# Theory of Wind Driven Sea

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## 1 Introduction

The wind-driven sea is one of the most common natural phenomena that we can observe by our own eyes without special devices. Development of reliable analytical theory of wind-driven sea is a really challenging task for a physicist; one can compare it with another fundamental problem of macroscopic physics: the problem of atmospheric boundary layer. Both problems are naturally connected because sea waves are driven by wind; both problems are very difficult because the wind is badly turbulent and so far we don't have any satisfactory theory of the strong turbulence. However, the case of wind-driven sea is easier. In this case we have in our possession a natural small parameter: the ratio of atmospheric and water density. The density of air depends on temperature and the level of moisture. It is reasonable to put

$$\epsilon = \frac{\rho_{atm}}{\rho_w} \sim 1.2 \cdot 10^{-3}.$$

Due to smallness of $\epsilon$, the influence of wind to the surface is relatively weak and can be parameterized by complexification of the dispersion law. Let $\eta(\vec{r}, t)$, $\vec{r} = (x, y)$ be elevation. In the linear approximation one can put

$$\eta(r,t) = R_e\, \Psi\, e^{-i\,\omega_k\, t + i\,\vec{k}\,\vec{r}}. \tag{1.1}$$

Here $\Psi$ is an arbitrary complex amplitude, $\omega_k = \sqrt{g\,k + \sigma\,k^3}$ is the dispersion law, $g$ is the gravity acceleration, and $\sigma$ is the surface tension. In the presence of wind the horizontal sea surface is unstable, and one should perform the replacement

$$\omega_k \to \omega_k + \frac{i}{2}\,\gamma_k, \tag{1.2}$$

where $\gamma_k$ is the growth-rate of instability (if $\gamma_k > 0$), or decrement (if $\gamma_k < 0$). In the presence of wind, at least in some domain on the $k$-plane, $\gamma_k > 0$.

Analytic description of $\gamma_k$ is possible if the boundary layer is laminar and its velocity profile

$$V_x = U = U_0(z) \tag{1.3}$$

is known [4]. In reality the boundary layer is badly turbulent. However, the standard theory of turbulent boundary layer over rigid rough plane is hardly applicable for the sea-air case. The waves exited by wind disturb atmosphere, creating the "radiation stress" that causes deviation of $U_0(z)$ from logarithmic law. As a result, the analytical derivation of $\gamma_k$ is still an unresolved problem. At the moment, there are a dozen heuristic models of $\gamma_k$; a review can be found in [1]. In all these models

$$\gamma_k = \epsilon\, \omega_k\, \beta_k,$$

where $\beta_k$ is a dimensionless growing function on $k$.

Let $U$ be the wind speed on some reference height (usually oceanographers measure it at $h = 10\, m$). Most authors [2, 3] believe that $\beta_k$ grows at large $k$ as follow:

$$\beta_k = m \left(\frac{U}{c}\right)^2 \quad \text{if} \quad \frac{U}{c} \geq 1. \tag{1.4}$$

Here $c = \sqrt{g/k}$ is the phase velocity of wind-driven waves, $m$ is a dimensionless parameter that varies in different models in a pretty wide range: $0.04 < m < 0.2$. On our opinion, the lowest value of $m$ is more realistic.

If the sea is smooth, the instability takes place for all small scales until is arrested by viscosity in the capillary region at wave length $\lambda \simeq 2 \div 3\, mm$. Notice, that capillary effects are essential, if the wave length $\lambda \leq 10\, cm$; for $\lambda < 1.7\, cm$ they are predominant. If the surface is not smooth, the situation is much more complicated. Wave breaking events create drops and microscale turbulence. In this case the wave motion is strongly contaminated by the vorticity. We can speak about long enough waves only, with the wave length more than the characteristic scale of white-capping.

Let $U_{10} \simeq 15\, m/sec$; then the critical wave length is

$$\lambda_{max} \simeq \frac{2\pi}{k_0} = \frac{2\pi\, U_{10}^2}{g} \sim 144\, m.$$

If we accept that the characteristic size of wave-breaker, $\lambda_{min} \simeq 1\, m$, we can speak about pure potential waves in the range $\lambda_{max} < \lambda < \lambda_{min}$. Even for $\lambda \sim 1\, m$, we have $\gamma/\omega \simeq 7 \cdot 10^{-3} \ll 1$. Study of more short waves is important if we are interested in microwave or optical images of ocean surface. Moreover, short waves realize the lion share of stress or momentum transport from air to water. However, short waves with $\lambda \leq \lambda_{min}$ do not contribute essentially to the energy balance.

The impact of wind on the sea surface is of two kinds. First, the presence of wind makes the sea surface unstable. This effect is universal. It takes place for the laminar (Miles mechanism [4]) as well as for turbulent boundary layers. In the last case turbulent fluctuations of air pressure create also "seeds" of unstably growing waves (Phillips mechanism [5]). The central question of the wind-driven sea theory is the following: what is the mechanism of instability arresting? As far as instability is a linear process, this mechanism must be nonlinear.

It was Phillips again who in 1958 offered the mechanism of instability arresting [6]. To describe his scenario we must mention first that the characteristic feature of the wind-driven sea kinetics is the downshift of the spectral peak. When wind starts blowing, it excites first the short waves, such that the spectral peak is posed at $k_{max} \gg k_0$. Then the spectral peak drifts to small wave numbers, approaching the critical value $k_0$. This downshift, which is the increasing of the mean wave length, one can easily observe. When $k$ approaches $k_0$, the downshift is arrested, and the sea reaches the stage of "maturity".

