Lecture 2

Theory of weak turbulence

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{2.8}$$

After non-symmetric Fourier transform,

$$\Psi(r) = \int \Psi(k)\, e^{ikr}\, dk, \qquad \Psi(k) = \frac{1}{(2\pi)^2}\int \Psi(r)\, e^{-ikr}\, dr, \tag{2.9}$$

equation (2.8) reads:

$$\frac{\partial \eta}{\partial t} = \frac{\delta \tilde{H}}{\delta \Psi_k^*}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta \tilde{H}}{\delta \eta_k^*}, \tag{2.10}$$

$$\tilde{H} = \frac{1}{4\pi^2} H = H_0 + H_1 + H_2 + \cdots \tag{2.11}$$

In [41, 42] was shown that Hamiltonian $\tilde{H}$ can be expanded in Taylor series in powers of η:

$$\begin{aligned} H_0 &= \frac{1}{2}\int \left\{ k|\Psi_k|^2 + g\,|\eta_k|^2 \right\} dk \\ H_1 &= \frac{1}{2}\int L^{(1)}(k_1,k_2)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\,\delta(\vec{k_1}+\vec{k_2}+\vec{k_3})\,dk_1\,dk_2\,dk_3 \\ H_2 &= \frac{1}{2}\int L^{(2)}(k_1,k_2,k_3,k_4)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\eta_{k_4}\,\delta(k_1+k_2+k_3+k_4)\,dk_1 dk_2 \eta_{k_3}\eta_{k_4} \end{aligned} \tag{2.12}$$

$$\begin{aligned} L^{(1)}(k_1,k_2) &= -(k_1,k_2) - |k_1|\,|k_2| \\ L^{(2)}(k_1,k_2,k_3,k_4) &= \frac{1}{4}\,|k_1||k_2|\,\{-2|k_1| - 2|k_2| + |k_1+k_3| + |k_1+k_4| + |k_2+k_3| + |k_2+k_4|\}. \end{aligned} \tag{2.13}$$

Canonical transformation

$\eta, \psi \to \xi, R$

Transformed normal variables are defined as follows

Let us denote the new variables as ξ, R. In new variables the cubic terms in the Hamiltonian vanish and one can introduce the normal variables b_k,

$$\begin{aligned} \xi_k &= \frac{1}{\sqrt{2}} \left(\frac{k}{g}\right)^{1/4} (b_k + b^*_{-k}), \\ R_k &= \frac{i}{\sqrt{2}} \left(\frac{g}{k}\right)^{1/4} (b_k - b^*_{-k}). \end{aligned} \tag{2.14}$$

In new variables equation (2.10) takes form

$$\frac{\partial b_k}{\partial t} + i \frac{\delta \tilde{H}}{\delta b_k^*} = 0, \tag{2.15}$$

where the Hamiltonian is expressed as

$$\begin{aligned} \tilde{H} &= \int \omega_k \, b_k \, b_k^* \, dk + \frac{1}{4} \int T_{k_1 k_2 k_3 k_4} \, b^*_{k_1} \, b^*_{k_2} \, b_{k_3} \, b_{k_4} \times \\ & \times \delta(k_1 + k_2 - k_3 - k_4) \, dk_1 \, dk_2 \, dk_3 \, dk_4, \end{aligned} \tag{2.16}$$

and the coupling coefficient $T_{k_1 k_2, k_4 k_3}$ satisfies the symmetry conditions:

$$T_{k_1 k_2, k_3 k_4} = T_{k_2 k_1, \, k_3 k_4} = T_{k_1 k_2, \, k_4 k_3} = T_{k_2 k_4, k_1 k_2}. \tag{2.17}$$

<u>Formal</u>!!! conservative quanties are

$\tilde{H}$, $N = \int |b|^2 d\vec{r}$ wave action $\qquad \vec{P} = \int \vec{k} |b|^2 d\vec{r}$ momentum

The explicit expression for T is complicated [32, 42]:

$$\begin{aligned}
T_{12,34} &= \frac{1}{2}\left(\tilde{T}_{12,34}+\tilde{T}_{21,34}\right),\\
\tilde{T}_{12,34} &= -\frac{1}{2}\frac{1}{(k_1k_2k_3k_4)^{1/4}}\Bigg\{-12k_1k_2k_3k_4-\\
&-2(\omega_1+\omega_2)^2\Big[\omega_3\omega_4\Big((\vec{k}_1\cdot\vec{k}_2)-k_1k_2\Big)+\omega_1\omega_2\Big((\vec{k}_3\cdot\vec{k}_4)-k_3k_4\Big)\Big]\frac{1}{g^2}\\
&-2(\omega_1-\omega_3)^2\Big[\omega_2\omega_4\Big((\vec{k}_1\cdot\vec{k}_3)+k_1k_3\Big)+\omega_1\omega_3\Big((\vec{k}_2\cdot\vec{k}_4)+k_2k_4\Big)\Big]\frac{1}{g^2}\\
&-2(\omega_1-\omega_4)^2\Big[\omega_2\omega_3\Big((\vec{k}_1\cdot\vec{k}_4)+k_1k_4\Big)+\omega_1\omega_4\Big((\vec{k}_2\cdot\vec{k}_3)+k_2k_3\Big)\Big]\frac{1}{g^2}\\
&+[(\vec{k}_1\cdot\vec{k}_2)+k_1k_2][(\vec{k}_3\cdot\vec{k}_4)+k_3k_4]+[-(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][-(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]\\
&+[-(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][-(\vec{k}_2\cdot\vec{k}_3)+k_2k_3]\\
&+4(\omega_1+\omega_2)^2\frac{[(\vec{k}_1\cdot\vec{k}_2)-k_1k_2][(\vec{k}_3\cdot\vec{k}_4)-k_3k_4]}{\omega_{1+2}^2-(\omega_1+\omega_2)^2}\\
&+4(\omega_1-\omega_3)^2\frac{[(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]}{\omega_{1-3}^2-(\omega_1-\omega_3)^2}\\
&+4(\omega_1-\omega_4)^2\frac{[(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][(\vec{k}_2\cdot\vec{k}_3)+k_2k_3]}{\omega_{1-4}^2-(\omega_1-\omega_4)^2}\Bigg\}.
\end{aligned}\tag{2.18}$$

Here $\omega_i=\sqrt{g\,|k_i|}$. Then equation (2.15) reads:

$$\frac{\partial b_k}{\partial t}+i\left(\omega_k\,b_k+\frac{1}{2}\int T_{kk_1k_2k_3}\,b^*_{k_1}b_{k_2}b_{k_3}\,\delta(k+k_1-k_2-k_3)\,dk_1dk_2dk_3\right)=0. \quad (2.19)$$

This equation is known as "Zakharov equation"
(since 1967)

$$T_k = T_{k,k,k,k} = 2k^3 \qquad k = |\vec{k}|$$

If $\quad k_1 \parallel k$

$$T_{k k_1} = k k_1 \left(|k + k_1| - |k - k_1| \right)$$

The miracle!

