly convex.

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In particular, any maximal norm on a separable reflexive space X can be made locally uniformly convex without changing the isometry group.

In fact, by a result of Becerra Guerrero and Rodíguez-Palacios, if the norm is also convex transitive, then X is super-reflexive and the norm uniformly convex.

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Therefore, averaging over the Haar measure produces an isometry-invariant inner product $\langle\cdot|\cdot\rangle$.

Thus, the induced Euclidean norm

$$\|\cdot\|_{\langle\cdot|\cdot\rangle} = \sqrt{\langle\cdot|\cdot\rangle}$$

is a transitive and hence maximal norm on X such that

$$\mathrm{Isom}(X, \|\cdot\|) \leqslant \mathrm{Isom}(X, \|\cdot\|_{\langle\cdot|\cdot\rangle}).$$

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Then $\langle\cdot|\cdot\rangle_G$ is a G-invariant equivalent inner product on $\mathcal{H}$ and hence G is contained in the maximal bounded subgroup

$$U(\mathcal{H}, \langle\cdot|\cdot\rangle_G).$$

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By the Ehrenpreis–Mautner example, one of the above must hold.

Wood's question

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- Does $GL(X)$ always contain a maximal bounded subgroup? (Wood 1982)

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In fact, even more restrictive questions have been open so far:

- Is every bounded subgroup contained in a maximal bounded subgroup? (Wood 2006)
- Do super-reflexive spaces admit equivalent (almost) transitive norms? (Deville, Godefroy, Zizler 1993)

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Operator ideals and the Fredholm group

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Let us first consider the isometries imposed on us.

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Therefore, we can renorm X so that T becomes an isometry.

In a more general framework, if $\mathcal{I}$ is an ideal in the algebra of bounded operators $\mathcal{L}(X)$, we let

$$GL_{\mathcal{I}}(X) = \{\mathrm{Id} + A \in GL(X) \mid A \in \mathcal{I}\}$$

denote the subgroup of $\mathcal{I}$-perturbations of the identity.

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The most important ideals are

$$\mathcal{F} \quad \subseteq \quad \mathcal{AF} = \overline{\mathcal{F}} \quad \subseteq \quad \mathcal{K} \quad \subseteq \quad \mathcal{SS}$$

of respectively finite-rank, approximately finite-rank, compact and strictly singular operators.

For example, it is known by work of W. T. Gowers and B. Maurey (1992) that if X is a complex HI space, then

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in fact, any operator on X is of the form

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However, isometries are even more restrictive, namely, F. Räbiger and W. J. Ricker (1998) showed that any isometry has the form

$$\lambda \mathrm{Id} + K,$$

where K is compact.

But these results have the following generalisation.

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But what about groups of isometries?

Nearly trivial actions

Definition

Suppose $G \leqslant GL(X)$. We say that G acts nearly trivially on X if there is a G-invariant decomposition

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such that F is finite-dimensional and $T|_H = \lambda \mathrm{Id}_H$ for every $T \in G$.

Nearly trivial actions

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such that F is finite-dimensional and $T|_H = \lambda \mathrm{Id}_H$ for every $T \in G$.

In other words, G acts by scalar multiplication on the cofinite-dimensional subspace H and thus the non-trivial part of the action occurs on F.

One reason for our interest in nearly trivial actions are their relation to Wood's problem.

Proposition

Suppose X is an infinite-dimensional Banach space and $G \leqslant GL(X)$ is a bounded subgroup acting nearly trivially on X. Then G is not maximal bounded in $GL(X)$.

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The idea is that if $X = H \oplus F$ is the G-invariant decomposition, where F is finite-dimensional and G acts trivially on H (i.e., by scalar multiplication), then we can further split X as

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and then properly extend G to the bounded subgroup

$$\mathrm{Isom}(E) \times G$$

inside of $GL(X)$.

Structure theory for small subgroups of $GL(X)$

It is possible to develop quite a significant structure theory for small subgroups $G \leqslant GL(X)$, that is, bounded subgroups of $GL_{\mathcal{F}}(X)$ or $GL_{\mathcal{AF}}(X)$, under various additional assumptions on X and on G.

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The additional assumptions can be the separability of X^*, reflexivity of X, or that G is norm or strongly closed in $GL(X)$.

We shall not go too much into this, but just mention some of the main techniques and results with a view towards Wood's problem.