In 1960, Phillips explained this process in the following way [7]. Initially waves are excited in the area, where $\gamma_k$ has maximum, $k \sim k_{max}$. The waves grow until their amplitude reaches the value allowed for the stationary Stokes wave of the wavelength $\lambda_m \simeq 2\pi/k_m$. At that moment all wave crests have wedge-like shape with the angle $120^o$. Further growth is not possible. The obtuse wedges turn to white-caps, which absorb all energy supplied to sea by the wind. When waves are saturated at $k \sim k_{max}$, they are not yet saturated for $k < k_{max}$. Thus more long waves continue to grow until saturation completes. As far as they are saturated on more high level, the downshift of the spectral peak is in progress. The downshift arrests only at the moment, when the wave number of the spectral peak reaches its critical value $k_0$. More long waves move faster than wind and do not take energy from the atmospheric boundary layer.

This is an attractive scenario but it is inconsistent for several reasons. First of all, the Stokes wave of a finite amplitude is unstable. The theory of this instability has a long history described in details in the article [8]. The analytical theory of this instability is developed for waves of small amplitude only. However, our recent numerical simulation shows [9] that waves of high amplitudes, comparable with the limiting, are also badly unstable. Thus the Stokes waves can not be used as a model for the real wind-driven sea.

Another reason of inconsistency is even more important. One of the main characteristics of the wind-driven sea is its average steepness $\mu$. It could be defined by several different ways; the most "scientific" definition is the following:

$$\mu^2 = \langle |\nabla\eta|^2 \rangle. \tag{1.5}$$

However, steepness defined by this way can not be easily measured. As a result, oceanographers prefer another definition of $\mu$ (thereafter we denote it as $\mu_p$):

$$\mu_p^2 = \langle \eta^2 \rangle \, k_p^2 \; = \; \langle \eta^2 \rangle \, \frac{\omega_p^4}{g^2} \tag{1.6}$$

In this equation, $k_p$ and $\omega_p$ are the wave number and the frequency of the spectral peak. For a typical sea, $\mu > \mu_p$, though for the limiting Stokes wave $\mu$ and $\mu_p$ are close to each other, $\mu \simeq \mu_p \simeq 0.3$.

According to Phillips theory we have to observe exactly this value of steepness. However, that high level of steepness was never observed. Being artificially created, these steep waves immediately break and lose their energy, making only a few oscillations. Field observations show that steepness essentially depends on the "wave age", $a = c_p/U$, which is the ratio of the phase and wind velocities. The younger waves are, the higher the steepness, though even for very young waves, when $a \simeq 0.1$, the steepness is limited: $\mu_p \leq 0.12$. The "mature" sea is essentially less steep: $\mu_p \simeq 0.06 - 0.07$. This is very serious discrepancy with Phillips theory. Wind-driven ocean waves are much more smooth than predicts the Phillips model.

One more weak point of Phillips theory is that it fails to predict with accuracy the energy spectrum. In the stationary sea the autocorrelation function of elevation,

$$\hat{F}(\tau) = \hat{F}(-\tau) = \langle \eta(t) \, \eta(t+\tau) \rangle, \tag{1.7}$$

does not depend on $t$. Its cosine Fourier transform, defined as

$$F(\omega) = \frac{1}{\pi} \int_0^\infty \hat{F}(\tau) \, \cos \omega \, \tau \, d\tau, \tag{1.8}$$

traditionally is called the energy spectrum of the surface. Apparently the mean squared elevation is

$$\langle \eta^2 \rangle = \hat{F}(0) = \int_0^\infty F(\omega) \, d\omega. \tag{1.9}$$

Spectrum $F(\omega)$ has dimension $L^2 T$. Why is it called the energy spectrum? Let us introduce function

$$E(\omega) = \rho_w \, g \, F(\omega), \tag{1.10}$$

that has dimension $M/T$, the same as the spectral distribution of energy density. Indeed, in an ensemble of non-interacting waves $E(\omega) \, d\omega$ is the spectral density of wave energy propagating in all directions and having frequencies within the spectral band $\omega \to \omega + d\omega$. Let us stress that the very concept of energy spectrum makes sense in the linear theory only. The real sea is nonlinear, and such quantity as energy spectrum just cannot be properly defined. It would be better to call $F(\omega)$ the "elevation spectrum", but in this article we will use the traditional language.

One can notice that the energy spectrum can be constructed from gravity acceleration $g$ and circular frequency $\omega$ in a unique way:

$$F(\omega) = \frac{\alpha\, g^2}{\omega^5} = F_{Ph}(\omega). \tag{1.11}$$

This is the famous Phillips spectrum, where $\alpha$ is a dimensionless "Phillips constant". It is widely considered that Phillips spectrum is automatically consistent with the Phillips scenario of wind-driven sea based on the concept of Stokes wave. Actually, this question is not that simple. Realization of Phillips spectrum presumes not only the domination of Stokes breakers – it presumes also equipartition of the breakers over their wave numbers as well as validity of linear dispersion law $\omega^2 = gk$. These questions are discussed in recent articles [36, 37]. Another problem is the divergence of steepness. In the case of Phillips spectrum, the integral in the expression for steepness,

$$\mu^2 = \frac{1}{g^2}\int_0^\infty \omega^4\, F(\omega)\, d\omega, \tag{1.12}$$

is divergent and must be cut off at certain $\omega = \omega_{max}$. Also, there is the lower border of Phillips spectrum applicability, $\omega = \omega_{min}$. As a result, the steepness will be:

$$\mu^2 = \alpha\, \ln \frac{\omega_{max}}{\omega_{min}}. \tag{1.13}$$

Here we suppose that spectrum (1.11) is realized within the range $\omega_{min} < \omega < \omega_{max}$. If $\omega_{max} \to \infty$, then $\mu \to \infty$. Meanwhile the average steepness of any Stokes wave is finite. This is a serious contradiction, showing that the theory of "Phillips sea" is far from being simple. We will discuss this subject in more details in Chapter 8.

