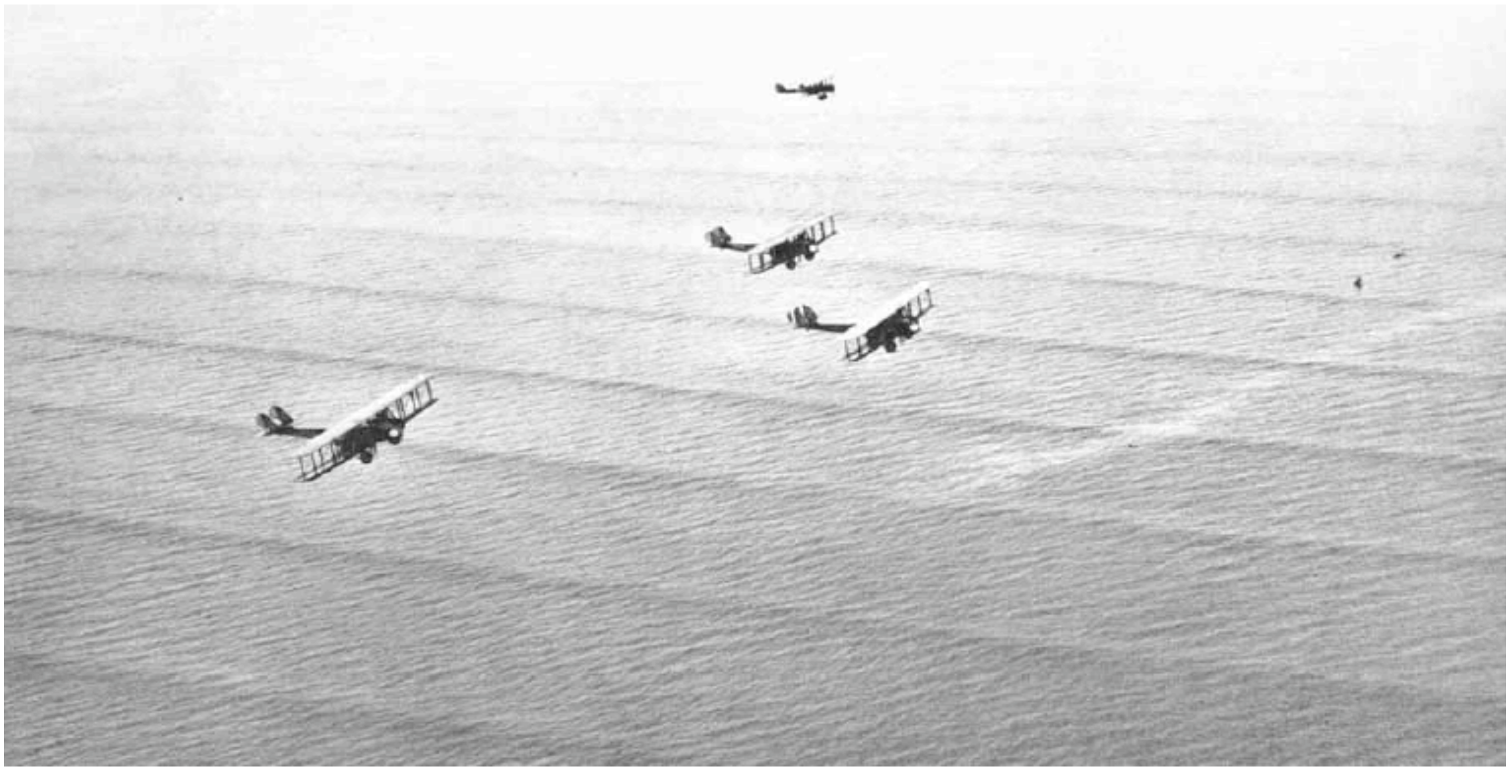
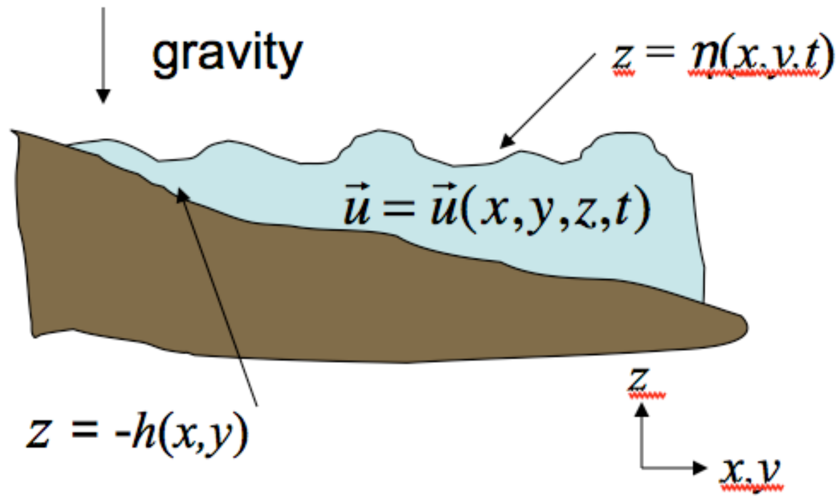
The nonlinear Schrödinger equation, dissipation and ocean swell



Workshop on Ocean Wave Dynamics - Fields

Diane Henderson, Harvey Segur
Penn State U U of Colorado

Preliminaries:
Stokes' equations of water waves (1847)



$$\partial_t \eta + \nabla\phi \cdot \nabla \eta = \partial_z \phi,$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g\eta = \frac{\sigma}{\rho} \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right), \qquad \text{on } z = \eta(x,y,t),$$

$$\Delta \phi = 0 \qquad -h(x,y) < z < \eta(x,y,t),$$

$$\partial_z \phi + \nabla \phi \cdot \nabla h = 0 \qquad \text{on } z = -h(x,y).$$

Overall objective:

Find a good (approximate) model, to predict accurately the evolution of ocean swell as it propagates over long distances in the ocean.

Candidate #1: nonlinear Schrödinger eq'n

Candidate #2: damped nonlinear Schrödinger eq'n

Candidate #3: ???

Chapter 1:

Nonlinear Schrödinger equation

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma |A|^2 A = 0$$

(Zakharov, 1968)

An approximate model for waves on deep water:

Chapter 1:

Nonlinear Schrödinger equation

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\, |A|^2\, A = 0$$

(Zakharov, 1968)

An approximate model for waves on deep water:

$$\eta(X,Y,T;\varepsilon) \sim \varepsilon[A(\varepsilon(X - c_g T), \varepsilon Y, \varepsilon^2 X)\cdot e^{i\theta} + A^* e^{-i\theta}] + O(\varepsilon^2)$$

surface elevation — slow modulation — fast oscillations

BIG discovery in the 1960s:

The modulational instability (or Benjamin-Feir instability) was discovered by several people, in different scientific disciplines, in different countries, using different methods:

Lighthill (1965), Whitham (1967), Zakharov (1967, 1968), Ostrovsky (1967), Benjamin & Feir (1967), Benjamin (1967), Benney & Newell (1967),...

Modulational instability

- Dispersive medium: waves at different frequencies travel at different speeds

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- In a dispersive medium without dissipation, a uniform train of plane waves of finite amplitude is likely to be unstable

Modulational instability

- Dispersive medium: waves at different frequencies travel at different speeds
- In a dispersive medium without dissipation, a uniform train of plane waves of finite amplitude is likely to be unstable
- Maximum growth rate of (nonlinear) instability:

$$\Omega = K\left|A_0\right|^2$$

$\left|A_0\right|$ = amplitude of carrier wave

Experimental evidence of modulational instability in deep water - Benjamin (1967)

near the wavemaker
"uniform" wavetrain

60 m downstream
"disintegrated" mess

frequency = 0.85 Hz, wavelength = 2.2 m,
water depth = 7.6 m

Experimental evidence of modulational instability of EM waves in an optical fiber

Effect of fiber loss 287

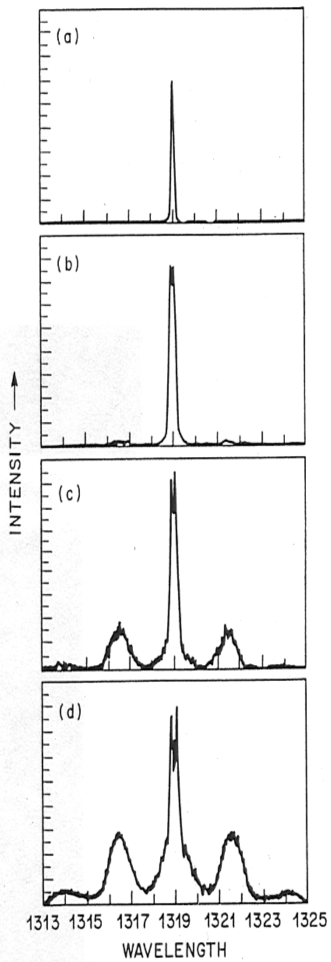


Fig.15.1 Experimental observation of modulational instability (Tai *et al.* 1986a). Input power level low (a); 5.5 *W* (b); 6.1 *W* (c); 7.1 *W* (d). For details see text.

