Simple C^*-algebras of generalized tracial rank one

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Joint work with Guihua Gong and Zhuang Niu –in progress

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$$\begin{aligned}(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A) \\ \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B), r_B).\end{aligned} \quad (e\,0.1)$$

Here r_C is an affine map from $T(C)$ into $S_1(K_0(C))$, the state space of $K_0(C)$, such that $r_C(\tau)([p]) = \tau(p)$ for all projections in $M_k(C)$, $k \geq 1$.

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for all $x \in K_0(A)$ and $\tau \in T(A)$. We will write

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A).$$

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If $TR(A) \le k$ but $TR(A) \not\le k-1$, we say A has tracial rank k and write $TR(A) = k$.

Theorem

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$$\mathrm{Ell}(A) = (G_0, (G_0)_+, u, G_1, \Delta, r).$$

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Jiang-Su algebra $\mathcal{Z}$ is not an AH-algebra.

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(L-Zhuang Niu 2008) Let A and B be two unital amenable separable simple $\mathcal{Z}$-stable C^*-algebras which satisfies the UCT. Suppose that $TR(A \otimes M_p) = 0$ *and* $TR(B \otimes M_p) = 0$

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(L-Zhuang Niu 2008) Let A and B be two unital amenable separable simple $\mathcal{Z}$-stable C^*-algebras which satisfies the UCT. Suppose that $TR(A \otimes M_p) = 0$ and $TR(B \otimes M_p) = 0$ for all UHF-algebras M_p of infinite type.

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$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

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Moreover, A can be constructed to be locally approximated by subhomogeneous C^*-algebras.

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Let G be a countable weakly unperforated simple partially ordered group with an order unit u. Then G has the rationally Riesz property if and only if $S_u(G)$ is a metrizable Choquet simplex.

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Denote

$$D_{X,k,\{X_j\},\{m_j\}} = \{f \in C(X.M_k) : f(x) \in B_j \quad \text{for all} \quad x \in X_j, 1 \leq j \leq n\}.$$

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Theorem

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Theorem

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Denote by $\mathcal{J}_1$ the class of all unital C^*-algebras of the form $A = A(F_1, F_2, \phi_0, \phi_1)$.

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(ii) If $u, v \in U(B)$ with $\mathrm{cel}((u^*)^k v^k) \leq L$ for some integer $k > 0$, then

$$\mathrm{cel}(u^* v) \leq 6\pi + L/k.$$

Theorem

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$$0 \to K_1(B) \xrightarrow{\imath_*} K_0(M_{\phi,\psi}) \xrightarrow{(\pi_0)_*} K_0(A) \to 0 \text{ and} \qquad \text{(e 0.6)}$$

$$0 \to K_0(B) \xrightarrow{\imath_*} K_1(M_{\phi,\psi}) \xrightarrow{(\pi_0)_*} K_1(A) \to 0. \qquad \text{(e 0.7)}$$

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$$K_1(M_{\phi,\psi}) = K_0(B) \oplus K_1(A). \qquad \text{(e 0.11)}$$

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In particular, if $\overline{R}_{\phi,\psi} = 0$, there exists $\Theta \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(M_{\phi,\psi}))$ such that $[\pi_0] \circ \Theta = [\mathrm{id}_A]$ and

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$$\mathrm{ker} R_{\phi,\psi} = \mathrm{ker} \rho_B \oplus K_1(A).$$

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$$\mathrm{Ell}(A) = (G_0, (G_0)_+, u, G_1, \Delta, r).$$

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Moreover, A can be constructed as an inductive limit of inductive limits of subhomogenuous C^*-algebras with dimension of base spaces no more than three and A is $\mathcal{Z}$-stable.