The Phillips spectrum does not include the wind velocity. Since 1972, starting with the article of Toba [11], experimentalists found that equation (1.11) does not explain the results of laboratory and field observations. The "rear faces" of the spectra, $\omega > \omega_p$, are described much better by the less steep spectrum

$$F(\omega) = \frac{\beta_1\, g\, V}{\omega^4} = F_1(\omega). \tag{1.14}$$

Here $\beta_1$ is a small dimensionless parameter of order $\epsilon \sim 10^{-3}$, $V$ is some characteristic velocity composed of the wind velocity $U$ and the phase velocity of the spectral peak $c_p$. In seria of articles [38, 39] Resio and Long claim that

$$\beta \simeq 0.05, \quad V = \left(U^2\, c_p\right)^{1/3}.$$

The more detailed study of experimental data shows that just after the spectral peak function $F(\omega)$ is even less steep:

$$F(\omega) = F_2(\omega) = \frac{\beta_2\, g\, V}{\omega^{11/3}\, \omega_p^{1/3}}. \tag{1.15}$$

In many experiments the subject of measurements is the spatial spectrum

$$I_k = 2\pi \langle |\eta_k|^2 \rangle k, \tag{1.16}$$

with a typical behavior right after the spectral peak described as:

$$I_k^{(2)} = \frac{\beta_2}{2} \frac{V}{g^{1/2}\, k^{11/3}\, k_p^{1/6}}, \tag{1.17}$$

$$I_k^{(1)} = \frac{\beta_1}{2} \frac{V}{g^{1/2}\, k^{5/2}}. \tag{1.18}$$

Spectra (1.17), (1.18) are "spacial analogs" of frequency spectra (1.16) and (1.15). Phillips spectrum (1.11) also has the spatial analog:

$$I_{Ph}(k) = \frac{1}{2} \frac{\alpha}{k^3}. \tag{1.19}$$

Factor 1/2 appears from the identity $\omega/g\, d\omega = 1/2\, dk$.

Spectra (1.14 – 1.18) are often called Zakharov-Filonenko (ZF) spectra. They were predicted theoretically in 1966 [10, 34], later on were observed in many field and laboratory experiments (see, for instance [11–13, 18]) and found in numerical experiments [14-16]. In 1985, Phillips acknowledged [17] that spectrum (1.14) is much closer to reality than (1.11). Theoretical explanation of spectra (1.14) - (1.18) is very simple: they are Kolmogorov-Zakharov (KZ) spectra carrying constants of motion – the energy and the wave action – along the $k$-space. Spectra (1.14), (1.18) describe the direct cascade – the transport of energy from large to small scales, while spectra (1.16), (1.17) describe the inverse cascade – the transport of wave action to large scales. These spectra appeared first in PhD thesisis of V. Zakharov in 1966 [34] and were published in the refereed scientific journal much later, in 1982 [35].

Ubiquitity of ZF-spectra, $F(\omega) \simeq \omega^{-4}$, does not mean that Phillips spectra $F_{Ph} \simeq \omega^{-5}$ and $I_p(\omega) \simeq k^{-3}$ are not observed in the ocean and in the wave tanks. They are systematically noticed [19–21] as high-frequency, in other words, short scale asymptotics. Also, they were observed in numerical experiments [22]. Appearance of these asymptotics is inevitable: KZ-spectra decay too slowly to provide convergence of average steepness. Meanwhile, the average steepness should be small, $\mu^2 < 0.02$, otherwise the intensive wave-breaking will immediately occur to consume any excessive energy.

The Phillips spectrum does not arrest the divergence of steepness completely, but the logarithmic divergence is very slow, and the factor $\ln(\omega_{max}/\omega_{min}) \simeq 2.5$. From (1.13) we can estimate the Phillips constant $\alpha$ to be of order $10^{-2}$. In reality, $\alpha$ varies in the limit $5 \cdot 10^{-3} < \alpha < 0.01$. We should stress that with increasing of steepness the onset of wave-breaking is very sharp. This is a threshold phenomenon, like the second-order phase transition. We will discuss this point in Chapter 8.

As far as $\mu$ is small, one can use expansion in powers of $\mu$ as a basic analytic technique for study of nonlinear wave interaction. Performing this expansion we realize that we have to deal with resonant interactions of certain amount of waves that form "a resonant group". For gravity waves on two-dimensional plane the most important groups are quadruplets of waves with wave vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4$, satisfying resonant conditions

$$\begin{aligned} \vec{k}_1 + \vec{k}_2 &= \vec{k}_3 + \vec{k}_4, \\ \omega_{k_1} + \omega_{k_2} &= \omega_{k_3} + \omega_{k_4}. \end{aligned} \tag{1.20}$$

We should stress that the study of these resonance processes makes sense only if the basin where waves propagate is large enough. If $\lambda$ is the largest wave-length in the quartet and $\mu$ is the characteristic steepness, the size of basin must be at least $L \sim \frac{1}{\mu^2}\lambda$. If $\lambda \sim 100\ m$ and $\mu \simeq 0.1$, one gets $L > 10\ km$. This condition is fulfilled in the ocean but hardly can be satisfied in wave-tanks. There is another type of nonlinear wave interaction, localized in space: the white-capping. In some situations, especially for very young waves, this phenomenon can play a very important role, in experimental wave tanks as well. However for the well-developed wind-driven sea the white-capping is an effect of secondary importance.

There is no doubts that the wind-driven sea needs a statistical description. It can be presented as an ensemble of gravity waves of different scales. These waves take energy from wind, lose energy in the white-capping event, and exchange energy in four-wave resonant interactions. All these three processes can be described by a single kinetic equation, first derived by Hasselmann in 1962 [23, 24], named after him, and written for the spectrum of "wave action", $N(\vec{k}, \vec{r}, t)$:

$$\frac{\partial N}{\partial t} + \frac{\partial \omega}{\partial \vec{k}} \nabla N = S_{nl} + S_{in} + S_{diss}. \tag{1.21}$$

In this equation $\omega = \sqrt{gk}$ is the dispersion law for gravity waves, $S_{in}$ is the input from wind, $S_{diss}$ is the dissipation due to white-capping, and $S_{nl}$ is the collision term that describes four-wave resonant interaction. The last term can be derived from the "first principles" and we will perform the brief derivation in Chapter 2. In fact, Hasselmann equation is just a kinetic equation for bosonic particles known in theoretical physics since 1928 [40]. Kinetic equations have standard thermodynamic solutions, however we will concentrate our attention on completely different, Kolmogorov-like solutions. Theory of these solutions is called weak turbulence.