Resonant conditions

$$\vec{k} + \vec{k}_1 = \vec{k}_2 + \vec{k}_3$$

$$k^{1/2} + k_1^{1/2} = k_2^{1/2} + k_3^{1/2} \quad \text{or} \quad \boxed{\omega + \omega_1 = \omega_2 + \omega_3}$$

In 1-D case

$$T(A, \xi) = A^3 T(\xi) = 0 \; !!!$$

(1) Trivial resonanses

$$k_2 = k \qquad k_3 = k_1$$

$$k_2 = k_1 \qquad k_3 = k$$

Dyachenko, Zakharov 1994

(3) nontrivial resonanses

$$k = A^2(\xi^2 + \xi + 1)^2 \qquad k_2 = A^2 \xi^2 (\xi+1)^2$$

$$k_1 = -A^2 \xi^2 \qquad k_3 = A^2(\xi+1)^2$$

$$\omega = A(\xi^2 + \xi + 1)$$

$$\omega_1 = A\xi$$

$$\omega_2 = A\xi(\xi+1)$$

$$\omega_3 = A(\xi+1)$$

Due to this miracle one can replace

$$T_{kk_1k_2k_3} \rightarrow \frac{1}{2}\left(T_{kk_2} + T_{kk_3} + T_{k_1k_2} + T_{k_1k_3}\right) - \frac{1}{4}\left(T_k + T_{k_1} + T_{k_2} + T_{k_3}\right)$$

Or

$$T_{kk_1k_2k_3} = \frac{1}{2}\,\theta(k)\,\theta(k_1)\,\theta(k_2)\,\theta(k_3)\times$$

$$\times\Big[kk_1(k+k_1) + k_2k_3(k_2+k_3) + kk_3|k-k_3| + kk_2|k-k_2| + k_1k_3|k_1-k_3| + k_1k_2|k_1-k_2|\Big]$$

The „compact" equation, very convinient for numerical simulation.

All $k_i > 0$

Conjecture!

In all order of perturbation waves propagating in one direction do not generate waves propagating in backward direction.

Supported by numerical solutions of exact Euler equations in conformal coordinates

Statistical description:

$\langle \eta_k \eta_{k'} \rangle = I_k \, \delta(k+k')$

$\langle b_k b^*_{k'} \rangle = g N_k \, \delta(k-k')$

$I(k) = \frac{\omega(k)}{2} (N_k + N_{-k})$ + small terms, containg "slave harmonics"

N_k obeys the kinetic equation (Hasselmann equation

$$\frac{dN_k}{dt} = S_{nl} = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \, \delta(k+k_1-k_2-k_3^*) \, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times$$
$$\times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - N_k N_{k_1} N_{k_3}) dk_1 dk_2 dk_3. \qquad (2.28)$$

can be found, for instance, in [41, 42]. Here

$$\frac{dN_k}{dt} = \frac{\partial N_k}{\partial t} + \frac{\partial \omega}{\partial k} \nabla N_k \qquad (2.29)$$

and $T_{kk_1k_2k_3}$ is a homogenous function of order 3:

$$T_{\lambda k, \lambda k_1, \lambda k_2, \lambda k_3} = \lambda^3 T_{kk_1k_2k_3}. \qquad (2.30)$$

Simple calculation shows that $T_{k,k,k,k} = T = 2k^3$.

For smooth spectra S_{nl} can be estimated as follow:

$$S_{nl} \simeq 4\pi \, \omega_p \, \mu_p^4 \, N_k. \qquad (2.31)$$

$\mu_p^2 \simeq \langle \eta^2 \rangle \frac{\omega_p^4}{g^2}$ μ_p steepness of the spectral peak. Another definition of steepness is

$\mu^2 = \langle \nabla \eta^2 \rangle$ $\mu^2 > \mu_p^2$!!

Consider the widely used Hasselmann equation:

$$\frac{\partial N}{\partial t} + \frac{\partial \tilde{\omega}}{\partial \vec{k}} \frac{\partial N}{\partial \vec{r}} = S_{nl}, \tag{2.1}$$

$$\begin{aligned} S_{nl} = \pi g^2 \int & |T_{kk_1,k_2k_3}|^2 \, \delta(k + k_1 - k_2 - k_3) \\ & \times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \\ & \times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} \\ & - N_k N_{k_1} N_{k_2} - N_k N_{k_1} N_{k_3}) \, \mathrm{d}k_1 \, \mathrm{d}k_2 \, \mathrm{d}k_3. \end{aligned} \tag{2.2}$$

Here $\omega_k = \sqrt{g\, k \tanh kH}$, H is the depth, $T_{kk_1k_2k_3} = T_{k_1kk_2k_3} = T_{k_2k_3kk_1} = T_{kk_1k_3k_2}$ are the coupling coefficients, and

$$\tilde{\omega}(k) = \omega(k) + 2g \int T_{kk_1,kk_1} \, N_{k_1} \, \mathrm{d}k_1 \tag{2.3}$$

is the renormalized frequency.

As mentioned earlier, the nonlinear interaction term S_{nl} can be presented in the form

$$S_{nl} = F_k - \Gamma_k \, N_k, \tag{2.4}$$

where

$$F_k = \pi g^2 \int |T_{kk_1k_2k_3}|^2 \delta(k + k_1 - k_2 - k_3)$$
$$\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) N_{k_1} N_{k_2} N_{k_3}\, dk_1\, dk_2\, dk_3. \quad (2.5)$$

and Γ_k, the dissipation rate due to the presence of four-wave processes, is the following:

$$\Gamma_k = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \delta(k + k_1 - k_2 - k_3)$$
$$\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})$$
$$\times (N_{k_1} N_{k_2} + N_{k_1} N_{k_3} - N_{k_2} N_{k_3})\, dk_1\, dk_2\, dk_3. \quad (2.6)$$

2

Let us study equation

$S_{nl} = 0$ Let $k = |k|$ and

$$N_k = k^{-x}. \quad (3.3)$$

By plugging (3.3) into (3.1) we find that each particular term in S_{nl} is diverging, but in different terms the divergence can be cancelled, thus there is a "window of opportunity" for the exponent x. As a result,

$$S_{nl} = g^{3/2}\, k^{-3x+19/2}\, F(x). \quad (3.4)$$

Here $F(x)$ is a dimensionless function, defined inside interval $x_1 < x < x_2$. The edges of the window, x_1 and x_2, are the subject for determination.

