A surface-aware projection basis for oceanic flows

K S Smith[1] J Vanneste[2]

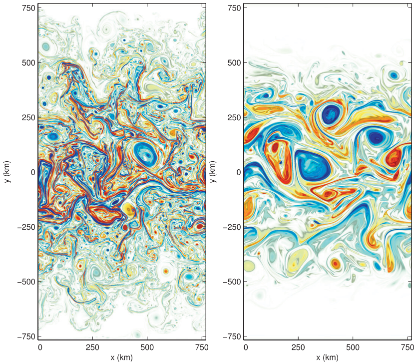
[1]Courant Institute, New York University

[2]School of Mathematics and Maxwell Institute
University of Edinburgh

Motivation

High-resolution numerical modelling and satellite observations suggest ocean turbulence is in a surface quasi-geostrophic regime near the surface.

Surface vorticity



Baroclinic instability with $b_y \neq 0$ (left) and with $b_y = 0$ (right) (Roullet et al, JPO, 2012)

Interior and surface motion

Recall quasi-geostrophic model:

$$\partial_t q + \partial(\psi, q) = 0, \quad \text{and} \quad \partial_t b + \partial(\psi, b) = 0 \ \text{ at } \ z = z^{\pm},$$

with the inversion

$$\partial_{xx}\psi + \partial_{yy}\psi + \partial_z \left(\frac{f^2}{N^2} \partial_z \psi \right) = q \quad \text{and} \quad \partial_z \psi = b/f \ \text{ at } \ z = z^{\pm}.$$

Three dynamical variables:

- potential vorticity $q(x, y, z, t)$,
- surface and bottom buoyancy $b(x, y, z^{\pm}, t)$.

Simplified models:

- QG turbulence: $b(x, y, z^{\pm}, t) = \mathrm{const.}$,
- SQG turbulence: $q = 0$.

Interior and surface motion

Recall quasi-geostrophic model:

$$\partial_t q + \partial(\psi, q) = 0, \quad \text{and} \quad \partial_t b + \partial(\psi, b) = 0 \quad \text{at} \quad z = z^{\pm},$$

with the inversion

$$\partial_{xx}\psi + \partial_{yy}\psi + \partial_z \left(\frac{f^2}{N^2} \partial_z \psi \right) = q \quad \text{and} \quad \partial_z \psi = b/f \quad \text{at} \quad z = z^{\pm}.$$

Three dynamical variables:

- potential vorticity $q(x, y, z, t)$,
- surface and bottom buoyancy $b(x, y, z^{\pm}, t)$.

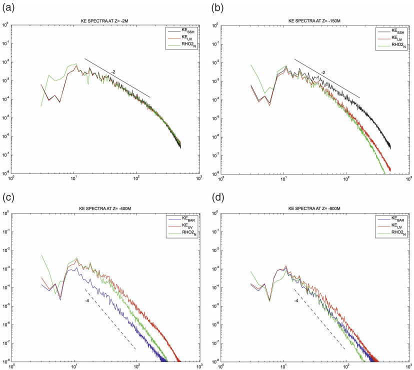
Simplified models:

- QG turbulence: $b(x, y, z^{\pm}, t) = \mathrm{const.}$,
- SQG turbulence: $q = 0$.

Interior and surface motion

Predictions:

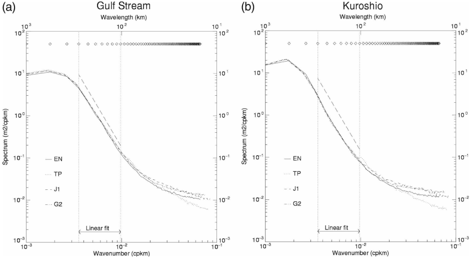
	QG	SQG
energy spectrum	k^{-3}	$k^{-5/3}$
SSH spectrum	k^{-5}	$k^{-11/3}$
Rossby number	k^{0}	$k^{2/3}$



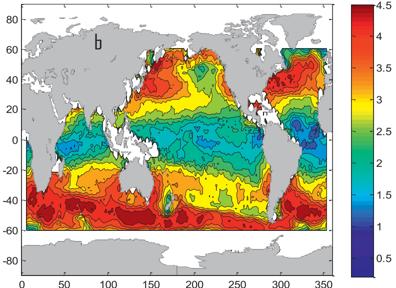
Spectra in primitive equation simulations (Klein et al, JPO, 2009)

Interior and surface motion

Observed SSH: SQG $k^{-11/3}$ spectrum in energetic regions.