It is astonishing what a large amount of information could be extracted from a careful study of pure conservative kinetic equation

$$\frac{\partial N}{\partial t} + \frac{\partial \omega}{\partial k} \nabla N = S_{nl} \tag{1.22}$$

and even from the ultimately simple equation

$$S_{nl} = 0. \tag{1.23}$$

It would not be correct to think that (1.23) is similar to other integral equations. In fact, it is much closer to the nonlinear elliptic equation of second order and has a rich family of solutions parameterized by an arbitrary function of one variable. In particulary, it has the family of solutions parameterized by three parameters: fluxes of energy, wave action, and momentum. The simplest ones, the parametric solutions, are powerlike (1.17)–(1.18). As we have mentioned before, they describe rare faces of observed ocean spectra perfectly well. We will discuss this fact in Chapter 7.

From the analytical view-point equation (1.22) is even more interesting. It has a rich family of self-similar solutions [1, 26, 27], which pretty well describe the dependance of main characteristics of wind-driven sea – the mean energy and the frequency of spectral peak – on fetch (distance from the shore) and duration (time since the start of wind blowing). These solutions have free parameters depending on the choice of model for $S_{in}$. The dissipative term $S_{diss}$ for well-developed sea plays a role of universal sink of energy at high wave numbers and does not affect essentially the dynamics of spectral peak (see Chapter 8). In other words, we strongly disagree with presumption that $S_{diss}$ makes an essential contribution to the energy balance. Actually, in the case of $S_{diss}$ we even don't have such idealized but accurate model that gives the Miles theory. All existing models are purely heuristic; they are not supported by theoretical considerations or experimental observations. There is one thing only that we know for sure: $S_{diss}$ should not surpass $S_{in}$, otherwise the wind will not excite waves. In Chapter 7 we offer the first scientifically justified model of $S_{diss}$. In Chapter 9 we discuss shortly the results of numerical simulation of the wind-driven sea.

We are ready now to formulate the central message of this article. The wind-driven sea is a very complicated object. Such complex and difficult for study phenomena as excitation of waves by wind and the wave-breaking make the development of self-consistent theory of wind-driven sea an arduous problem. Nevertheless, the situation is not hopeless. The leading physical process that dominates in the ocean and in the certain wave-tanks is the nonlinear four-wave interaction; this process can effectively be studied analytically and numerically. It is widely accepted that on the "rear face" of the spectrum, in the equilibrium range of wave numbers, the Hasselmann equation should be reduced to the stationery equation

$$S_{nl} + S_{in} + S_{diss} = 0. \tag{1.24}$$

Some researches suppose that all three terms in this equation are equally important, while some others believe that $S_{nl}$ is the term of secondary importance, comparable with $S_{in}$. Our viewpoint is completely different: the leading term in the balance equation (1.24) is the collision term $S_{nl}$. Thus the study of equation (1.23) is a matter of key importance. It has plenty solutions, which can be used for description of observed spectra and are characterized by constant fluxes of wave action, energy and momentum. In the real sea in presence of $S_{in}$ and $S_{diss}$ the fluxes are not constant: they are slowly varying functions on wave vector $k$. If $S_{in}$ and $S_{diss}$ are known, the solution behind the spectral peak can be efficiently found by combination of analytic and numeric approaches.

In the same way, the conservative equation (1.22), though not being complete, is a very good approximation of the more exact equation (1.21). It has self-similar solution that depends on two arbitrary parameters. In the real sea these constants are slow varying functions on wave vector and time. They can be found from the average equation

$$\frac{\partial}{\partial t}\langle N\rangle + \langle\frac{\partial\omega}{\partial k}\,\nabla N\rangle = \langle S_{in} + S_{diss}\rangle, \tag{1.25}$$

where averaging means integration over $k$. In Chapter 6 we discuss equation (1.25) in more details and just mention now that to get it we have assumed $\langle S_{nl}\rangle = 0$. What is formulated above sounds like a program for future researchers, however this program is rather far advanced, and in this article we present some aspects of its current status.

# 2 Kinetic Hasselmann equation

We study the weakly nonlinear waves on the surface of an ideal fluid on infinite depth in an infinite basin. The vertical coordinate is

$$-\infty < z < \eta(r,t), \qquad r = (x,y), \tag{2.1}$$

the fluid is incompressible,

$$div\, V = 0, \tag{2.2}$$

and velocity $V$ is a potential field

$$V = \nabla \Phi, \tag{2.3}$$

where potential $\Phi$ satisfies the Laplace equation

$$\Delta \Phi = 0 \tag{2.4}$$

under boundary conditions

$$\Phi|_{z=\eta} = \Psi(r,t), \qquad \Phi_z|_{z=-\infty} = 0. \tag{2.5}$$

The total energy of the fluid, $H = T + U$, has the following terms:

$$T = \frac{1}{2}\int d\vec{r}\int_{-\infty}^{\eta}(\nabla \Phi)^2\,dz = \frac{1}{2}\int \Psi\,\Phi_n\,dS, \tag{2.6}$$

$$U = \frac{1}{2}\,g\int \eta^2\,d\vec{r}. \tag{2.7}$$

The Dirichlet-Neumann boundary problem (2.4), (2.5) is uniquely resolved; thus the flow is defined by fixation of $\eta$ and $\Psi$. This pair of variables is canonical; thus the evolution equations for $\eta$, $\Psi$ take the form [28]:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{2.8}$$