Let us study the quadruplet of waves with wave vectors $k, k_1, k_2, \vec{k}_3$, satisfying resonant conditions (1.12). Suppose that $|k_1| \ll |k|$. The three-wave resonant condition,

$$\vec{k} = \vec{k}_2 + \vec{k}_3, \qquad \omega_k = \omega_{k_2} + \omega_{k_3}, \tag{3.5}$$

has no nontrivial solutions, thus one of vectors $\vec{k}_2, \vec{k}_3$ must be small. If $|k_3| \ll |k_2|$, then

$$\begin{aligned} \vec{k}_2 &= \vec{k} + \vec{k}_1 - \vec{k}_3, \\ \omega(k_2) &= \sqrt{gk}\left(1 + \frac{1}{2}\frac{(k, \vec{k}_1 - \vec{k}_3)}{k^2} + \cdots\right), \end{aligned} \tag{3.6}$$

and we can put $|k_3| = |k_1|$. Vectors $\vec{k}_1, \vec{k}_3$ are small and have approximately the same length k_1. If vector k is directed along axis x, the coupling coefficient $T_{kk_1k_2k_3}$ depends on four parameters $k, k_1, \theta_1, \theta_3$. Here θ_1, θ_3 are angles between $\vec{k}_1, \vec{k}_3$ and $\vec{k}$. Remembering that $k_1 \ll k$, we calculate the coupling coefficient in this asymptotic domain. A tedious calculation presented in article [25] leads to the following compact result:

$$\begin{aligned} T_{kk_1k_2k_3} &\simeq \frac{1}{2} k\, k_1^2\, T_{\theta_1,\theta_3}, \\ T_{\theta_1,\theta_2} &= 2(\cos\theta_1 + \cos\theta_3) - \sin(\theta_1 - \theta_3)(\sin\theta_1 - \sin\theta_3). \end{aligned} \tag{3.7}$$

Suppose $N = N_0 + N_1$ N_0 – long wave component, N_1 – short wave component. Take into account the interaction between N_0 and N_1 only. One can see that N_1 satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_i} D_{ij}\, k^2 \frac{\partial}{\partial k_j} N_1, \tag{3.9}$$

where D_{ij} is the diffusion tensor

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq\, q^{17/2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_3 |T(\theta_1, \theta_3)|^2 \, p_i p_j N(\theta, q) N(\theta_3, q) \quad (3.10)$$

$$p_1 = \cos\theta_1 - \cos\theta_3, \quad p_2 = \sin\theta_1 - \sin\theta_3$$

If spectrum is isotropic and does not depend on angle θ, we get the further simplification:

$$D_{ij} = D\,\delta_{ij}, \qquad D = \frac{5}{8}\pi^3 g^{3/2} \int_0^\infty q^{17/2} N^2(q) dq. \quad (3.11)$$

Taking into account (3.3), we find that diffusion coefficient D diverges at $k \to 0$ if $x > 19/4$. Thus $x_2 = 19/4$.

Let us find behavior of function $F(x)$ near $x = x_2$. In the isotopic case equation (3.9) reads

$$\frac{\partial N_1}{\partial t} = \frac{D}{k}\frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} N_1. \quad (3.12)$$

If $k \to 19/4$, we get the following estimate:

$$F(x) = \frac{19}{4} \cdot \frac{11}{4} \cdot \frac{5\pi^3}{16} \frac{1}{19/4 - x} \simeq \frac{126.4}{19/4 - x} \quad (3.13)$$

To find x_1, the lower end of window, we should study the influence of short waves to the long ones. Let us suppose that $|k_1|, |k_2| \gg k$. In the first approximation $|k_3| = |k|$, and the resonant interaction S_{nl} can be separated into two groups of terms: $S_{nl} = S_{nl}^{(1)} + S_{nl}^{(2)}$. For $S_{nl}^{(1)}$ the integrand includes product $N_{k_1} N_{k_2}$. If we put $k_1 = k_2$, we get the following expression for the low-frequency tail of spectrum:

$$S_{nl}^{(1)} = 2\pi g^2 \int |T_{kk_1,k_1,k_3}|^2 \, \delta(\omega - \omega_{k_3}) \, (N_{k_3} - N_k) \, N_{k_1}^2 \, dk_1. \quad (3.14)$$

Notice, if $|k_1| \gg |k|$, then $|T_{kk_1,k_1,k_3}|^2 \simeq k_1^2$ and integrand in (3.14) is proportional to $k_1^2 N_{k_1}^2$. If $x < 2$, integral (3.14) diverges.

The group of terms linear with respect to the high-frequency tail of spectrum is more complicated:

$$\begin{aligned} S_{nl}^{(2)} &= 2\pi g^2 N_k \int |T_{kk_1k_2k_3}|^2 N_{k_3}(N_{k_1} - N_{k_2}) \times \\ &\times \ \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})\, \delta(k + k_1 - k_2 - k_3)\, dk_1\, dk_2\, dk_3. \end{aligned} \tag{3.15}$$

We can perform expansion

$$N_{k_1} - N_{k_3} = p_i \frac{\partial N}{\partial k_{1i}}, \qquad p_i = (k - k_3)_i. \tag{3.16}$$

In the general anisotropic case the integrand is proportional to $k_1^2(p\,\nabla N_{k_1})$ and the divergence occurs if $x = x_1 = 2$. However, in the isotropic case this term, the most divergent one, is cancelled after integration by angles. In this case we should study quadratic terms in expansion of the integrand in powers of parameter $(P, k_1)/k_1^2$. The most aggressive term appears from the expansion of δ-function on frequencies $\delta(\omega_{k_1} - \omega_{k_1+p} + \omega_k - \omega_{k_3})$. Performing integration by angles we end up with the equation

$$\begin{aligned} \frac{\partial N_k}{\partial t} &= q\, k^7 N_k \frac{\partial N}{\partial k}, \\ q &= \frac{25}{16}\pi^3 g^{3/2} E = \frac{25}{8}\pi^3 g^{3/2} \int_0^\infty k^{3/2} N_k\, dk. \end{aligned} \tag{3.17}$$