Tai, Hasegawa & Tomita (1986)

L = 1.3*10^{-6} m,

T = 4*10^{-15} s

Recall:

$$\Omega = K|A_0|^2$$

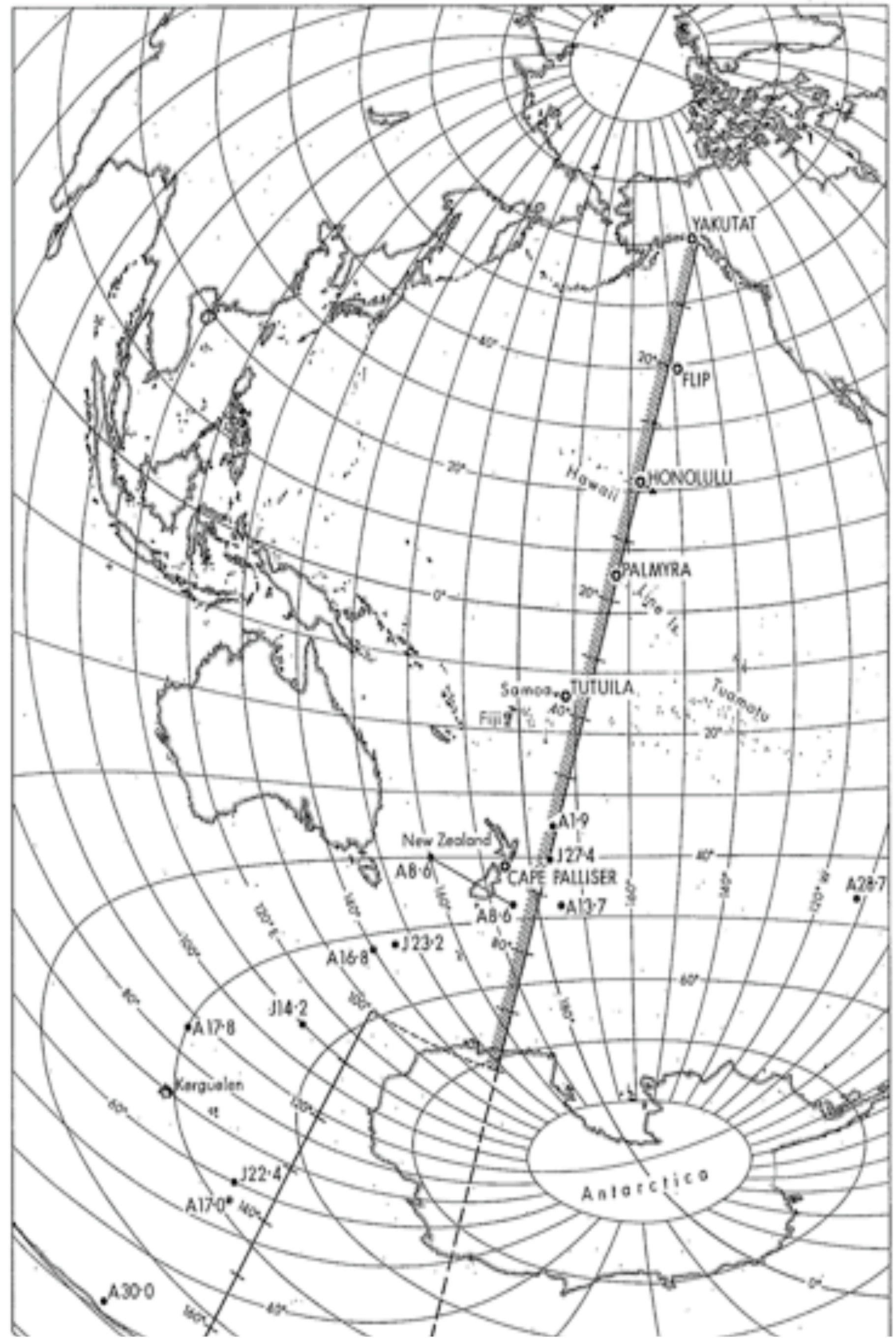
Questions:

map from

Snodgrass *et al*, 1966

Storms near Antarctica generated ocean swell that propagated 13,000 km across the Pacific.

Q1: If ocean swell is unstable, how do waves travel coherently over 13,000 km?



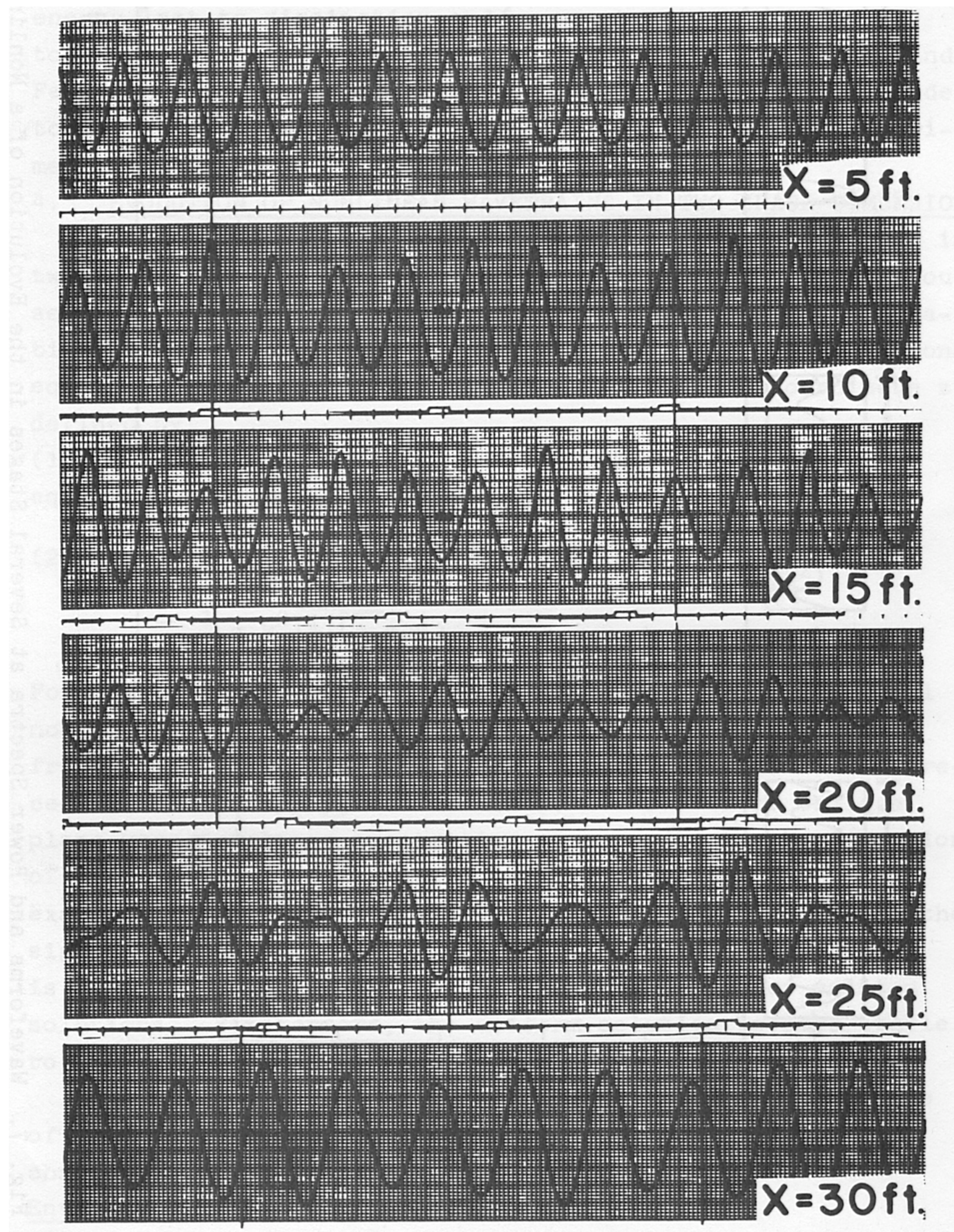
Question 2:

Lake et al (1977) sought experimental evidence of FPU recurrence on deep water

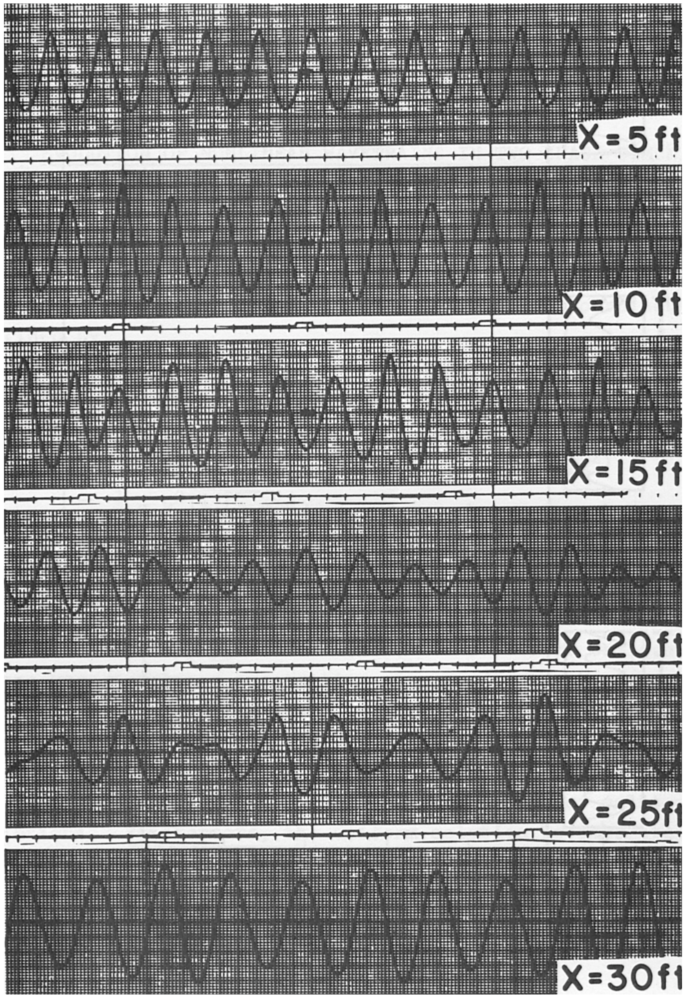
Initial frequency:

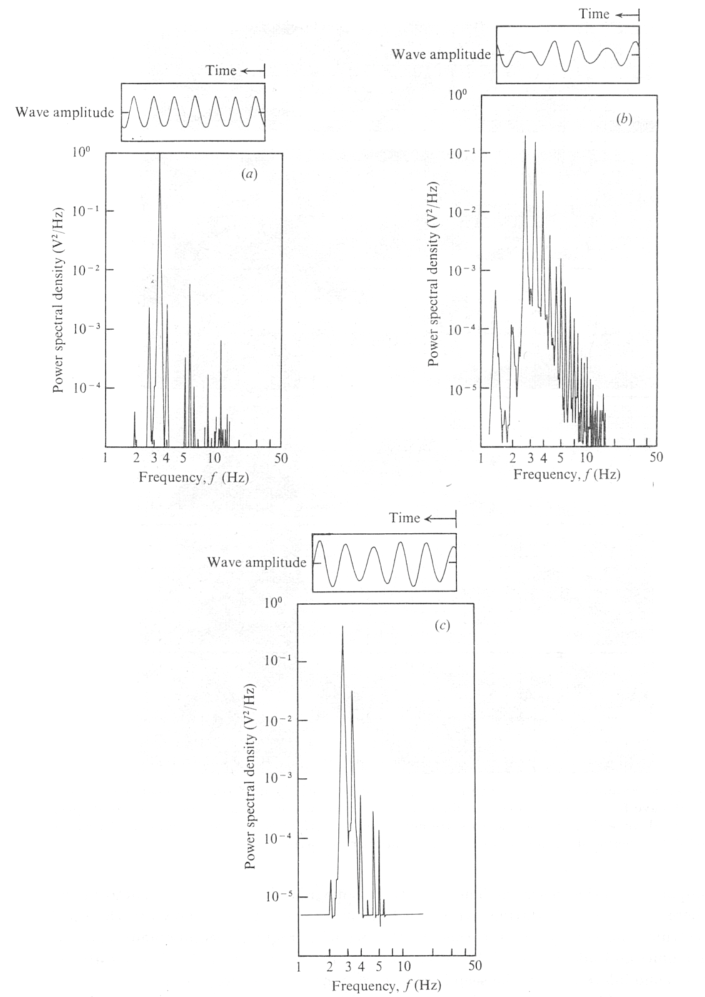
ω = 3.6 Hz

λ = 12 cm



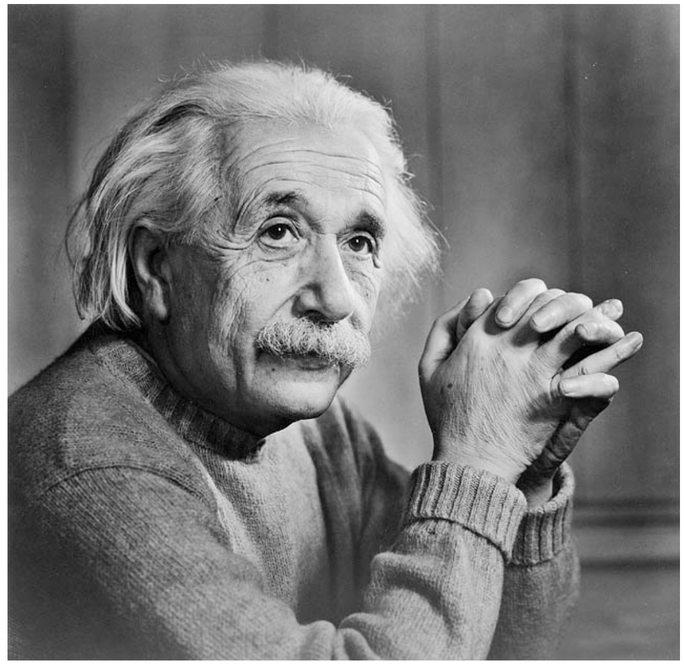
Lake, Yuen, Rungaldier, Ferguson (1977)





Frequency downshifting, which is impossible in NLS

Albert Einstein



"Everything should be made as simple as possible, but not simpler."

Chapter 2:

Damped nonlinear Schrödinger equation

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\,|\,A\,|^2\,A + i\delta A = 0$$

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\,|A|^2\,A + i\delta A = 0$$

Mathematical results (Segur *et al*, 2005)

- A uniform train of oscillatory plane waves of finite amplitude on deep water is unstable if $\delta = 0$.
- But the same wave train is stable for any $\delta > 0$.

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma |A|^2 A + i\delta A = 0$$

Mathematical results (Segur *et al*, 2005)

- A uniform train of oscillatory plane waves of finite amplitude on deep water is unstable if $\delta = 0$.
- But the same wave train is stable for any $\delta > 0$.
- For any $\delta \geq 0$, there is no downshifting, according to damped NLS.

=> This mathematical model has the potential to answer one of the two questions.

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\,|A|^2\, A + i\delta A = 0$$

Mathematical results (Segur *et al*, 2005)

- A uniform train of oscillatory plane waves of finite amplitude on deep water is unstable if $\delta = 0$.
- But the same wave train is stable for any $\delta > 0$.

Q: What makes this instability so unusual?

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\,|A|^2\,A + i\delta A = 0$$

Q: What makes this instability so unusual?

Standard situation: A non-dissipative model predicts an instability with growth rate Ω.

With physical dissipation (not in model), expect:

Observed growth rate =

Predicted growth rate – physical decay rate

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma \mid A \mid^2 A = 0$$

Q: What makes this instability so unusual?

NLS: Predicted growth rate $\Omega = K\left|A_0\right|^2$

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\,|A|^2\,A + i\delta A = 0$$

Q: What makes this instability so unusual?

NLS: Predicted growth rate $\Omega = K\left|A_0\right|^2$

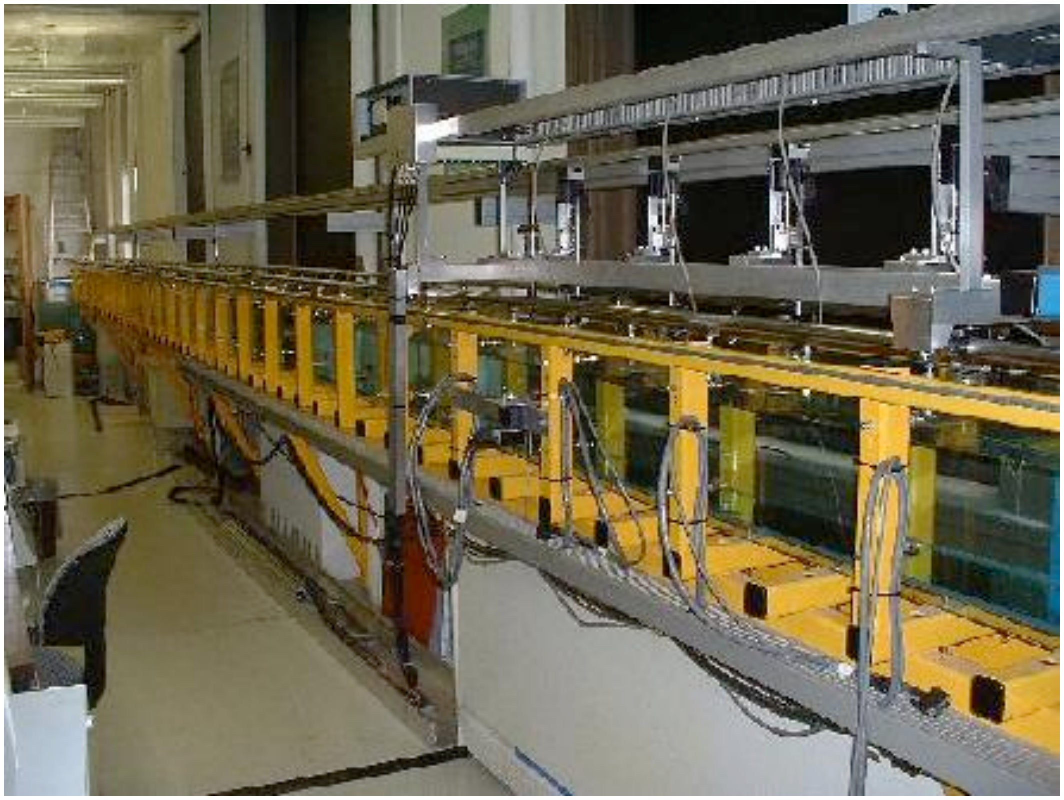
Damped NLS:

$$\Omega = K\left|A_0\right|^2 \cdot e^{-2\left|A_0\right|^2 \tau}$$

Observed growth rate =

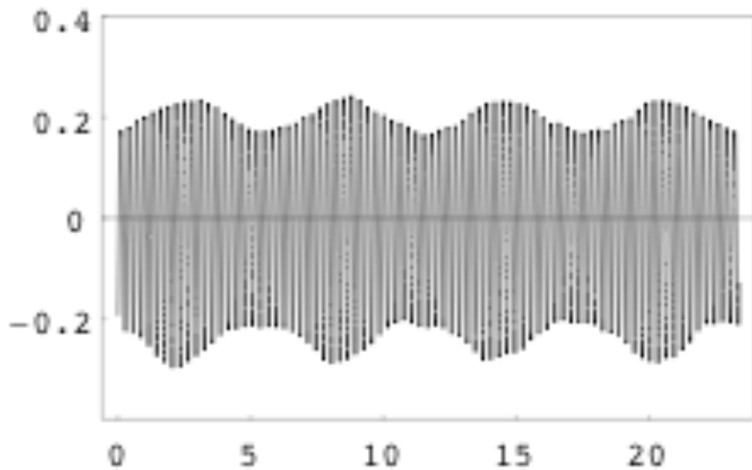
Predicted growth rate – physical decay rate