Le Traon et al (JPO, 2009)



Xu and Fu (JPO, 2011, 2012)

Interior and surface motion

Vertical structure of SQG motion:

$$\hat{q} = 0 \quad \Rightarrow \quad \partial_z \left(\frac{f^2}{N^2} \partial_z \hat{\psi} \right) - \kappa^2 \hat{\psi} = 0 \quad \Rightarrow \quad \hat{\psi} \propto \mathrm{e}^{N\kappa z/f}$$

for Fourier mode (k, l) with $\kappa^2 = k^2 + l^2$.

- Exponential decay from surface,
- non-zero surface buoyancy $b(z^{\pm}) = f \partial_z \hat{\psi}(z^{\pm}) \neq 0$.

A difficulty:

Vertical structure of SQG motion is poorly represented by standard basis of barotropic + baroclinic modes.

Interior and surface motion

Vertical structure of SQG motion:

$$\hat{q} = 0 \quad \Rightarrow \quad \partial_z \left(\frac{f^2}{N^2} \partial_z \hat{\psi} \right) - \kappa^2 \hat{\psi} = 0 \quad \Rightarrow \quad \hat{\psi} \propto \mathrm{e}^{N\kappa z/f}$$

for Fourier mode (k, l) with $\kappa^2 = k^2 + l^2$.

- Exponential decay from surface,
- non-zero surface buoyancy $b(z^{\pm}) = f \partial_z \hat{\psi}(z^{\pm}) \neq 0$.

A difficulty:

Vertical structure of SQG motion is poorly represented by standard basis of barotropic + baroclinic modes.

Modal expansion

Standard basis

Standard basis of baroclinic modes:

Eigenfunctions of

$$\left(\frac{f^2}{N^2}\psi_n'\right)' = -\lambda_n^2\psi_n, \quad \text{with} \quad \psi_n' = 0 \text{ at } \quad z = 0, \ -H.$$

For constant N: $\psi_n \sim \cos(n\pi z/H)$, $n = 0, 1, \cdots$.

Advantages:

- orthogonal basis, $\int_{-H}^{0} \psi_n \psi_m \, \mathrm{d}z \propto \int_{-H}^{0} \nabla\psi_n \cdot \nabla\psi_m \, \mathrm{d}z \propto \delta_{mn}$,
- diagonalise energy,
- describes (interior) QG dynamics with a few modes,
- mode structure independent of κ.

Heavily used: projection of data, basis for simplified models. . .

Modal expansion
Standard basis

Standard basis of baroclinic modes:

Eigenfunctions of

$$\left(\frac{f^2}{N^2}\psi_n'\right)' = -\lambda_n^2\psi_n, \quad \text{with} \quad \psi_n' = 0 \text{ at } \quad z = 0, -H.$$

For constant N: $\psi_n \sim \cos(n\pi z/H)$, $n = 0, 1, \cdots$.

Advantages:

- orthogonal basis, $\int_{-H}^{0} \psi_n\psi_m \,\mathrm{d}z \propto \int_{-H}^{0} \nabla\psi_n \cdot \nabla\psi_m \,\mathrm{d}z \propto \delta_{mn}$,
- diagonalise energy,
- describes (interior) QG dynamics with a few modes,
- mode structure independent of κ.

Heavily used: projection of data, basis for simplified models...

Modal expansion
Standard basis

Difficulty:

- basis unsuitable to describe SQG-like motion since $f\psi_n' = b = 0$ at $z = 0, -H$,
- non-uniform convergence for surface modes $\mathrm{e}^{N\kappa z/f} = \sum_n A_n \cos(n\pi z/H)$,
- many modes needed to represent motion with surface activity.

Need to find an alternative, 'surface-aware' basis.

Some attempts:

- Tulloch & Smith (JAS, 2009), Lapeyre (JPO, 2009): add SQG mode $\mathrm{e}^{-N\kappa z/f}$ to standard basis,
- Scott & Furnival (JPO, 2012): add barotropic mode to 'Dirichlet basis' satisfying $\psi_n = 0$ at $z = 0$.

But, non-orthogonal, overcomplete bases.

Modal expansion

Standard basis

Difficulty:

- basis unsuitable to describe SQG-like motion since $f\psi_n' = b = 0$ at $z = 0, -H$,
- non-uniform convergence for surface modes $\mathrm{e}^{N\kappa z/f} = \sum_n A_n \cos(n\pi z/H)$,
- many modes needed to represent motion with surface activity.

Need to find an alternative, 'surface-aware' basis.

Some attempts:

- Tulloch & Smith (JAS, 2009), Lapeyre (JPO, 2009): add SQG mode $\mathrm{e}^{-N\kappa z/f}$ to standard basis,
- Scott & Furnival (JPO, 2012): add barotropic mode to 'Dirichlet basis' satisfying $\psi_n = 0$ at $z = 0$.

But, non-orthogonal, overcomplete bases.