Here E is the total energy. Thus in the isotropic case $x_1 = 5/2$ and we get for function $F(x)$ the following estimate:

$$F = \frac{5}{2}\frac{25}{8}\pi^3 \frac{1}{5/2 - x} = \frac{241.86}{5/2 - x}. \tag{3.18}$$

Finally function F(x) is presented by plots

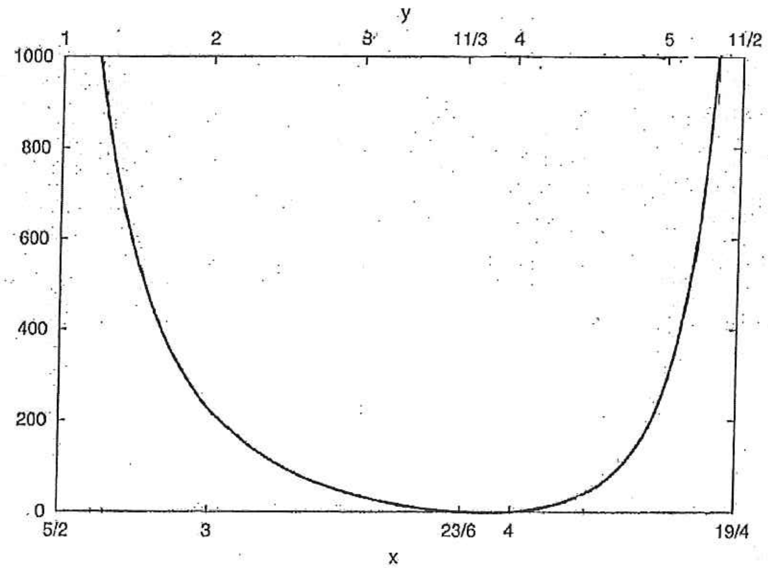

Figure 1: Plot of function $F(x)$.

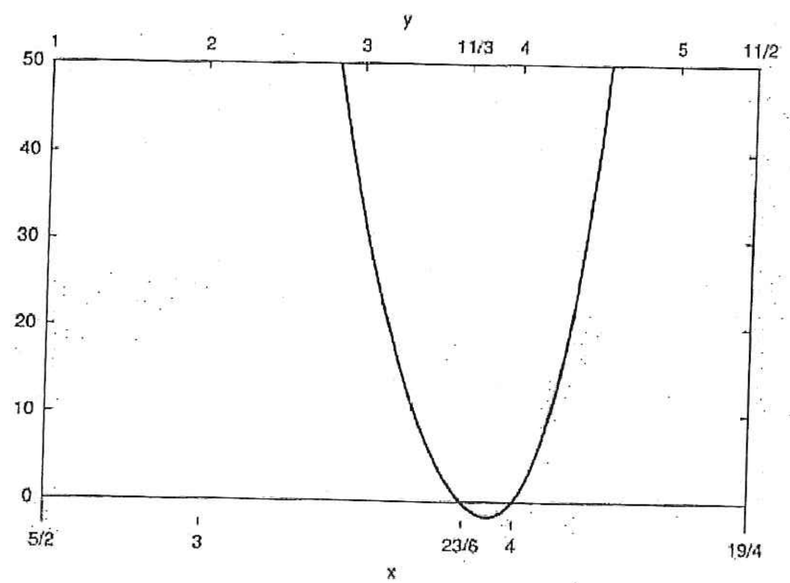

Figure 2: Plot of function $F(x)$; zoom in the vertical direction.

Theorem

$F(x)$ has only two zeros

$x_1 = \frac{23}{6}$ $x_2 = 4$

(The proof is long)

To prove this result, let us consider that spectra are isotropic and present conservation laws of energy and wave action in the differential form:

$$\frac{\partial I_k}{\partial t} = 2\pi k\omega_k \frac{\partial N_k}{\partial t} = -\frac{\partial P}{\partial k}, \tag{3.20}$$

$$P = 2\pi \int_0^k k\omega_k S_{nl}\, dk, \tag{3.21}$$

$$2\pi k \frac{\partial N_k}{\partial t} = \frac{\partial Q}{\partial k}, \tag{3.22}$$

$$Q = 2\pi \int_0^k k\, S_{nl}\, dk. \tag{3.23}$$

Here P is the flux of energy directed to high wave numbers, while Q is the flux of wave action directed to small wave numbers. Equations

$$P = P_0 = const, \quad Q = Q_0 = const \tag{3.24}$$

apparently are solutions of stationary equation $S_{nl} = 0$. We will look for the solution in the powerlike form $N = \lambda\, k^{-x}$; then equations (3.24) read

$$P_0 = 2\pi g^2 \lambda^3 \frac{F(x)}{3(x-4)} k^{-3(x-4)} \tag{3.25}$$

$$Q_0 = -2\pi g^{3/2} \lambda^3 \frac{F(x)}{3(x-26/3)} k^{-3(x-26/3)} \tag{3.26}$$

One can see that P_0 and Q_0 are finite only if $F(4) = 0$ and $F(26/3) = 0$, moreover, if $F'(4) > 0$ and $F'(26/3) < 0$. We conclude that equation $S_{nl} = 0$ has the following solutions:

$$N_k^{(1)} = c_p \left(\frac{P_0}{g^2}\right)^{1/3} \frac{1}{k^4}, \tag{3.27}$$

$$N_k^{(2)} = c_q \left(\frac{Q_0}{g^{3/2}}\right)^{1/3} \frac{1}{k^{23/6}}. \tag{3.28}$$

Here c_p, c_q are dimensionless Kolmogorov constants

$$c_p = \left(\frac{3}{2\pi\, F'(4)}\right)^{1/3}, \quad c_q = \left(\frac{3}{2\pi\, |F'(23/6)|}\right)^{1/3}.$$

On Figure 2 is presented the zoom of function $F(x)$ in vertical coordinate. The numerics gives $F'(4) = 45.2$ and $F'(23/6) = -40.4$. In the area of zeros $F(x)$ can be approximated by parabola,

$$F(x) \simeq 256.8(x - 23/6)(x - 4). \tag{3.29}$$

Thus we get

$$c_p = 0.219, \quad c_q = 0.227, \tag{3.31}$$

In the isotropic case, the energy spectrum ~~$F(\omega)$ defined by (1.8)~~ can be expressed through N_k,

$$F(\omega)d\omega = 2\pi\omega_k\, N_k\, k\, dk, \tag{3.32}$$

and the energy spectrum corresponding to solution (3.27) has the following form, called Zakharov-Filonenko spectrum:

$$F^{(1)}(\omega) = 4\pi c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^2}{\omega^4}. \tag{3.33}$$