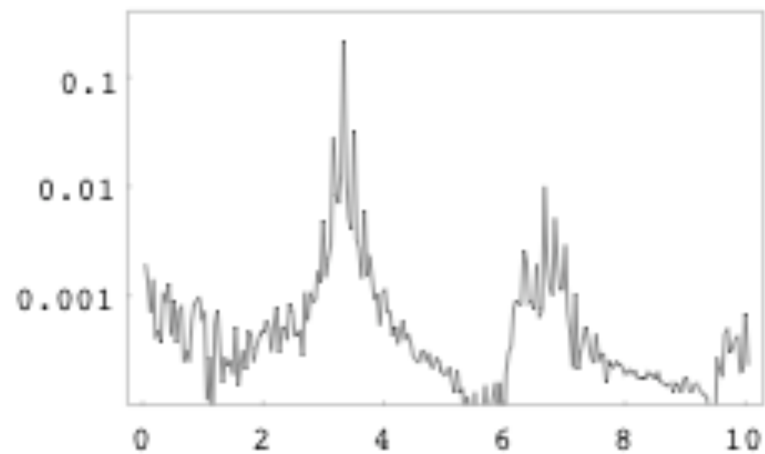
Experimental verification of theory



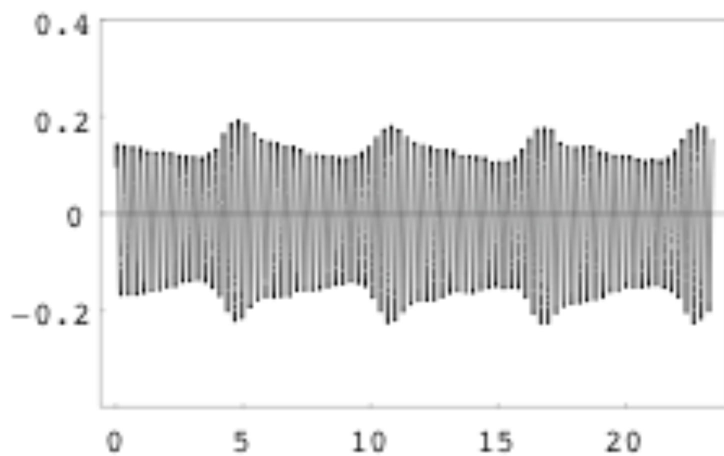
(former) 1-D tank at Penn State

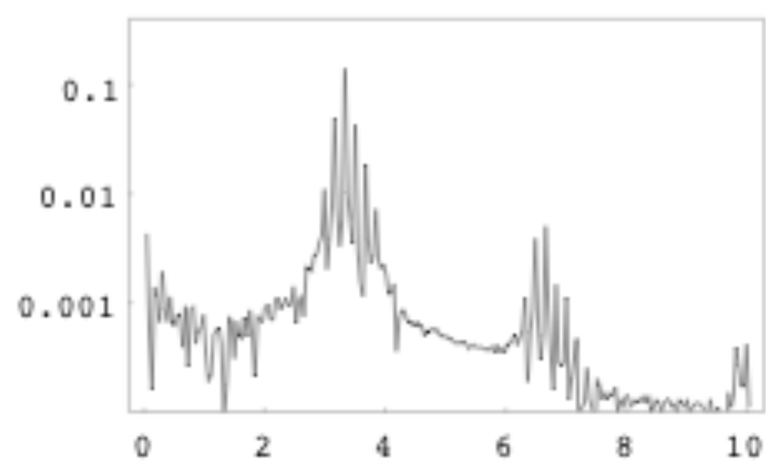
Experimental wave records





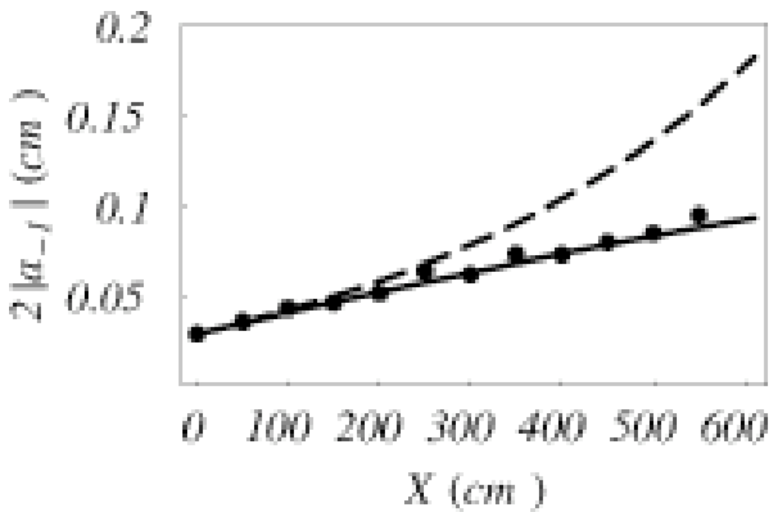
x_1

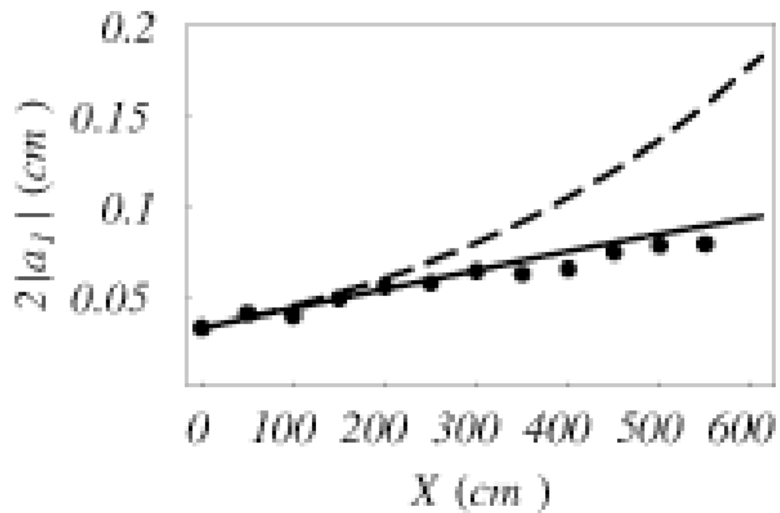




x_6

Amplitudes of seeded sidebands
(damping factored out of data)





(with overall decay factored out)

___ damped NLS theory

- - - Benjamin-Feir growth rate

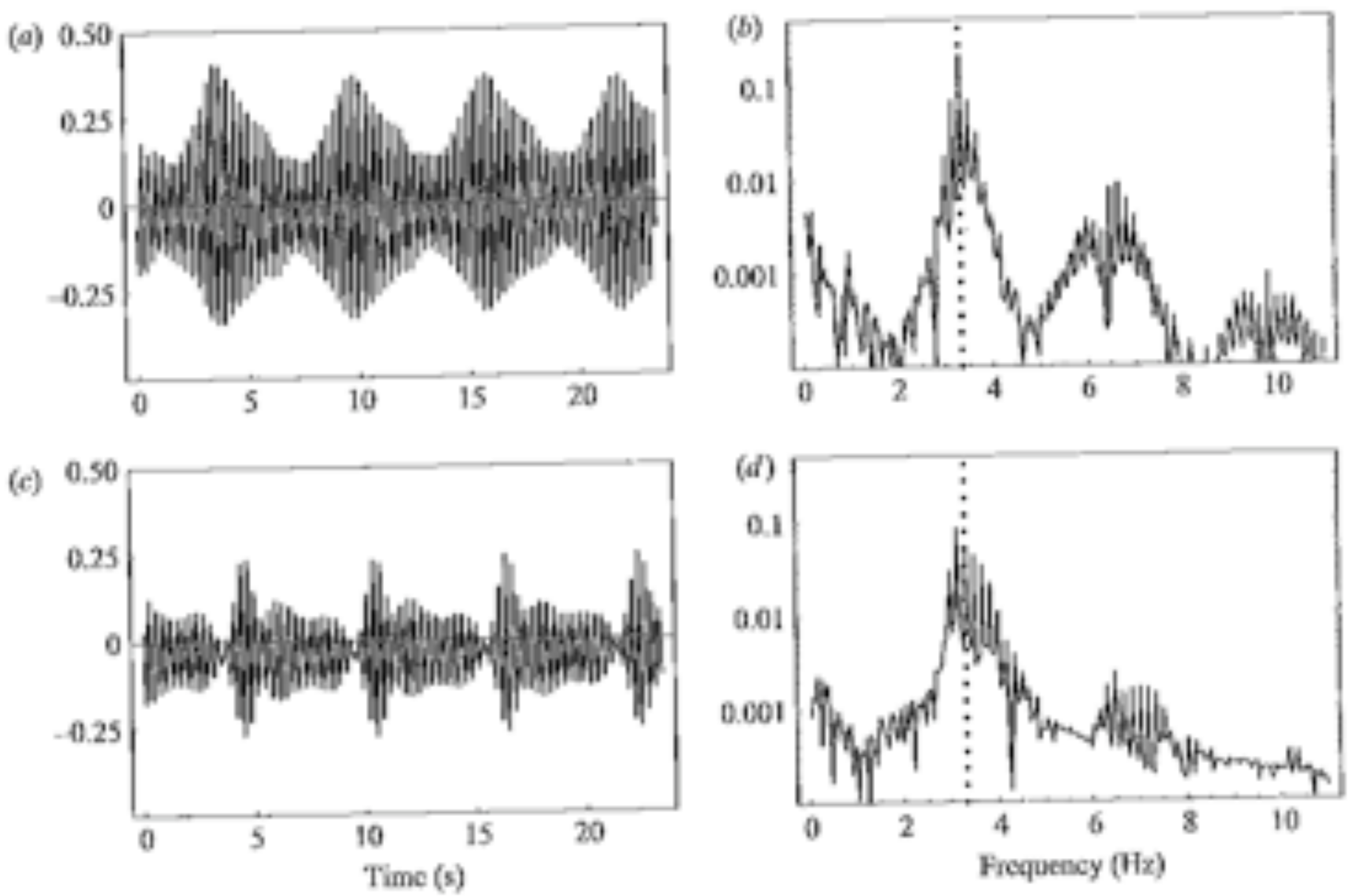
• • • experimental data

Q: What if nonlinearity >> dissipation?