After non-symmetric Fourier transform,

$$\Psi(r) = \int \Psi(k)\,e^{ikr}\,dk, \qquad \Psi(k) = \frac{1}{(2\pi)^2}\int \Psi(r)\,e^{-ikr}\,dr, \tag{2.9}$$

equation (2.8) reads:

$$\frac{\partial \eta}{\partial t} = \frac{\delta \tilde{H}}{\delta \Psi_k^*}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta \tilde{H}}{\delta \eta_k^*}, \tag{2.10}$$

$$\tilde{H} = \frac{1}{4\pi^2}\,H = H_0 + H_1 + H_2 + \cdots \tag{2.11}$$

In [41, 42] was shown that Hamiltonian $\tilde{H}$ can be expanded in Taylor series in powers of $\eta$:

$$\begin{aligned}
H_0 &= \frac{1}{2}\int \left\{k|\Psi_k|^2 + g\,|\eta_k|^2\right\} dk \\
H_1 &= \frac{1}{2}\int L^{(1)}(k_1,k_2)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\,\delta(\vec{k_1}+\vec{k_2}+\vec{k_3})\,dk_1\,dk_2\,dk_3 \\
H_2 &= \frac{1}{2}\int L^{(2)}(k_1,k_2,k_3,k_4)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\eta_{k_4}\,\delta(k_1+k_2+k_3+k_4)\,dk_1 dk_2 \eta_{k_3}\eta_{k_4}
\end{aligned} \tag{2.12}$$

Here

$$L^{(1)}(k_1,k_2) = -(k_1,k_2) - |k_1|\,|k_2| \tag{2.13}$$
$$L^{(2)}(k_1,k_2,k_3,k_4) = \frac{1}{4}\,|k_1||k_2|\,\{-2|k_1| - 2|k_2| + |k_1+k_3| + |k_1+k_4| + |k_2+k_3| + |k_2+k_4|\}$$

Variables $\eta, \Psi$ are not the optimal canonical ones. We can perform a canonical transformation to new variables in such a way that cubic terms in the Hamiltonian will be cancelled. The details of this transformation are given in Appendix 1.

Let us denote the new variables as $\xi, R$. In new variables the cubic terms in the Hamiltonian vanish and one can introduce the normal variables $b_k$,

$$\begin{aligned} \xi_k &= \frac{1}{\sqrt{2}}\left(\frac{k}{g}\right)^{1/4}(b_k + b^*_{-k}), \\ R_k &= \frac{i}{\sqrt{2}}\left(\frac{g}{k}\right)^{1/4}(b_k - b^*_{-k}). \end{aligned} \tag{2.14}$$

In new variables equation (2.10) takes form

$$\frac{\partial b_k}{\partial t} + i\,\frac{\delta \tilde{H}}{\delta b^*_k} = 0, \tag{2.15}$$

where the Hamiltonian is expressed as

$$\begin{aligned} \tilde{H} &= \int \omega_k\, b_k\, b^*_k\, dk + \frac{1}{4}\int T_{k_1k_2k_3k_4}\, b^*_{k_1}\, b^*_{k_2}\, b_{k_3}\, b_{k_4} \times \\ &\quad \times \delta(k_1+k_2-k_3-k_4)\, dk_1\, dk_2\, dk_3\, dk_4, \end{aligned} \tag{2.16}$$

and the coupling coefficient $T_{k_1k_2,k_4k_3}$ satisfies the symmetry conditions:

$$T_{k_1k_2,k_3k_4} = T_{k_2k_1,\,k_3k_4} = T_{k_1k_2,\,k_4k_3} = T_{k_2k_4,k_1k_2}. \tag{2.17}$$

The explicit expression for $T$ is complicated [32, 42]:

$$\begin{aligned} T_{12,34} &= \frac{1}{2}\left(\tilde{T}_{12,34} + \tilde{T}_{21,34}\right), \\ \tilde{T}_{12,34} &= -\frac{1}{2}\,\frac{1}{(k_1k_2k_3k_4)^{1/4}}\Bigg\{ -12k_1k_2k_3k_4 - \\ &\quad -2(\omega_1+\omega_2)^2\Big[\omega_3\omega_4\Big((\vec{k}_1\cdot\vec{k}_2) - k_1k_2\Big) + \omega_1\omega_2\Big((\vec{k}_3\cdot\vec{k}_4) - k_3k_4\Big)\Big]\,\frac{1}{g^2} \\ &\quad -2(\omega_1-\omega_3)^2\Big[\omega_2\omega_4\Big((\vec{k}_1\cdot\vec{k}_3) + k_1k_3\Big) + \omega_1\omega_3\Big((\vec{k}_2\cdot\vec{k}_4) + k_2k_4\Big)\Big]\,\frac{1}{g^2} \end{aligned}$$

$$-2(\omega_1-\omega_4)^2\Big[\omega_2\omega_3\Big((\vec{k}_1\cdot\vec{k}_4)+k_1k_4\Big)+\omega_1\omega_4\Big((\vec{k}_2\cdot\vec{k}_3)+k_2k_3\Big)\Big]\,\frac{1}{g^2}$$
$$+[(\vec{k}_1\cdot\vec{k}_2)+k_1k_2][(\vec{k}_3\cdot\vec{k}_4)+k_3k_4]+[-(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][-(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]$$
$$+[-(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][-(\vec{k}_2\cdot\vec{k}_3)+k_2k_3]$$
$$+4(\omega_1+\omega_2)^2\,\frac{[(\vec{k}_1\cdot\vec{k}_2)-k_1k_2][(\vec{k}_3\cdot\vec{k}_4)-k_3k_4]}{\omega_{1+2}^2-(\omega_1+\omega_2)^2}$$
$$+4(\omega_1-\omega_3)^2\,\frac{[(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]}{\omega_{1-3}^2-(\omega_1-\omega_3)^2}$$
$$\left.+\,4(\omega_1-\omega_4)^2\,\frac{[(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][(\vec{k}_2\cdot\vec{k}_3)+k_2k_3]}{\omega_{1-4}^2-(\omega_1-\omega_4)^2}\right\}. \tag{2.18}$$

Here $\omega_i=\sqrt{g\,|k_i|}$. Then equation (2.15) reads:

$$\frac{\partial b_k}{\partial t}+i\left(\omega_k\,b_k+\frac{1}{2}\int T_{kk_1k_2k_3}\,b^*_{k_1}b_{k_2}b_{k_3}\;\delta(k+k_1-k_2-k_3)\,dk_1dk_2dk_3\right)=0. \tag{2.19}$$

This equation was derived and studied in [28, 41, 42].