This spectrum was found in 1966 as a solution of equation $S_{nl} = 0$ [10]. For the spatial spectrum

$$I_k\, dk = 2\pi\omega_k\, N(k)\, k\, dk, \tag{3.34}$$

Hence

$$I_k^{(1)} = 2\pi\, c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^{1/2}}{k^{5/2}} \simeq k^{-2.5}. \tag{3.35}$$

Spectra (3.27), (3.33), (3.25) are realized if we have a source of energy that is concentrated at small wave number and generates the amount of energy P in a unit of time. For the spectrum (3.28), first reported by Zakharov in 1966 [34],

$$I_k^{(2)} = 2\pi\, c_q\, Q^{1/3}\, k^{-7/3} \simeq 2\pi\, c_q\, Q^{1/3}\, k^{2.33}, \tag{3.36}$$

$$F^{(2)}(\omega) = 4\pi\, c_q\, Q^{1/3}\, \frac{g^{4/3}}{\omega^{11/3}}. \tag{3.37}$$

Spectra (3.30) and (3.36) can be realized in the case of a small source of wave action in the high wave numbers area.

The described spectra exhaust all powerlike isotropic solutions of the stationary kinetic equation $S_{nl} = 0$. It is important to stress that thermodynamical solutions $N = const$ and $N = c/k^{1/2}$ are not the solutions of this equation, because their exponents $x = 0$ and $x = 1/2$ are far below the lower end of the "window of possibility" $x_1 = 5/2$. This fact means that thermodynamics has nothing in common with the theory of wind-driven sea.

Solutions (3.29) and (3.30) are not the unique stationary solutions of $S_{nl} = 0$. The general isotropic solution describes the situation when both the energy source at small wave numbers and the wave action source exist simultaneously and have the following form:

$$N_k^{(3)} = c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{1}{k^4}\, L\left(\frac{g^{1/2}\, Q\, k^{1/2}}{P}\right). \tag{3.38}$$

Here L is an unknown function of one variable,

$$L \to 1 \quad \text{at} \quad k \to 0, \qquad L(\xi) \to \frac{c_q}{c_p}\, \xi^{1/3} \quad \text{at} \quad k \to \infty. \tag{3.39}$$

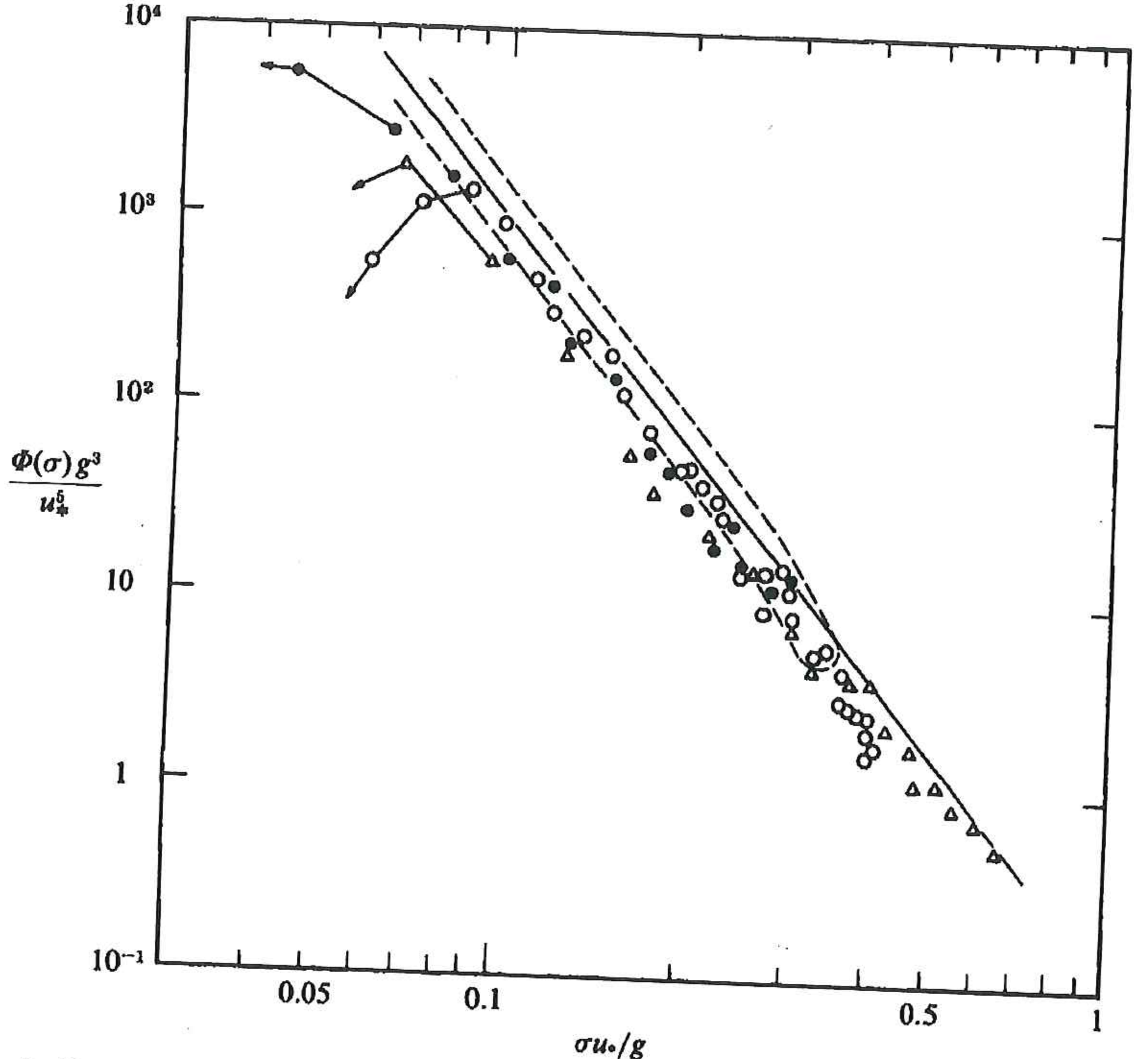

FIGURE 3. A dimensionless plot of the frequency spectrum in the equilibrium range. The broken lines enclose the extensive measurements of Forristall (1981) and the continuous line represents (3.4) with $\alpha = 0.11$. The open circles indicate results from Kawai *et al.* (1977) at $u_* = 37.2$ cm/s; results from Kondo *et al.* (1973) are shown as solid circles at $u_* = 49.6$ cm/s and the triangles at $u_* = 24.4$ cm/s.