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\,|A|^2\,A + i\delta A = 0$$

Q: What if nonlinearity >> dissipation?
A: Frequency downshifting

not predicted by either NLS ($\delta = 0$ or $\delta > 0$)



x_1

x_{12}

Recall the title of talk:

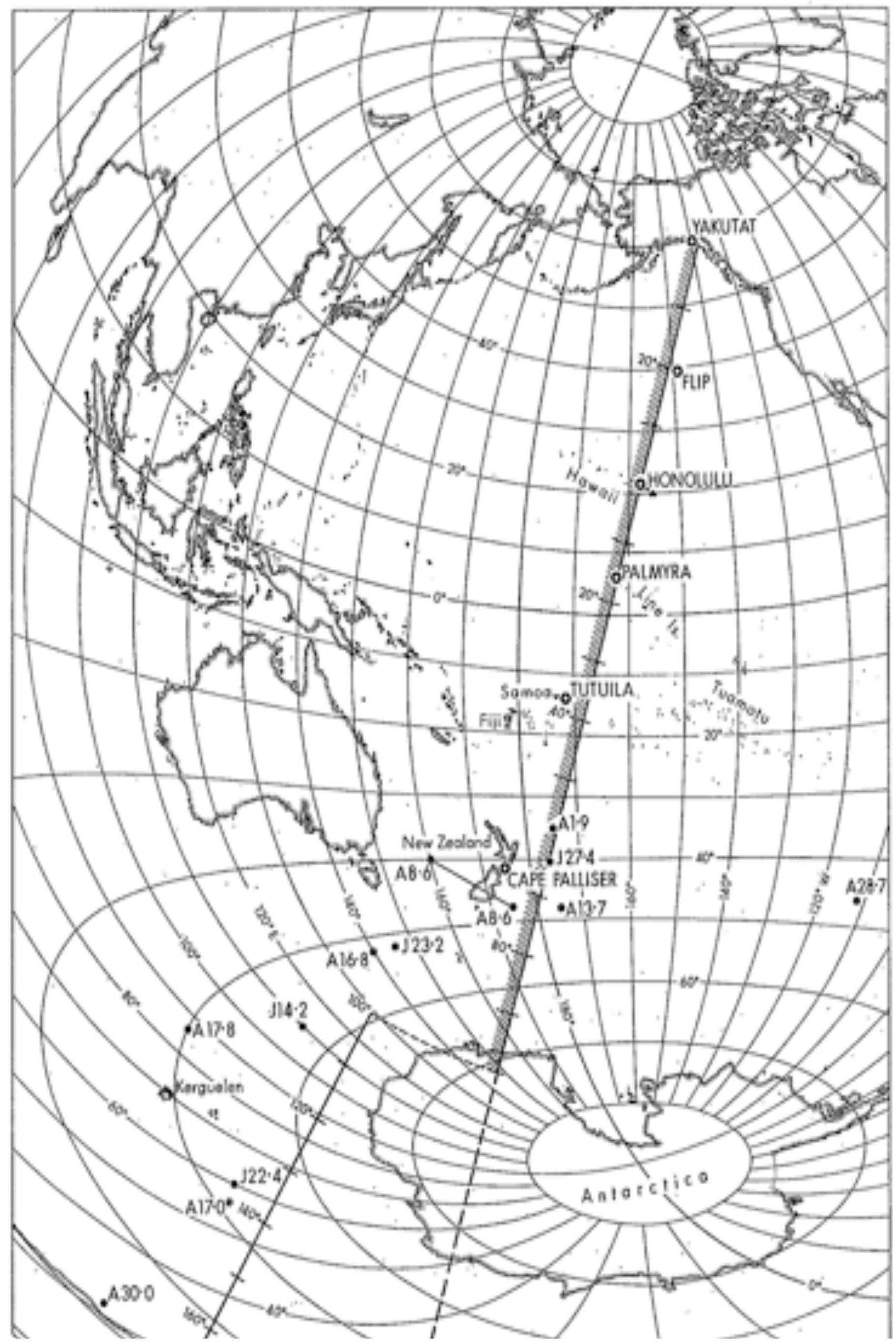
The nonlinear Schrödinger equation, dissipation and ocean swell

Q: Do the theory and the laboratory experiments actually predict what happens to ocean swell?

Recall
Snodgrass *et al*, 1966

Storms near Antarctica generated ocean swell that propagated 13,000 km across the Pacific.

Q: How much dissipation did the swell tracked by Snodgrass *et al* experience?



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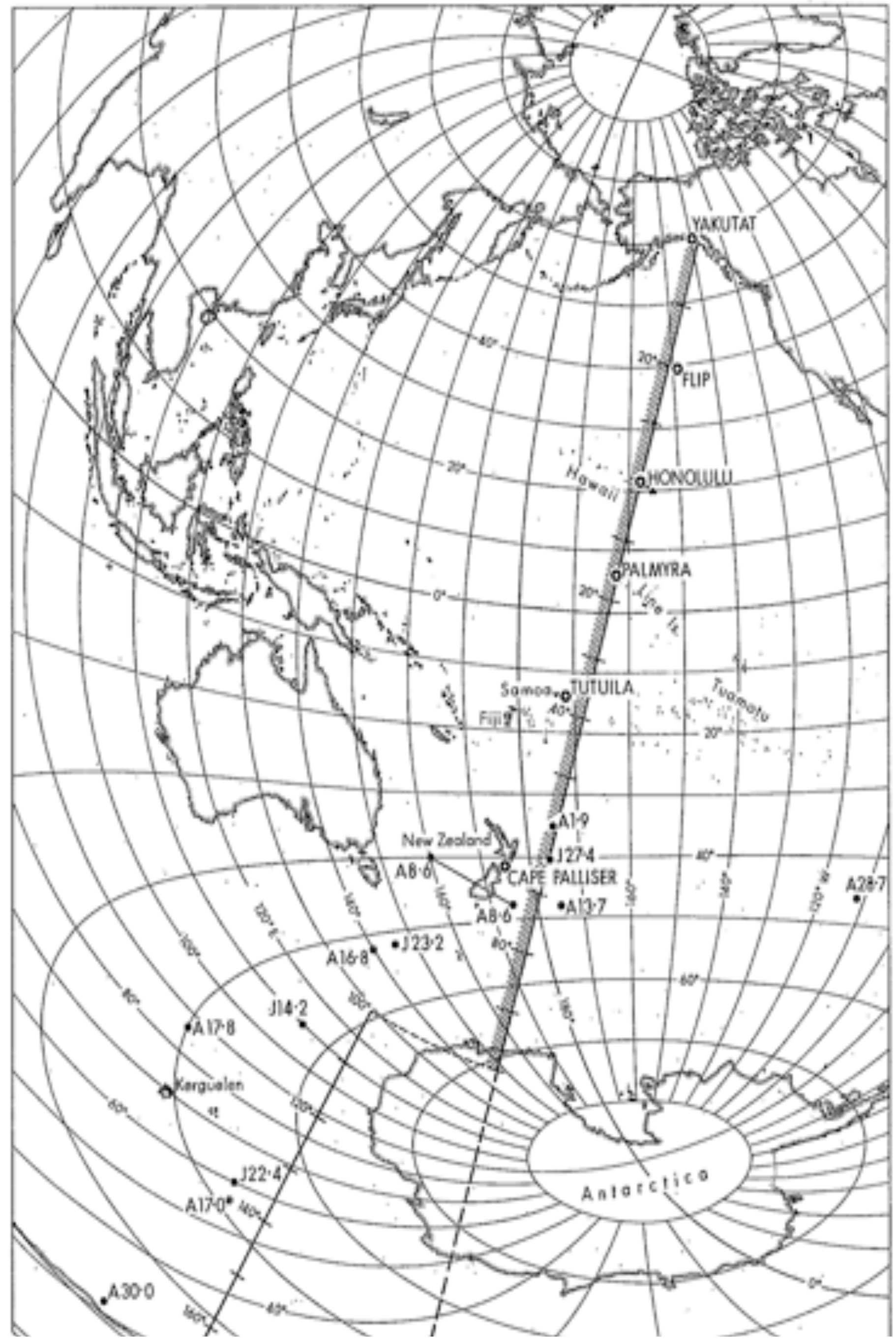
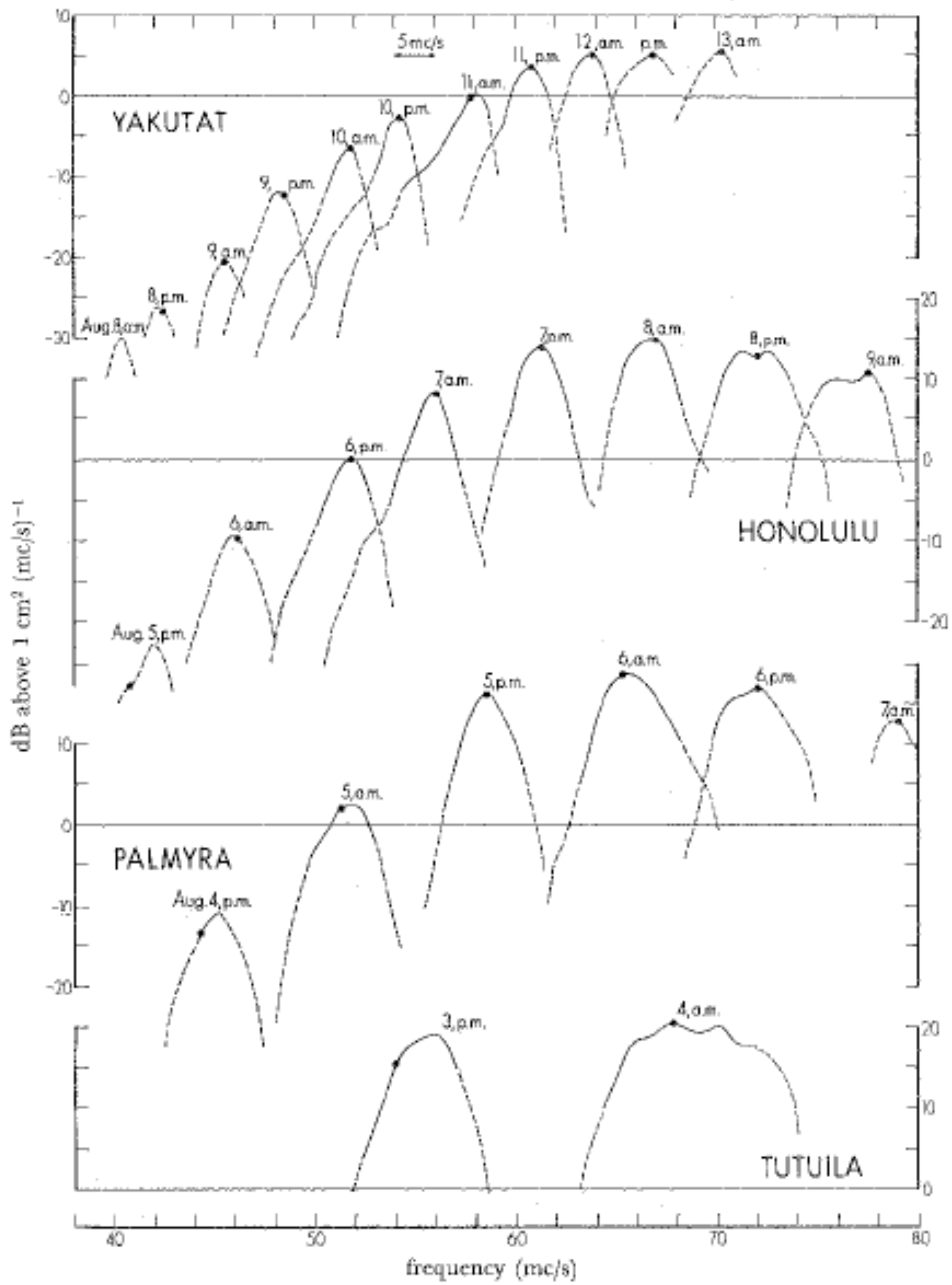


Q: How much dissipation did the swell tracked by Snodgrass *et al* experience?