Equation (2.10) has natural motion constants of energy and momentum,

$$\tilde{H}=const,\qquad \tilde{P}=\int\vec{k}\,b_k\,b_k^*\,dk=const. \tag{2.20}$$

while equation (2.19) conserves one additional constant $N$:

$$\tilde{N}=\frac{1}{g}\int|b_k|^2\,dk. \tag{2.21}$$

Thereafter we call (2.21) the "wave action". The energy and momentum are exact motion constants, while $\tilde{N}$ is only an approximate integral. Next-order resonant wave interactions, including five-wave interaction, destroy conservation of wave action. However, this process is slow (see [43]).

Let us start now the statistical description of the basic dynamic equation. Since this moment we assume that $\eta(\vec{r},t),\Psi(\vec{r},t)$ are random fields and in the first approximation consider them statistically homogenous. One can introduce the following correlation functions:

$$\langle\eta(r)\,\eta(r+R)\rangle = I(R) \tag{2.22}$$
$$\langle\eta_k\,\eta_{k'}\rangle = I_k\,\delta(k+k') \tag{2.23}$$
$$\langle b_k\,b^*_{k'}\rangle = g\,N_k\,\delta(k-k') \tag{2.24}$$
$$\langle\xi_k\,\xi_{k'}\rangle = \tilde{I}_k\,\delta(k+k') \tag{2.25}$$
$$I(R) = \int I_k\,e^{ikR}\,dk \tag{2.26}$$

Functions $I(k)$ and $\tilde{I}(k)$ are close to each other. In the area of spectral maximum

$$\Delta(k) = \frac{\tilde{I}(k) - I(k)}{I(k)} \sim \mu^2$$

is small, however it grows fast at $k \to \infty$. This subject is discussed in details in the forecoming article [43]. By definition

$$\tilde{I}(k) = \frac{\omega(k)}{2}\,(N_k + N_{-k}). \tag{2.27}$$

Thereafter we assume that $N = N(r,k,t)$ is also a slowly varying function on coordinate $r$ and accept that $N = N(r,k,t)$ satisfies the Hasselmann kinetic equation. The derivation of the resulting equation

$$\frac{dN_k}{dt} = S_{nl} = \pi g^2 \int |T_{kk_1,k_2k_3}|^2\,\delta(k + k_1 - k_2 - k_3^*)\,\delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times$$
$$\times(N_{k_1}N_{k_2}N_{k_3} + N_kN_{k_2}N_{k_3} - N_kN_{k_1}N_{k_2} - N_kN_{k_1}N_{k_3})dk_1dk_2dk_3. \tag{2.28}$$

can be found, for instance, in [41, 42]. Here

$$\frac{dN_k}{dt} = \frac{\partial N_k}{\partial t} + \frac{\partial\omega}{\partial k}\,\nabla N_k \tag{2.29}$$

and $T_{kk_1k_2k_3}$ is a homogenous function of order 3:

$$T_{\lambda k,\lambda k_1,\lambda k_2,\lambda k_3} = \lambda^3\,T_{kk_1k_2k_3}. \tag{2.30}$$

Simple calculation shows that $T_{k,k,k,k} = T = 2k^3$.

For smooth spectra $S_{nl}$ can be estimated as follow:

$$S_{nl} \simeq 4\pi\,\omega_p\,\mu_p^4\,N_k. \tag{2.31}$$

Notice that $\mu_p^4 \simeq \langle\eta^2\rangle^2\omega_p^8/g^4$, thus $S_{nl} \simeq \omega_p^9$. That makes $S_{nl}$ very sensitive to the accurate calculation of $\omega_p$. Moreover, the experimental spectra usually have "peakedness", when the essential part of wave energy is concentrated in a narrow spectral band $\delta k$ near the spectral peak wave number $k_p$. In this case the resonant interaction term is much more powerful. To estimate it, let us change the variables, $k_i = k_p + \kappa_i$, and perform the expansion

$$\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3} = (\vec{V}, \vec{\kappa} + \vec{\kappa}_1 - \vec{\kappa}_2 - \vec{\kappa}_3) + \Delta\omega_k - \Delta\omega_{k_1} - \Delta\omega_{k_2} - \Delta\omega_{k_3} \tag{2.32}$$

$$\vec{V} = \frac{\partial\omega}{\partial\vec{k}}, \qquad \Delta\omega_k = \frac{1}{2}\,\frac{\partial^2\omega}{\partial k_i\partial k_j}\bigg|_{k=k_p}\,\kappa_i\kappa_j$$

After replacement $T_{kk_1k_2k_3} \to T_{kp,kp,kp,kp} = 2k_p^3$, the nonlinear term $S_{nl}$ reads:

$$S_{nl} = 4\pi k_p^6 \int (N_{\kappa_1} N_{\kappa_2} N_{\kappa_3} - N_\kappa N_{\kappa_2} N_{\kappa_3} - N_\kappa N_{\kappa_1} N_{\kappa_2} - N_\kappa N_{\kappa_1} N_{\kappa_3}) \times$$
$$\times \delta(\kappa + \kappa_1 - \kappa_2 - \kappa_3)\, \delta(\Delta\omega_\kappa + \Delta\omega_{\kappa_1} - \Delta\omega_{\kappa_2} - \Delta\omega_{\kappa_3})\, d\kappa_1\, d\kappa_2\, d\kappa_3 \quad (2.33)$$

If $\delta k$ is characteristic width of the spectrum, the characteristic value of $\Delta\omega$ is the following:

$$\Delta\omega \simeq \frac{1}{8}\omega_p \frac{(\delta\, k)^2}{k_p^2}. \quad (2.34)$$