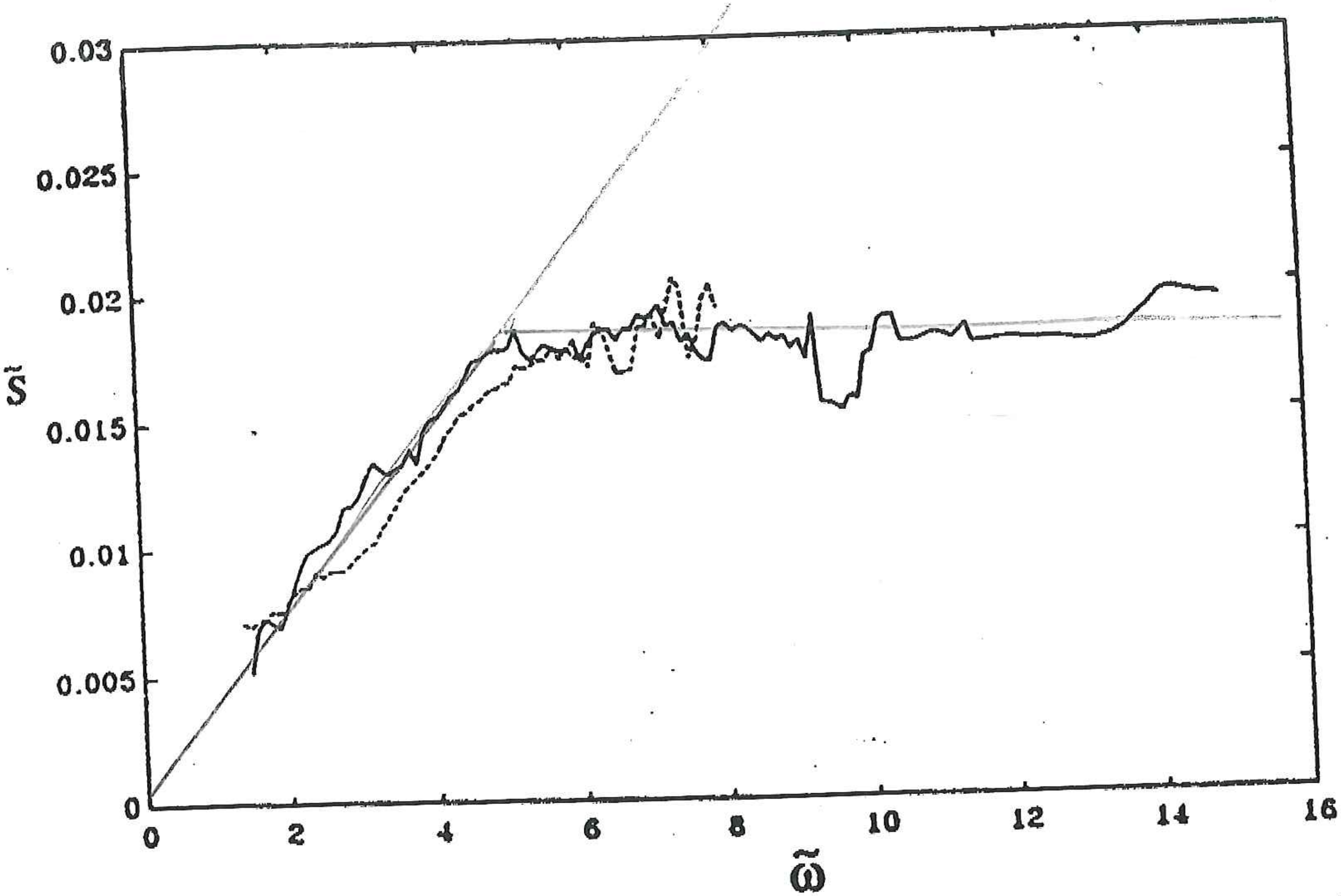

FIG. 1 "Grand average" of the dimensionless spectra in the saturation range. Lake Marken data (the dashed line) is shown separately because it represents shallow-water conditions.

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Experimental ~~eveti~~ evidence of weak turbulent KZ spectra.

This evidence is huge!

Dozen of measurements reveles $F(\omega) \simeq \omega^{-4}$

$I_k \simeq k^{-2.5}$.

A person who visited Zoo did not mention on elephant!

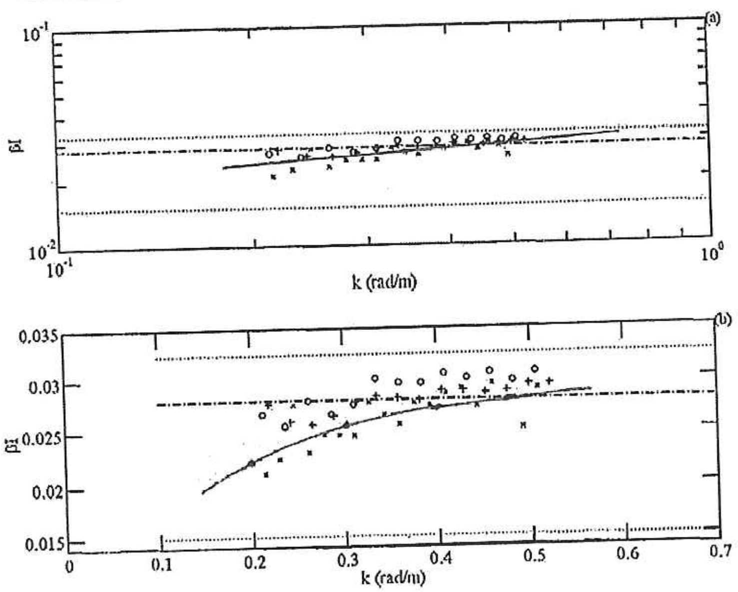

Figure 2. Dimensionless wave number spectral coefficient β_i plotted on logarithmic scales (a) and linear scales (b), taken from [20]. Here crosses represent the omnidirectional (averaged by angles) spectrum and dots correspond to $\xi(k) = 2\beta_I\, u_*\, g^{-0.5}\, k^{-2.5}$. The solid line in (a) and solid curve in (b) correspond to $\xi(k) \simeq k^{-7/3}$.

Dominance of nonlinear interactions

$$\frac{\partial N}{\partial T} = S_{nl} + S_{in} + S_{diss}$$

6. Damping due to nonlinear interaction

How must we compare S_{nl} and S_{in}?