Modal expansion

Standard basis

Difficulty:

- basis unsuitable to describe SQG-like motion since $f\psi_n' = b = 0$ at $z = 0, -H$,
- non-uniform convergence for surface modes $\mathrm{e}^{N\kappa z/f} = \sum_n A_n \cos(n\pi z/H)$,
- many modes needed to represent motion with surface activity.

Need to find an alternative, ‘surface-aware’ basis.

Some attempts:

- Tulloch & Smith (JAS, 2009), Lapeyre (JPO, 2009): add SQG mode $\mathrm{e}^{-N\kappa z/f}$ to standard basis,
- Scott & Furnival (JPO, 2012): add barotropic mode to ‘Dirichlet basis’ satisfying $\psi_n = 0$ at $z = 0$.

But, non-orthogonal, overcomplete bases.

New bases

We derive new surface-aware, orthogonal bases.

Ideas:

- Think of $Q = (q, b^+, b^-)$ not ψ as the dynamical variable to be expanded,
- Recall linear algebra: a unique basis diagonalises **2** quadratic forms $x^\mathrm{T} A x$ and $x^\mathrm{T} B x$ (solve $Ax = \lambda Bx$),
- Choose as quadratic form conserved quantities: energy and 'generalised enstrophy',

$$\int_{-H}^{0} |\nabla\psi|^2 \,\mathrm{d}z \quad \text{and} \quad \int_{-H}^{0} q^2 \,\mathrm{d}z + \alpha_+ (b^+)^2 + \alpha_- (b^-)^2.$$

Family of bases parameterised by $\alpha_\pm$.

New bases

We derive new surface-aware, orthogonal bases.

Ideas:

- Think of $Q = (q, b^+, b^-)$ not ψ as the dynamical variable to be expanded,
- Recall linear algebra: a unique basis diagonalises **2** quadratic forms $x^{\mathrm{T}}Ax$ and $x^{\mathrm{T}}Bx$ (solve $Ax = \lambda Bx$),
- Choose as quadratic form conserved quantities: energy and 'generalised enstrophy',

$$\int_{-H}^{0} |\nabla\psi|^2 \,\mathrm{d}z \quad \text{and} \quad \int_{-H}^{0} q^2 \,\mathrm{d}z + \alpha_+ (b^+)^2 + \alpha_- (b^-)^2.$$

Family of bases parameterised by $\alpha_\pm$.

New bases

We derive new surface-aware, orthogonal bases.

Ideas:

- Think of $Q = (q, b^+, b^-)$ not ψ as the dynamical variable to be expanded,
- Recall linear algebra: a unique basis diagonalises **2** quadratic forms $x^{\mathrm{T}}Ax$ and $x^{\mathrm{T}}Bx$ (solve $Ax = \lambda Bx$),
- Choose as quadratic form conserved quantities: energy and 'generalised enstrophy',

$$\int_{-H}^{0} |\nabla\psi|^2 \, \mathrm{d}z \quad \text{and} \quad \int_{-H}^{0} q^2 \, \mathrm{d}z + \alpha_+ (b^+)^2 + \alpha_- (b^-)^2.$$

Family of bases parameterised by $\alpha_\pm$.

New bases

We derive new surface-aware, orthogonal bases.

Ideas:

- Think of $Q = (q, b^+, b^-)$ not ψ as the dynamical variable to be expanded,
- Recall linear algebra: a unique basis diagonalises **2** quadratic forms $x^{\mathrm{T}}Ax$ and $x^{\mathrm{T}}Bx$ (solve $Ax = \lambda Bx$),
- Choose as quadratic form conserved quantities: energy and ‘generalised enstrophy’,

$$\int_{-H}^{0} |\nabla\psi|^2 \, \mathrm{d}z \quad \text{and} \int_{-H}^{0} q^2 \, \mathrm{d}z + \alpha_+ (b^+)^2 + \alpha_- (b^-)^2.$$

Family of bases parameterised by $\alpha_\pm$.

New bases

We derive new surface-aware, orthogonal bases.

Ideas:

- Think of $Q = (q, b^{+}, b^{-})$ not ψ as the dynamical variable to be expanded,
- Recall linear algebra: a unique basis diagonalises **2** quadratic forms $x^{\mathrm{T}}Ax$ and $x^{\mathrm{T}}Bx$ (solve $Ax = \lambda Bx$),
- Choose as quadratic form conserved quantities: energy and 'generalised enstrophy',

$$\int_{-H}^{0} |\nabla\psi|^2 \,\mathrm{d}z \quad \text{and} \int_{-H}^{0} q^2 \,\mathrm{d}z + \alpha_{+}(b^{+})^2 + \alpha_{-}(b^{-})^2.$$

Family of bases parameterised by $\alpha_{\pm}$.