Weak almost periodicity and equivariant decompositions

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Theorem (Alaoglu – Birkhoff and Jacobs – de Leeuw – Glicksberg)

Let X be a Banach space and $G \leqslant GL(X)$ a weakly almost periodic subgroup, i.e., such that any orbit $G \cdot x$ is relatively weakly compact in X. Then X admits a G-invariant decomposition

$$X = X_1 \oplus X_2 \oplus X_3,$$

where

- *X_1 is the set of G-invariant vectors,*
- *any orbit on X_2 is relatively norm compact,*
- *no non-zero orbit on X_3 is relatively norm compact.*

Though weakly almost periodic representations occur naturally in ergodic theory and dynamical systems, the main reason why the isometry group of a Banach space X should be weakly almost periodic is that X is reflexive (and hence any bounded subset if relatively weakly compact).

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And, in fact, in the setting of separable reflexive spaces, we can give very natural proofs of the aforementioned decomposition theorems based on the refinement of Lancien's renorming theorem.

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And, in fact, in the setting of separable reflexive spaces, we can give very natural proofs of the aforementioned decomposition theorems based on the refinement of Lancien's renorming theorem.

The other known proofs of these are either based on Ryll-Nardzewski's fixed point theorem or Namioka's joint continuity theorem.

Decompositions by groups of finite-dimensional isometries

Theorem

Let X be separable reflexive, $G \leqslant GL_{\mathcal{F}}(X)$ a bounded subgroup and $X = X_1 \oplus X_2 \oplus X_3$ the canonical decomposition.
Then X_2 and X_3 admit G-invariant Schauder decompositions

$$X_i = Y_1 \oplus Y_2 \oplus \ldots$$

(possibly with finitely many summands) so that each Y_i has a (possibly finite) Schauder basis.

Norm closed subgroups of $GL_{\mathcal{F}}(X)$

By the decomposition theorem, studying isometry groups of a reflexive Banach space X largely reduces to studying its action separately on X_2 and X_3.

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Now on X_2, as a consequence of the Peter-Weyl theorem, we have the following well-known equivalence.

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Now on X_2, as a consequence of the Peter-Weyl theorem, we have the following well-known equivalence.

Proposition

Let X be a Banach space. Then the following are equivalent for a bounded subgroup $G \leqslant GL(X)$.

- *every orbit $G \cdot x$ is relatively norm compact,*
- *G is relatively compact in the strong operator topology,*
- *X is the closed linear span of its finite-dimensional G-invariant subspaces.*

On the other hand, of relevance to X_3, we have

Theorem

Let X be a Banach space with separable dual and $G \leqslant GL_{\mathcal{F}}(X)$ a bounded subgroup, norm closed in $GL(X)$, so that no non-zero G-orbit is relatively norm compact.

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Theorem

Let X be a Banach space with separable dual and $G \leqslant GL_{\mathcal{F}}(X)$ a bounded subgroup, norm closed in $GL(X)$, so that no non-zero G-orbit is relatively norm compact.
Then G is discrete and locally finite in the norm topology.

Strongly closed bounded subgroups of $GL_{\mathcal{F}}(X)$

Combining our various results on small subgroups, we obtain

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Theorem

Suppose X is a separable, reflexive Banach space and $G \leqslant GL_{\mathcal{F}}(X)$ is bounded and strongly closed in $GL(X)$.

Strongly closed bounded subgroups of $GL_{\mathcal{F}}(X)$

Combining our various results on small subgroups, we obtain

Theorem

Suppose X is a separable, reflexive Banach space and $G \leqslant GL_{\mathcal{F}}(X)$ is bounded and strongly closed in $GL(X)$. Then G is an amenable Lie group and the connected component of the identity, $G_0 \leqslant G$, acts nearly trivially on X.

Quick and dirty solution to Wood's problem and relatives

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Now, since X is HI, if $X = Y \oplus Z$, then one of Y and Z is finite-dimensional.

So, if

$$X = X_1 \oplus X_2 \oplus X_3$$

is the canonical isometry-invariant decomposition, then exactly one of the three summands is infinite-dimensional.

Moreover, for $i = 2, 3$, we have a decomposition

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It follows that either

$$X = X_1 \oplus F,$$

or

$$X = Z \oplus F,$$

where F is finite-dimensional, the isometry group acts trivially on X_1 (i.e., by scalar multiplication) and Z has a finite-dimensional decomposition.

Since the last option is absurd, as then also X would have a finite-dimensional decomposition, it follows that any bounded subgroup of $GL_{\mathcal{F}}(X)$ acts nearly trivially on X.

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With a little more care in the decompositions, we obtain

Theorem

Let X be a reflexive HI space without the approximation property. Then X admits an isometry-invariant decomposition

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with F finite-dimensional and where H is a closed subspace carrying no non-trivial isometry.

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Corollary

There is no maximal bounded subgroup of $GL(X)$ and so X has no equivalent maximal norm.

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