In this section, we show that S_{nl} is the leading term in the balance equation (1.11). In fact, the forcing terms S_{in} and S_{dis} are not sufficiently accurately known; thus it is reasonable to accept the simplest models of both terms assuming that they are proportional to the action spectrum:

$$S_{\text{in}} = \gamma_{\text{in}}(k)\, N(k), \tag{6.1}$$

$$S_{\text{dis}} = -\gamma_{\text{dis}}(k)\, N(k). \tag{6.2}$$

Hence

$$\gamma(k) = \gamma_{\text{in}}(k) - \gamma_{\text{dis}}(k). \tag{6.3}$$

$$S_{nl} + \gamma(k)\, N_k = 0 \tag{6.4}$$

and present the S_{nl} term as

$$S_{nl} = F_k - \Gamma_k\, N_k. \tag{6.5}$$

The definitions of Γ_k and F_k are given by equations (2.5) anc (2.6).

The solution of stationary equation (6.4) is the following

$$N_k = \frac{F_k}{\Gamma_k - \gamma_k}. \tag{6.6}$$

A positive solution exists if $\Gamma_k > \gamma_k$. The term Γ_k can b treated as the nonlinear damping that appears due to four-wav interaction. This damping has a very powerful effect A 'naive' dimensional consideration gives

$$\Gamma_k \simeq \frac{4\pi g^2}{\omega_k}\, k^{10}\, N_k^2; \tag{6.7}$$

however, this estimate works only if $k \simeq k_p$, with k_p being the wave number of the spectral maximum.

Let $k \gg k_p$. Now for Γ_k one gets

$$\Gamma_k = 2\pi g^2 \int |T_{kk_1,kk_3}|^2\, \delta(\omega_{k_1} - \omega_{k_3})\, N_{k_1} N_{k_3}\, dk_1 dk_2. \tag{6.8}$$

$$\Gamma_k = 8\pi g^{3/2} k^2 \cos^2\theta \int_0^\infty k_1^{13/2} \tilde{N}^2(k_1)\, dk_1 . \tag{6.9}$$

Even for the most mildly decaying KZ spectrum, $N_k \simeq k^{-23/6}$, the integrand behaves like $k_1^{-7/6}$ and the integral diverges. For steeper KZ spectra, the divergence is stronger.

Let us estimate Γ_k for the case of a 'mature sea', when the spectrum can be taken in the form

$$N_k \simeq \frac{3}{2} \frac{E}{\sqrt{g}} \frac{k_p^{3/2}}{k^4} \theta(k - k_p). \tag{6.10}$$

Here E is the total energy. By plugging (6.10) into (6.9), one gets the equation

$$\Gamma_\omega = 36\pi\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \cos^2\theta, \tag{6.11}$$

which includes a huge enhancing factor: $36\pi \simeq 113.04$. For a very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \times 10^{-4} \omega \left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta. \tag{6.12}$$

In the isotropic case, to find Γ_k for $\omega/\omega_p \gg 1$ we need to perform a simple integration over angles that yields

$$\int_0^{2\pi}\int_0^{2\pi} T^2_{\theta_1,\theta_2}\, d\theta_1\, d\theta_2 = \frac{5}{2}(2\pi)^2;$$

thus instead of equation (6.11) we get

$$\Gamma_k = 5\pi g^{3/2}k^2 \int_0^{\infty} k_1^{13/2}\tilde{N}(k_1)^2\, dk_1 \tag{6.13}$$

or

$$\Gamma_\omega = \frac{45\pi}{2} g^{3/2}\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4. \tag{6.14}$$

Finally, assuming that

$$N_{k_p} \simeq \frac{3}{2}\frac{E}{\sqrt{g}k_p^{5/2}},$$

we get from equation (6.8) the following estimate for $\Gamma_p = \Gamma|_{k=kp}$:

$$\Gamma_p \simeq 9\pi\omega_p\mu_p^4. \tag{6.15}$$

Even in this case, we have a pretty high enhancing factor: $9\pi \simeq 28.26$. In fact, in all known models, Γ_k surpasses $\tilde{\gamma}_k$ at least in order of magnitude even for these very smooth waves.

In the presence of peakedness

$$\Gamma_p \simeq \Lambda\,\omega_p\mu_p^4. \tag{6.16}$$

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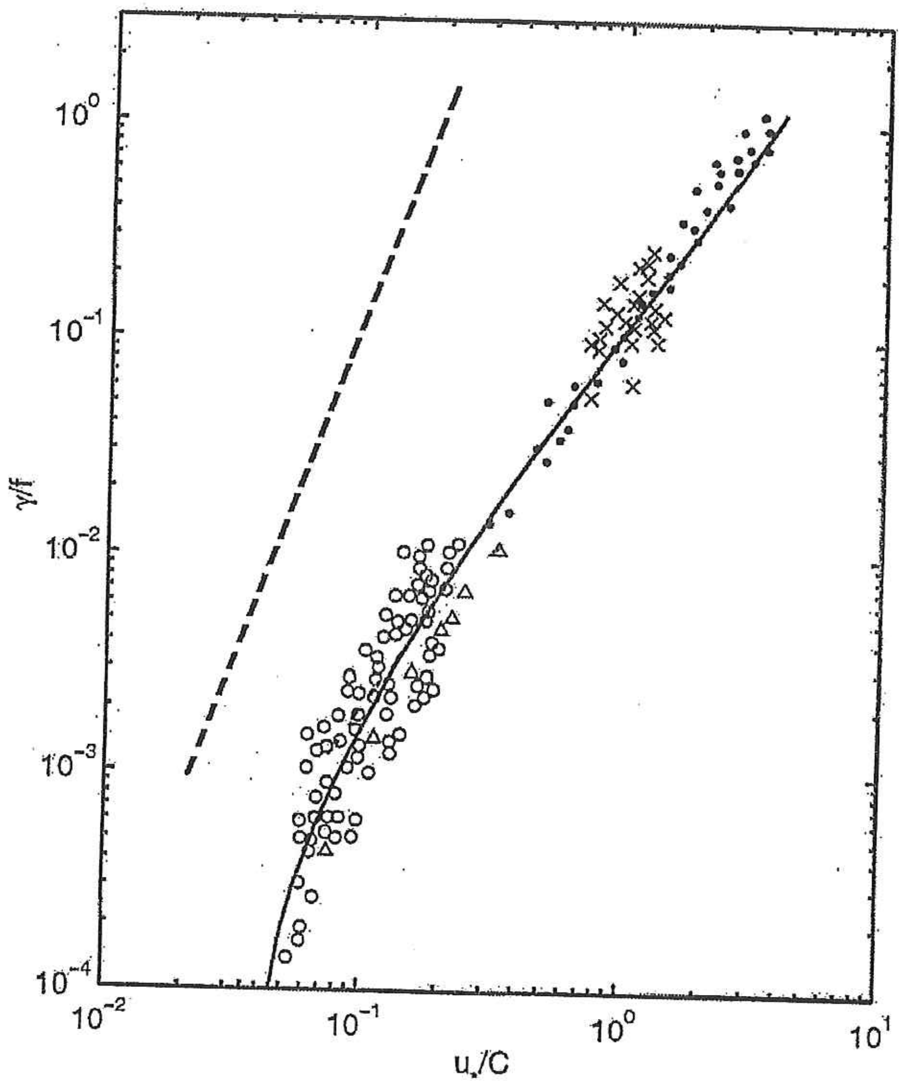