Replacement $\omega_p \to \Delta\omega$ leads to multiplication of $S_{nl}$ by factor $\omega_p/\Delta\omega \simeq 8k_p^2/(\delta k)^2$, and finally we get for $S_{nl}$ the following estimate:

$$S_{nl} \simeq \Lambda\, \omega_p\, \mu_p^4\, N_k^3 \quad (2.35)$$

Here $\Lambda$ is the "enhancement factor" that depending on peakedness varies within limits $10 < \Lambda < 10^3$. An accurate estimate of $\Lambda$ is a very delicate problem because of its strong susceptibility from detailed picture of the spectrum. Moreover, the enhancing factor could strongly depend on frequency $\omega$. Anyway, we insist that after a very short initial period of wind-sea development, $S_{nl}$ becomes the leading player in the wind-sea balance. The negligence of $\Lambda$ is the main reason that leads to underestimating of $S_{nl}$ role in the energy balance of wind-driven sea. Notice, that peakedness should not be too strong, otherwise the spectra become too narrow for regular statistical description. The narrow spectra are modulationally unstable and generate freak waves (see [44, 45]).

The next important question concerns the constants of motion. For infinite statistically homogenous sea all motion constants are infinite, however their spatial density is finite. Quantities

$$N = \int N_k\, dk, \quad E = \int \omega_k\, N_k\, dk = \langle \eta^2 \rangle, \quad \vec{P} = \int \vec{k}\, N_k\, d\vec{k}, \quad (2.36)$$

are finite. We should expect that they conserve energy but it is true only partly. This question was carefully studied in article [29]. In fact, only $N_k$ is the motion constant, while $E$ and $\vec{P}$ "escape" to the area of very short waves, $k \to \infty$, forming powerlike Kolmogorov type tails. They are just "formal" motion constants. In more details this question is described in Chapter 6.

Thereafter we will often use polar coordinates in $k$-space. More exactly speaking, we will use coordinates $\omega, \phi$, such that $|k| = \omega^2/g$. In these coordinates the action spectrum $N(\omega, \phi)$ is defined as

$$N(\omega, \phi)\, d\omega\, d\phi = N(\vec{k})\, d\vec{k}, \quad (2.37)$$

that actually means that

$$N(\omega,\ \phi) = \frac{2\omega^3}{g^2}\, N(\vec{k}). \tag{2.38}$$

Now we can introduce the energy spectrum

$$F(\omega, \phi) = \omega\, N(\omega, \phi) \tag{2.39}$$

and define

$$N(\omega) = \int_0^{2\pi} N(\omega, \phi)\, d\phi, \tag{2.40}$$

$$F(\omega) = \int_0^{2\pi} F(\omega, \phi)\, d\phi. \tag{2.41}$$

Notice that we already have defined $F(\omega)$ by equation (1.8). It is not obvious that equation (2.43) leads to the same $F(\omega)$, but they coincide in a certain approximation. Let us perform the time Fourier transform of $b_k(t)$ and introduce the pair correlation function

$$\langle b(k, \omega)\ b^*(k', \omega') \rangle = g\, N(k, \omega)\, \delta_{k-k'}\, \delta_{\omega-\omega'} \tag{2.42}$$

In the case of weak nonlinearity, $N(k, \omega)$ and $N(k)$ are connected by relation

$$N(k, \omega) = N(k)\, \delta(\omega - \omega_k), \tag{2.43}$$

which can be easily proved for the waves of very small amplitude. If (2.43) is satisfied, definitions of $F(\omega)$ by both equations (1.8) and (2.41) coincide.

Notice also that the spatial spectrum of elevation (1.16) can be expressed in terms of $N_k$ as follow:

$$I_k = \pi k (N_k + N_{-k}) \tag{2.44}$$

# 3 Kolmogorov-type spectra

In this chapter we address the following question: How to solve the stationery kinetic equation

$$S_{nl} \equiv 0\,? \tag{3.1}$$

Formally speaking, this equation has thermodynamically equilibrium solutions

$$N_k = \frac{T}{\omega_k + \mu}\,, \tag{3.2}$$

where temperature $T$ and $\mu$ are constants. However, we should be more careful. Suppose that $N_k$ depends on modulus $\vec{k}$ only. Let $k = |\vec{k}|$, and

$$N_k = k^{-x}. \tag{3.3}$$

By plugging (3.3) into (3.1) we find that each particular term in $S_{nl}$ is diverging, but in different terms the divergence can be cancelled, thus there is a "window of opportunity" for the exponent $x$. As a result,

$$S_{nl} = g^{3/2}\, k^{-3x+19/2}\, F(x). \tag{3.4}$$

Here $F(x)$ is a dimensionless function, defined inside interval $x_1 < x < x_2$. The edges of the window, $x_1$ and $x_2$, are the subject for determination.

Let us study the quadruplet of waves with wave vectors $\vec{k}, \vec{k}_1, \vec{k}_2, \vec{k}_3$, satisfying resonant conditions (1.12). Suppose that $|k_1| \ll |k|$. The three-wave resonant condition,

$$\vec{k} = \vec{k}_2 + \vec{k}_3, \qquad \omega_k = \omega_{k_2} + \omega_{k_3}, \tag{3.5}$$

has no nontrivial solutions, thus one of vectors $\vec{k}_2, \vec{k}_3$ must be small. If $|k_3| \ll |k_2|$, then

$$\begin{aligned} \vec{k}_2 &= \vec{k} + \vec{k}_1 - \vec{k}_3, \\ \omega(k_2) &= \sqrt{gk}\left(1 + \frac{1}{2}\frac{(k, \vec{k}_1 - \vec{k}_3)}{k^2} + \cdots\right), \end{aligned} \tag{3.6}$$

and we can put $|k_3| = |k_1|$. Vectors $\vec{k}_1, \vec{k}_3$ are small and have approximately the same length $k_1$. If vector $k$ is directed along axis $x$, the coupling coefficient $T_{kk_1k_2k_3}$ depends on four parameters $k, k_1, \theta_1, \theta_3$. Here $\theta_1, \theta_3$ are angles between $\vec{k}_1, \vec{k}_3$ and $\vec{k}$. Remembering that $k_1 \ll k$, we calculate the coupling coefficient in this asymptotic domain. A tedious calculation presented in article [25] leads to the following compact result:

$$\begin{aligned} T_{kk_1k_2k_3} &\simeq \frac{1}{2} k\, k_1^2\, T_{\theta_1,\theta_3}, \\ T_{\theta_1,\theta_2} &= 2(\cos\theta_1 + \cos\theta_3) - \sin(\theta_1 - \theta_3)(\sin\theta_1 - \sin\theta_3). \end{aligned} \tag{3.7}$$