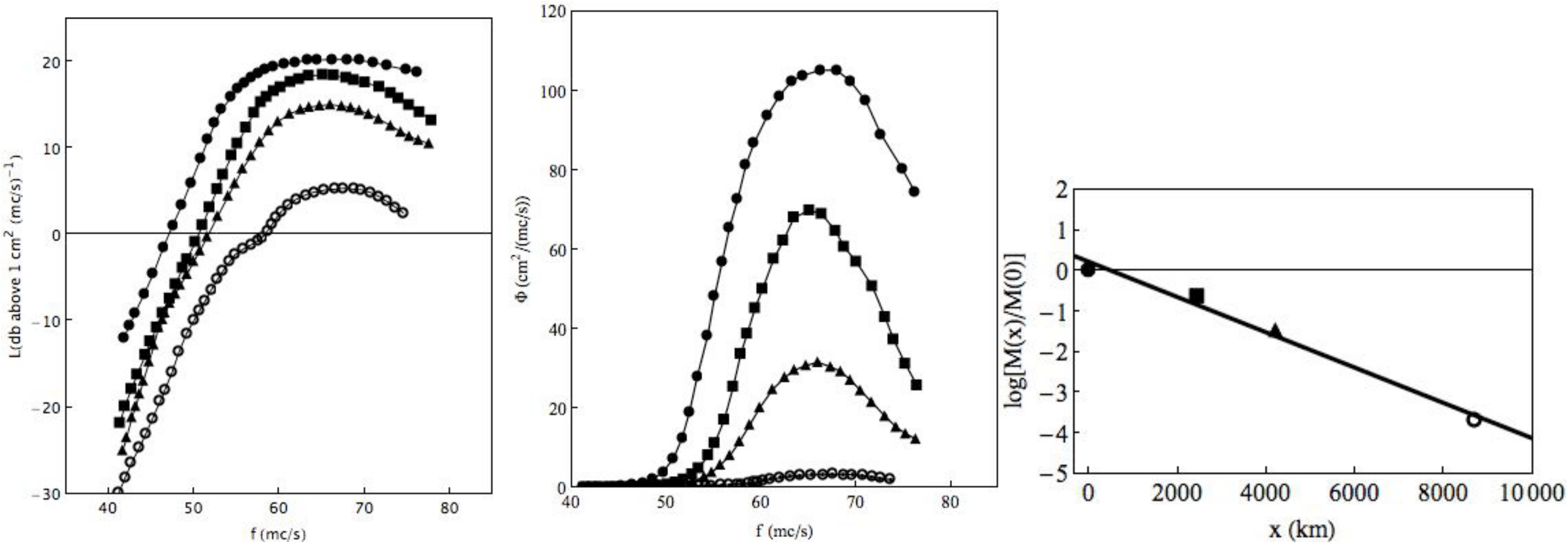
Snodgrass, p.432:
“negligible attenuation”

Figure 20 of Snodgrass et al (1966)

Wave spectra, measured at 12-hour intervals at 4 sequential measuring stations, are narrow at Tutuila, and become narrower at subsequent stations..

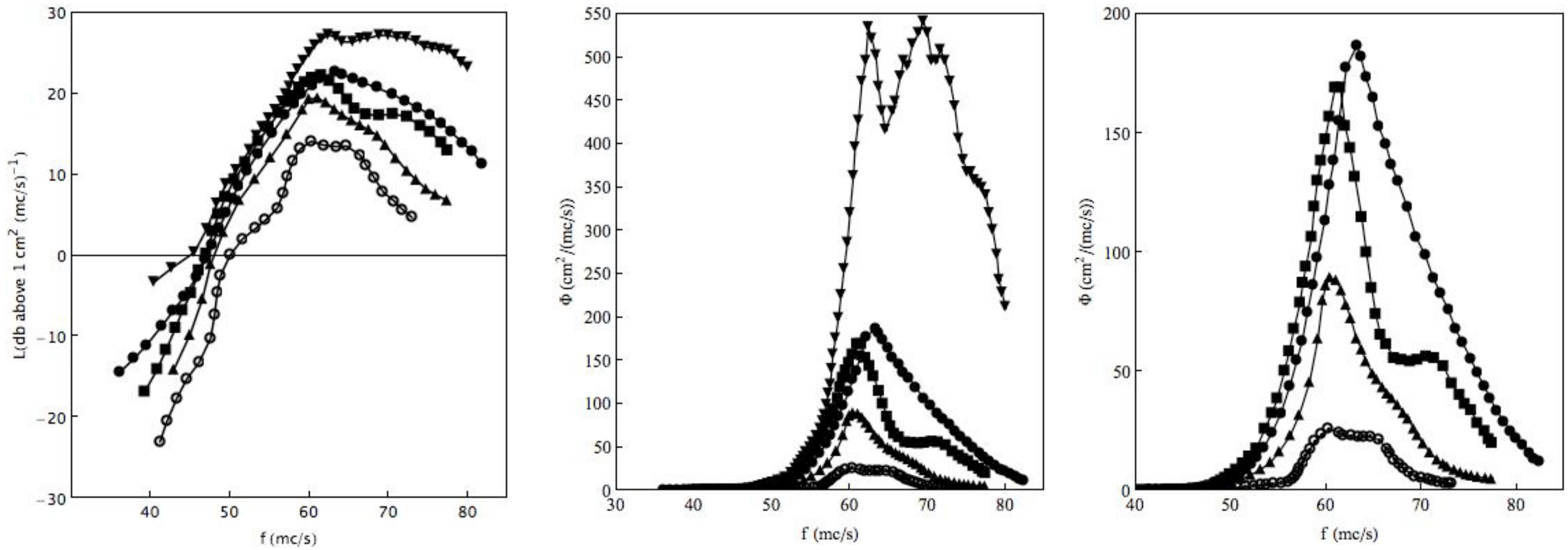


Data from Snodgrass et al (1966)
August 1.9 storm



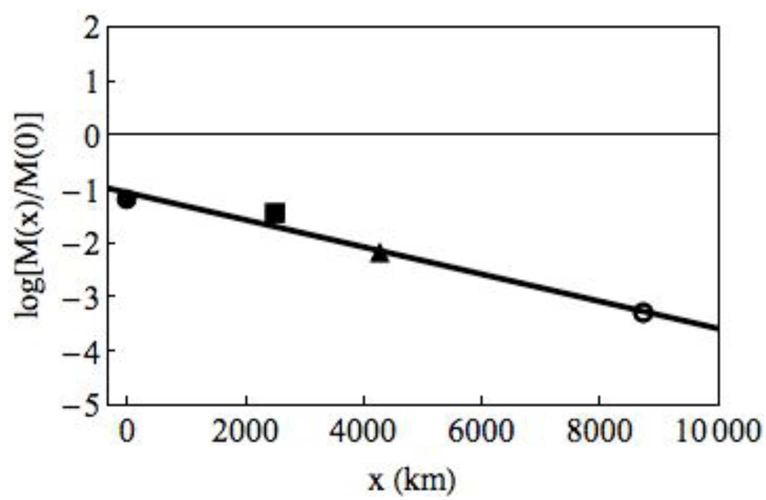
Energy decay rate: $\Delta = 0.43 \times 10^{-3}$ km^{-1}

Data from Snodgrass et al (1966)
August 13.7 storm



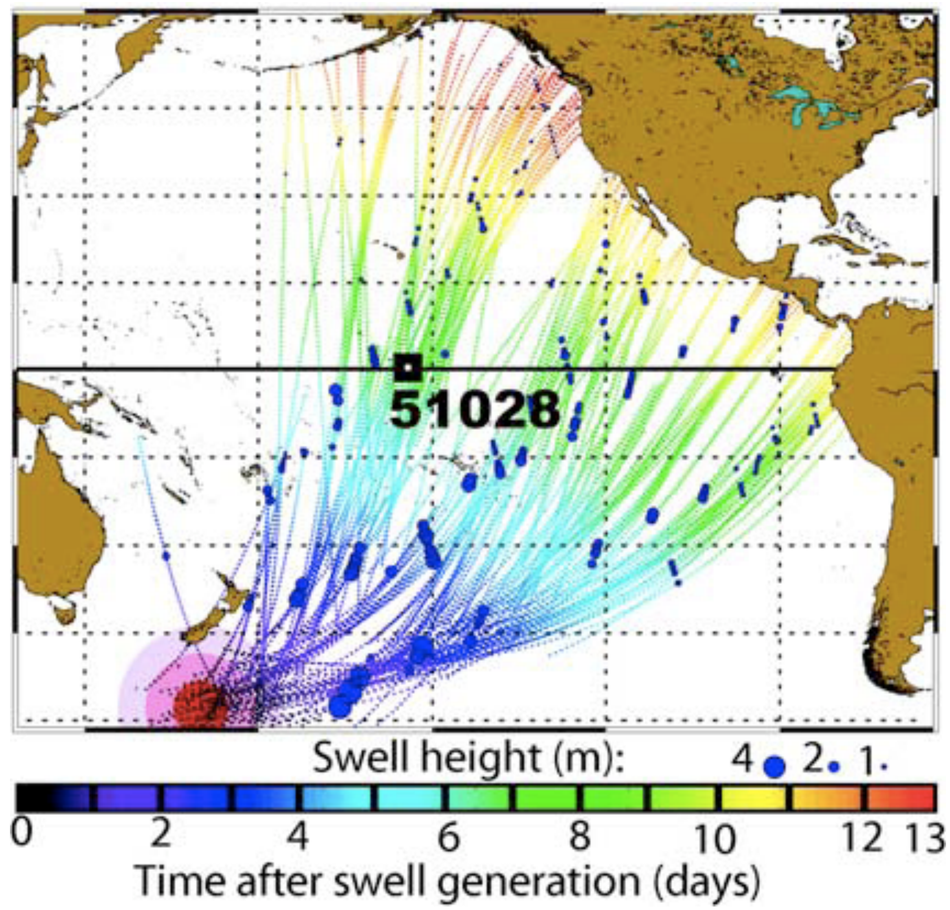
Energy decay rate:

$\Delta = 0.25 \times 10^{-3}$ km^{-1}



SAR data from Collard *et al*. (2009)

Statistical average
for 15-second waves,
over 35 swell tracks:



Energy decay rate:

$\Delta = 0.37 \times 10^{-3}\ \text{km}^{-1}$

Uncertainty:

$0.31 \times 10^{-3} < \Delta < 0.40 \times 10^{-3}\ \text{km}^{-1}$

Measured energy-decay rates of freely propagating waves

Event	k_0 (m^{-1})	Δ (m^{-1})
Aug 1.9 (S)	0.017	0.43×10^{-6}
Aug 13.7(S)	0.016	0.25×10^{-6}
Jul 23.2 (S)	0.014	0.23×10^{-6}
Collard	0.018	0.37×10^{-6}
PSU lab	44.1	0.22

How to relate Δ to δ ?

Recall dissipative NLS:

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma\,|A|^2\,A + i\delta A = 0$$

Derived using a small parameter:

$\varepsilon = 2|A_0|k_0 << 1,$

$\tau = \varepsilon^2 k_0 X$

=> to nondimensionalize Δ:

$$\delta = \frac{\Delta}{2\varepsilon^2 k_0}$$

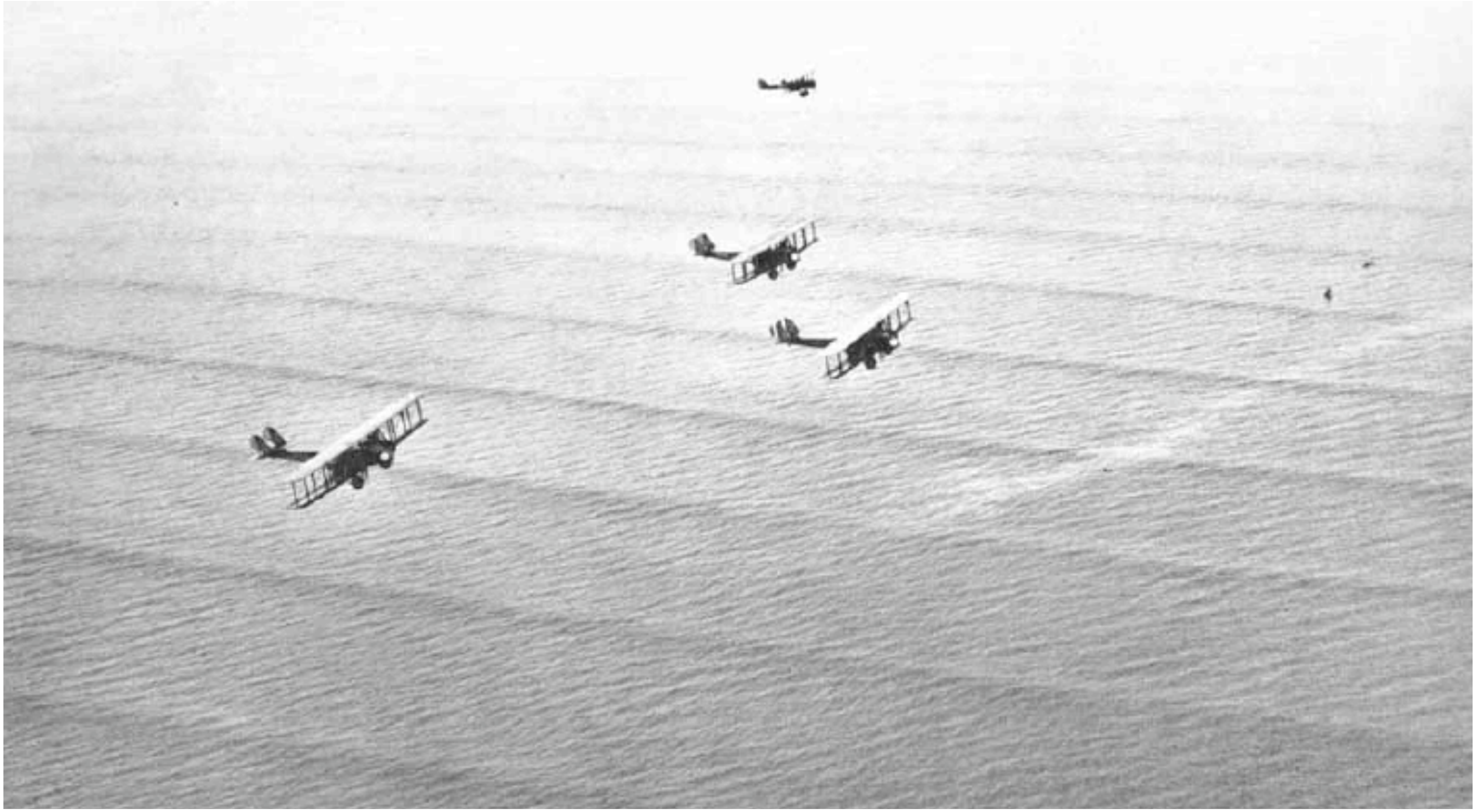
Dimensionless decay rates of freely propagating waves

Event	k_0 (m^{-1})	Δ (m^{-1})	ε	δ
Aug 1.9 (S)	0.017	0.43 x 10^{-6}	0.011	0.105
Aug 13.7(S)	0.016	0.25 x 10^{-6}	0.011	0.065
Jul 23.2 (S)	0.014	0.23 x 10^{-6}	0.0046	0.39
Collard	0.018	0.37 x 10^{-6}	0.029	0.012
PSU lab	44.1	0.22	0.10	0.25

Conclusions

- Dissipation is important in the evolution of surface waves, in the lab and in the ocean
- Dissipation can act on the same distance-scale as nonlinearity and dispersion
- Frequency downshifting occurs in the lab and in the ocean
- Open question: what causes the dissipation?
- Open question: what causes downshifting?

Thank you for your attention



Downshifting of wave trains

$$i\partial_\tau A + \alpha\partial_x^2 A + \beta\partial_y^2 A + \gamma |A|^2 A + i\delta A = 0$$

Define:

$$M(\tau) = \int_D \left|A(x,y,\tau)\right|^2 dxdy, \quad P_1(\tau) = i\int_D \left[A^*\partial_x A - A\partial_x A^*\right] dxdy$$

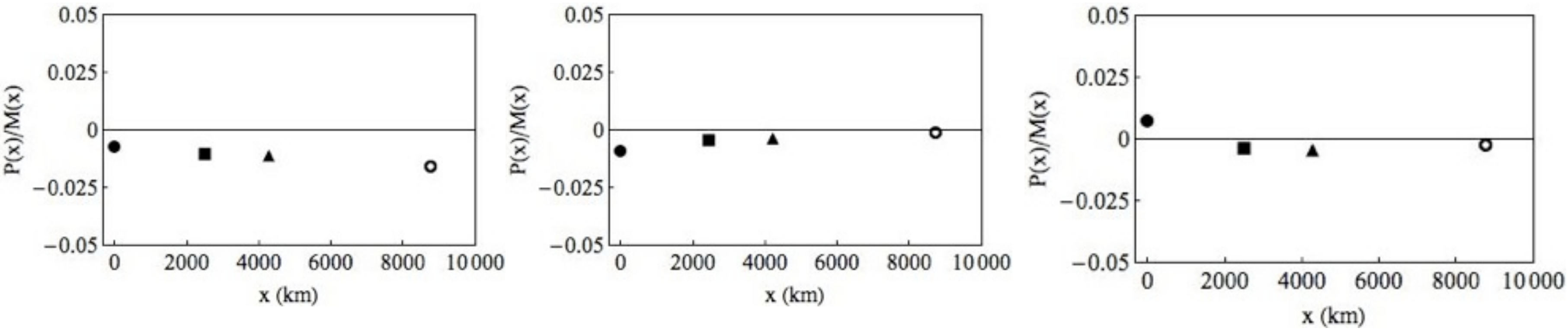
Show:

$$M(\tau) = M(0)\cdot e^{-2\delta\tau}, \quad P(\tau) = P(0)\cdot e^{-2\delta\tau},$$