Figure 3. Comparison of the experimental data on the wind-induced growth rate $2\pi\gamma_{in}(\omega)/\omega$ taken from [26] and the damping due to four-wave interactions $2\pi\Gamma(\omega)/\omega$, calculated for the narrow in angle spectrum at $\mu \simeq 0.05$ using equation (6.11) (dashed line).

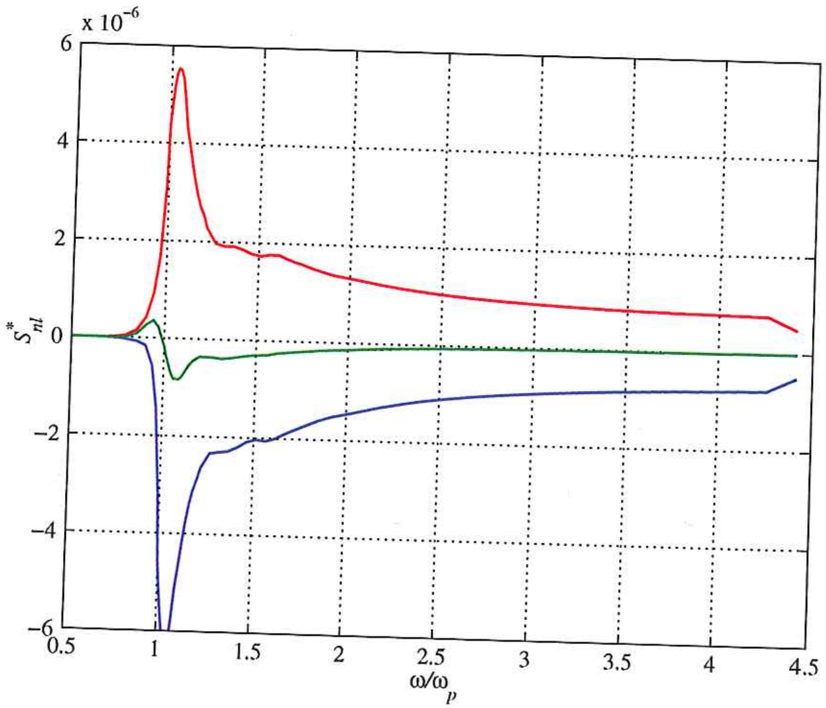


Figure 4. Split of the nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k N_k$ (lower curve).

data on γ_k shows that Γ_k surpasses γ_k at least by an order of magnitude. This fact is demonstrated in figure 3, where experimental data taken from [26] are presented.

As a result, we can conclude that S_{nl} is the leading term in the balance equation (1.11) and that the rear face of the spectrum is described by the solution of equation (4.1), which has a rich family of solutions. In particular, this equation describes the angular spreading.

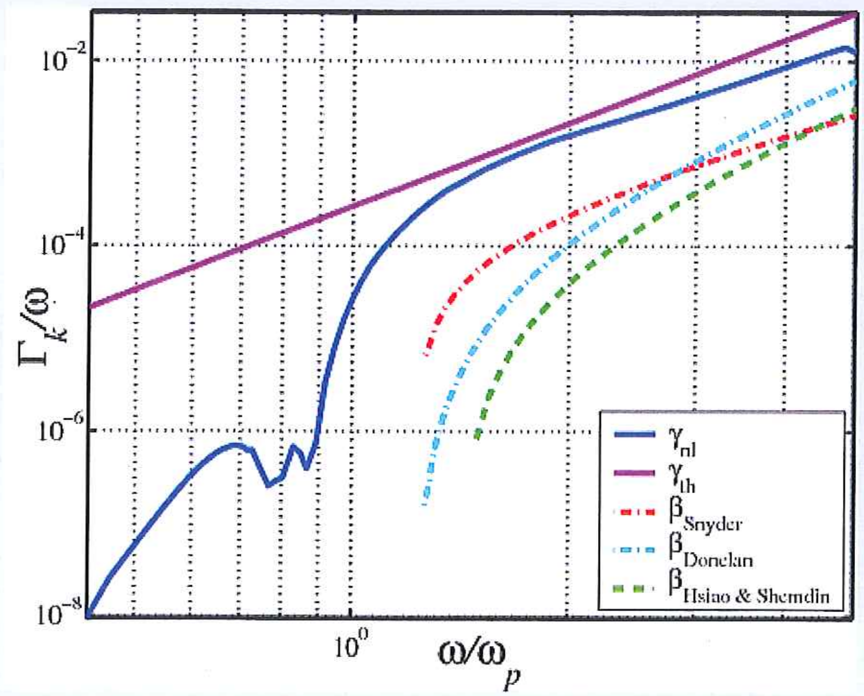
In figure 4, we demonstrate that for the nonlinear interaction term $S_{nl} = F_k - \Gamma_k N_k$, the magnitudes of constituents F_k and Γ_{kN_k} essentially exceed their difference. They are one order higher than the magnitude of S_{nl}

The dominance of S_{nl} was not apparent until now for two reasons. Firstly, it is not correct to compare S_{nl} and S_{in}; instead, one should compare Γ_k and γ_k. Secondly, the widely accepted models for S_{dis} essentially overestimate the dissipation due to white capping. As a result, the dominance of S_{nl} is masked. We offer an alternative model for S_{dis}, which will be published in a forthcoming article [27]. Preliminary results obtained in this direction are given in [28].

Compare nonlinear damping decrement and wind input increment

— 24 —

S_{nl} surpasses S_{in} and S_{diss} in order of magnitude !



Scales of altimeter measurements are larger than relaxation scales but they are smaller than scales of wave field variability due to wind input