New bases

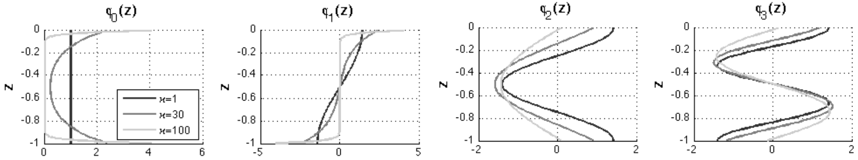
Basis vectors: eigenfunctions of

$$\left(\frac{f^2}{N^2}\psi_n'\right)' = -\lambda_n^2\psi_n, \quad \text{with} \quad \frac{f^2}{N^2 H}\psi_n' = \pm\frac{\lambda_n^2+\kappa^2}{\alpha_\pm}\psi_n \text{ at } z = 0, -H.$$

Limiting cases:

$\alpha_\pm \to \infty$: reduces to standard baroclinic basis for $n = O(1)$,

$\alpha_\pm \to 0$: 'Dirichlet basis' with $\psi_n = 0$ at $z = 0, -H$ + 2 SQG modes ($q = 0$) and imaginary λ_n.



New bases

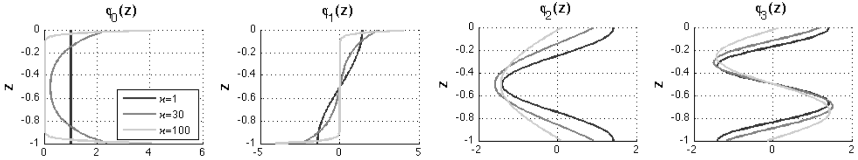
Basis vectors: eigenfunctions of

$$\left(\frac{f^2}{N^2}\psi_n'\right)' = -\lambda_n^2\psi_n, \quad \text{with} \quad \frac{f^2}{N^2 H}\psi_n' = \pm\frac{\lambda_n^2+\kappa^2}{\alpha_\pm}\psi_n \text{ at } z = 0, -H.$$

Limiting cases:

$\alpha_\pm \to \infty$: reduces to standard baroclinic basis for $n = O(1)$,

$\alpha_\pm \to 0$: 'Dirichlet basis' with $\psi_n = 0$ at $z = 0, -H$
+ 2 SQG modes ($q = 0$) and imaginary λ_n.

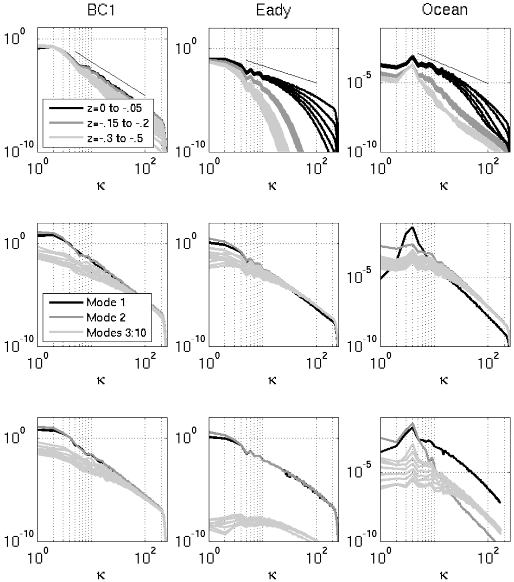


New bases

Application

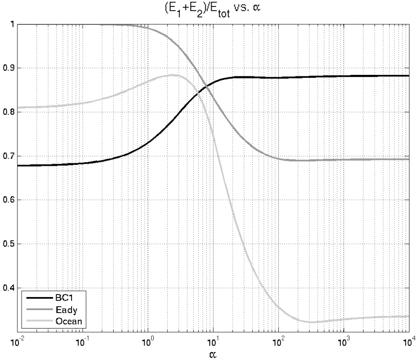
3 QG simulations of baroclinic instability:
(1) interior BC1,
(2) surface Eady,
(3) mixed Ocean.

$\alpha_{-} \to \infty$



New bases

Choosing α_+: maximise energy content of first 2 modes



Conclusion

- Effects of surface buoyancy gradients cannot be ignored in ocean turbulence,
- Eddies have rich, surface-intensified vertical structure that is not well-represented by standard vertical modes,
- New bases presented can capture most energy in such flows with a small truncation set,
- New bases can be very simple:

$$\psi_0 \propto \cosh\left[N\kappa(z+H)/f\right], \quad \psi_n \propto \sin\left[(n-1/2)\pi z/H\right].$$

- New bases depend on κ: coupling of horizontal and vertical structures.