Dynamics, dimension and classification of C^*-algebras

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Dimension and C*-algebraic regularity

Dynamic versions of dimension and regularity

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- $(U_\lambda)_{\lambda \in \Lambda}$ refines $\mathcal{V}$
- $\Lambda = \Lambda^{(0)} \cup \ldots \cup \Lambda^{(n)}$ and for each $i \in \{0, \ldots, n\}$, the $(U_\lambda)_{\Lambda^{(i)}}$ are pairwise disjoint.

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$$\varphi \circ \psi =_{\mathcal{F},\varepsilon} \mathrm{id}_A,$$

and such that F can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

with c.p.c. order zero maps

$$\varphi^{(i)} := \varphi|_{F^{(i)}}.$$

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$$\phi^{(i)} : M_k \oplus M_{k+1} \to A,\ i \in \{0, \ldots, n\},$$

such that

$$\sum_{i=0}^{n} \phi^{(i)}(1_k \oplus 1_{k+1}) \geq 1_A.$$

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A C*-algebra A is $\mathcal{Z}$-stable if and only if for every $k \in \mathbb{N}$ there are c.p.c. order zero maps

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such that

$$\Psi(e_{11}) = \mathbf{1} - \Phi(\mathbf{1}_{M_k})$$

and

$$\Phi(e_{11})\Psi(e_{22}) = \Psi(e_{22})\Phi(e_{11}) = \Psi(e_{22}).$$

DEFINITION

A unital simple C*-algebra A has tracial m-comparison, if whenever $0 \neq a, b \in M_\infty(A)_+$ satisfy

$$d_\tau(a) < d_\tau(b)$$

for all $\tau \in T(A)$, then

$$a \precsim b^{\oplus m+1}.$$

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$$\mathcal{E} = \{\mathcal{C}(X) \rtimes_\alpha \mathbb{Z} \mid X \text{ compact, metrizable, infinite,} \\ \alpha \text{ induced by a uniquely ergodic, minimal homeomorphism}\}.$$

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Moreover, the regularity properties ensure classification by ordered K-theory in this case. (Countable structures are sufficient for classification since $T(A)$ is a singleton for each A.)

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REMARK We think of $n+1$ as the number of colors, of $K^{(i)}$ as the the number of towers of color i, and of L as the length of the towers.

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- for each fixed i, the sets $V_k^{(i)}$ are pairwise disjoint
- the $U_k^{(i)}$ cover all of Y.

DEFINITION

We say $(X, \mathbb{Z}, \alpha)$ (α minimal) has dynamic m-comparison, if, whenever $U, V \subset X$ are open subsets with $\mu(U) < \mu(V)$ for any regular invariant Borel probability measure μ on X, then $U \precsim_m V$.

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If $(X, \mathbb{Z}, \alpha)$ is dynamically $\mathcal{Z}$-stable, then $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable.

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For the proof, one has to construct invariant measures from a system of open coverings of the form

$$(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{1, \ldots, L\})$$

(as in the definition of dynamic dimension), which become finer and finer, and for which L becomes larger and larger.

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For $V \subset X$ open, $\mu(V)$ is then defined as a limit along some ultrafilter of expressions like

$$\frac{\sharp\{l \mid U_{k,l}^{(i)} \subset V\}}{L}.$$

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and

$$\dim(X, \mathbb{Z}, \alpha) \leq 2(\dim X + 1)^2 - 1.$$

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We do not know, however, whether this ensures classifiability.

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(This plays a crucial role in their proof of the Farrell–Jones conjecture for hyperbolic groups.)

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