$$\Rightarrow \frac{P(\tau)}{M(\tau)} = \frac{P(0)}{M(0)} = average\_frequency$$

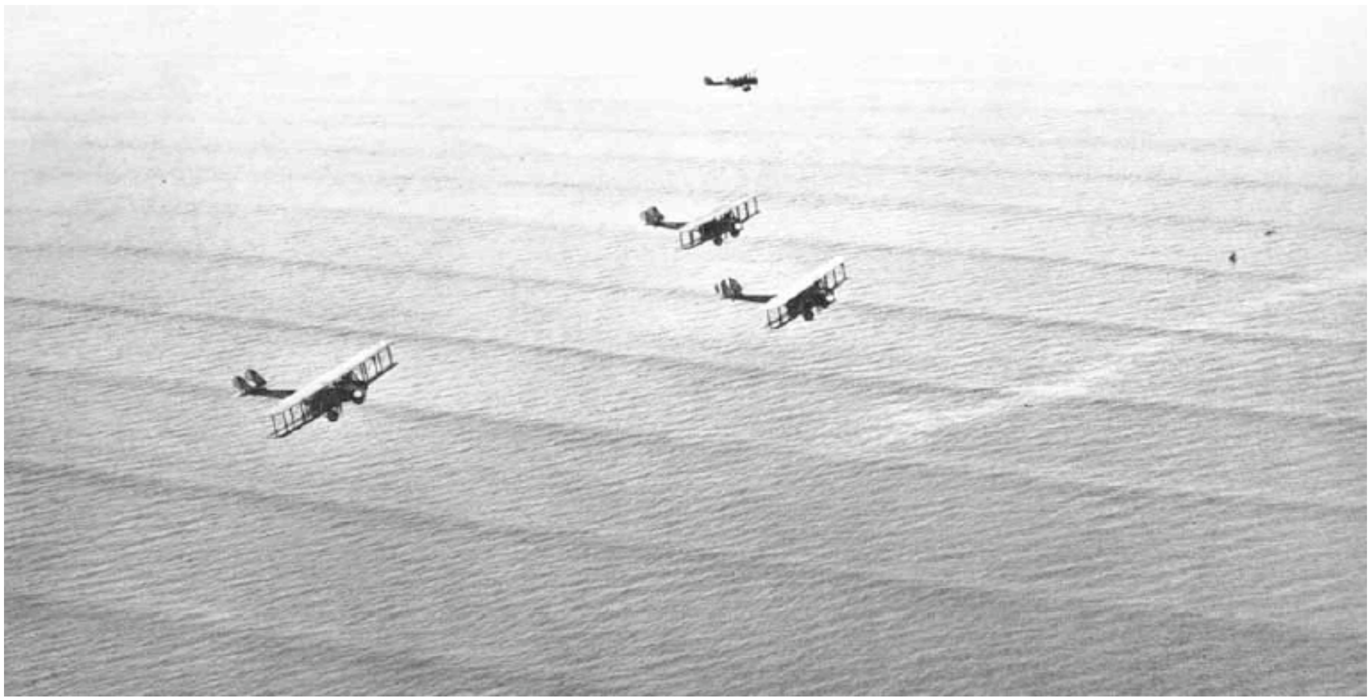
Downshifting in Snodgrass data?

Recall: dissipative NLS => $P(\tau)/M(\tau)$ = constant



Jul 23.2 | Aug 1.9 | Aug. 13.7

The role of dissipation in the evolution of ocean swell

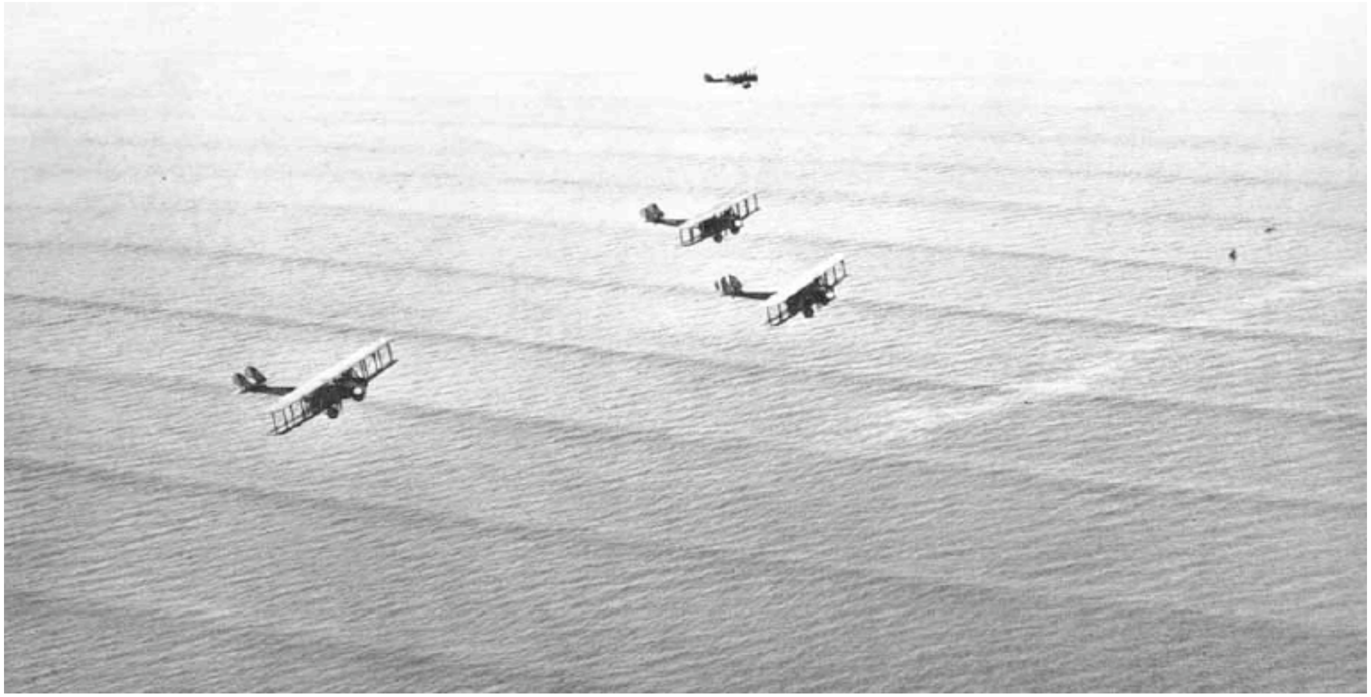


IMACS Conference – 2013

Diane Henderson, Penn State U

Harvey Segur, U of Colorado

The nonlinear Schrödinger equation, dissipation and ocean swell



AMS sectional meeting – Boulder, 2013

Diane Henderson, Harvey Segur
Penn State U U of Colorado

Conclusions

1. The damping rate for ocean swell is vastly smaller than that for laboratory water waves.

   But ocean swell is also less nonlinear than typical laboratory waves.

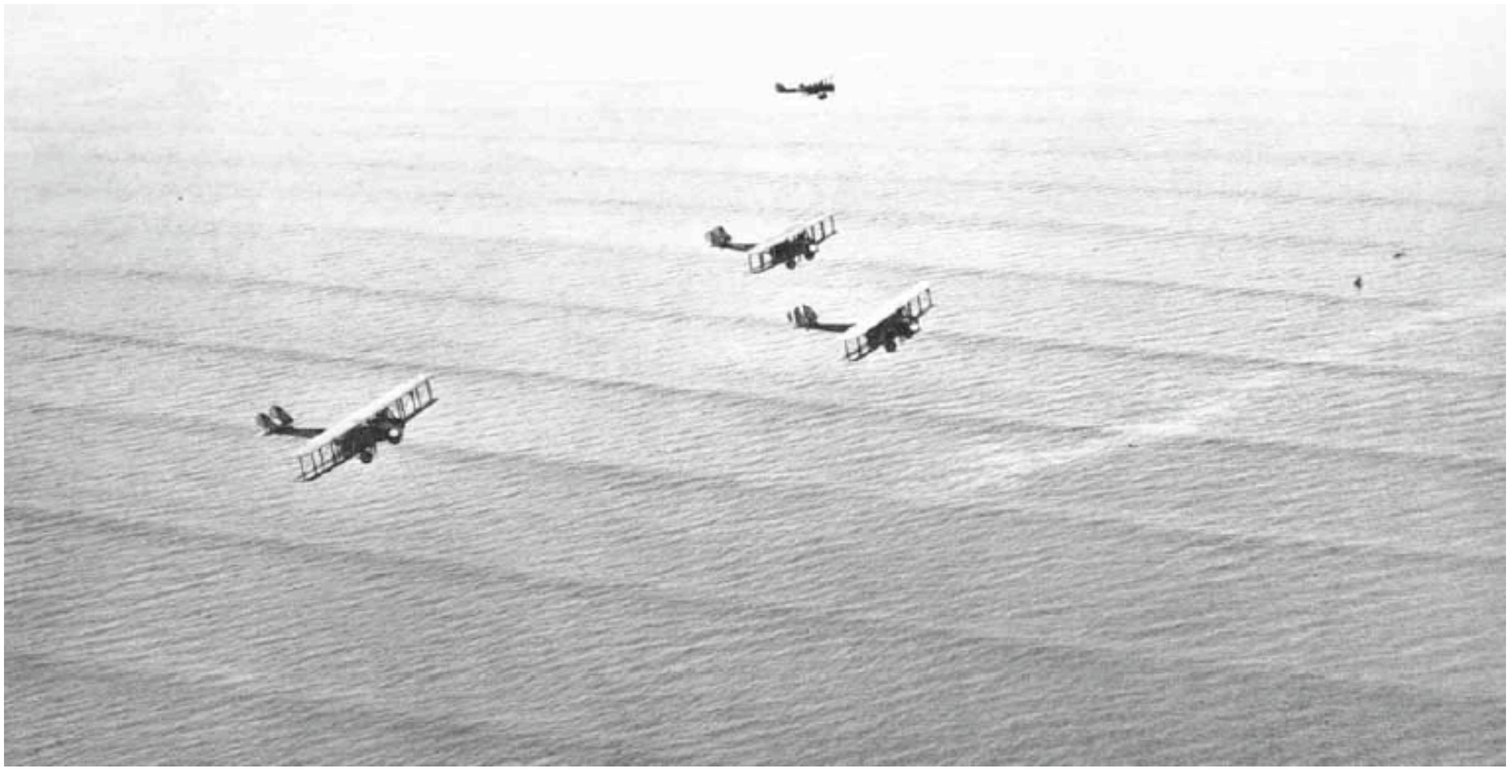
   The important parameter is δ, which compares distance-scales of damping and nonlinearity.

2. The range of values of δ for ocean swell overlaps the range of values for lab waves.

3. For ocean swell with small enough nonlinearity, dissipation impedes and can stop the modulational instability.

   Frequency downshifting occurs for lab waves and for ocean swell. It is not predicted by NLS, with or without damping.

The nonlinear Schrödinger equation, dissipation and ocean swell



Workshop on Ocean Wave Dynamics

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