On the diagonal $k_3 = k_1$, $\theta_3 = \theta_1$ we get a very simple expression published in 2003 [32]:

$$T_{kk_1} \simeq 2k_1^2 k\, \cos\theta_1. \tag{3.8}$$

Suppose that spectrum is separated to the low-frequency component $N_0(k)$ and the high-frequency component $N_1(k)$. We assume that $N_1 \ll N_0$ and take into account the interaction between $N_0$ and $N_1$ only. One can see that $N_1$ satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_i} D_{ij}\, k^2 \frac{\partial}{\partial k_j} N_1, \tag{3.9}$$

where $D_{ij}$ is the diffusion tensor

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq\, q^{17/2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_3 |T(\theta_1,\theta_3)|^2\, p_i p_j N(\theta,q) N(\theta_3,q) \tag{3.10}$$

$$p_1 = \cos\theta_1 - \cos\theta_3, \quad p_2 = \sin\theta_1 - \sin\theta_3$$

If spectrum is isotropic and does not depend on angle $\theta$, we get the further simplification:

$$D_{ij} = D\,\delta_{ij}, \qquad D = \frac{5}{8}\pi^3 g^{3/2} \int_0^\infty q^{17/2} N^2(q) dq. \tag{3.11}$$

Taking into account (3.3), we find that diffusion coefficient $D$ diverges at $k \to 0$ if $x > 19/4$. Thus $x_2 = 19/4$.

Let us find behavior of function $F(x)$ near $x = x_2$. In the isotopic case equation (3.9) reads

$$\frac{\partial N_1}{\partial t} = \frac{D}{k} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} N_1. \tag{3.12}$$

If $k \to 19/4$, we get the following estimate:

$$F(x) = \frac{19}{4} \cdot \frac{11}{4} \cdot \frac{5\pi^3}{16} \frac{1}{19/4 - x} \simeq \frac{126.4}{19/4 - x} \tag{3.13}$$

To find $x_1$, the lower end of window, we should study the influence of short waves to the long ones. Let us suppose that $|k_1|, |k_2| \gg k$. In the first approximation $|k_3| = |k|$, and the resonant interaction $S_{nl}$ can be separated into two groups of terms: $S_{nl} = S_{nl}^{(1)} + S_{nl}^{(2)}$. For $S_{nl}^{(1)}$ the integrand includes product $N_{k_1} N_{k_2}$. If we put $k_1 = k_2$, we get the following expression for the low-frequency tail of spectrum:

$$S_{nl}^{(1)} = 2\pi g^2 \int |T_{kk_1,k_1,k_3}|^2\, \delta(\omega - \omega_{k_3})\, (N_{k_3} - N_k)\, N_{k_1}^2\, dk_1. \tag{3.14}$$

Notice, if $|k_1| \gg |k|$, then $|T_{kk_1,k_1,k_3}|^2 \simeq k_1^2$ and integrand in (3.14) is proportional to $k_1^2 N_{k_1}^2$. If $x < 2$, integral (3.14) diverges.

The group of terms linear with respect to the high-frequency tail of spectrum is more complicated:

$$\begin{aligned} S_{nl}^{(2)} &= 2\pi g^2 N_k \int |T_{kk_1k_2k_3}|^2 N_{k_3} (N_{k_1} - N_{k_2}) \times \\ &\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})\, \delta(k + k_1 - k_2 - k_3)\, dk_1\, dk_2\, dk_3. \end{aligned} \tag{3.15}$$

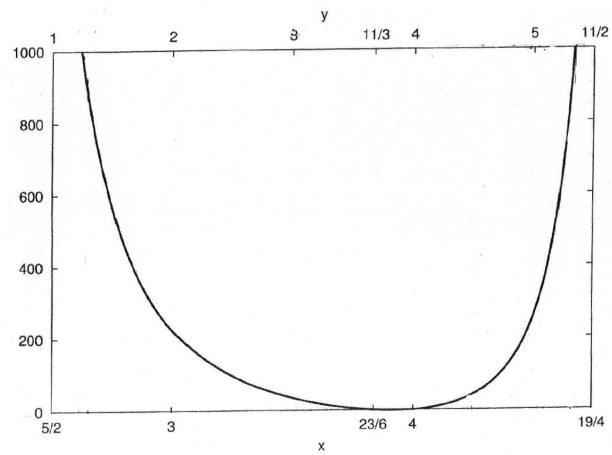


Figure 1: Plot of function $F(x)$.

We can perform expansion

$$N_{k_1} - N_{k_3} = p_i \frac{\partial N}{\partial k_{1i}}, \quad p_i = (k - k_3)_i. \tag{3.16}$$

In the general anisotropic case the integrand is proportional to $k_1^2 (p \nabla N_{k_1})$ and the divergence occurs if $x = x_1 = 2$. However, in the isotropic case this term, the most divergent one, is cancelled after integration by angles. In this case we should study quadratic terms in expansion of the integrand in powers of parameter $(P, k_1)/k_1^2$. The most aggressive term appears from the expansion of $\delta$-function on frequencies $\delta(\omega_{k_1} - \omega_{k_1+p} + \omega_k - \omega_{k_3})$. Performing integration by angles we end up with the equation

$$\begin{aligned} \frac{\partial N_k}{\partial t} &= q\, k^7 N_k \frac{\partial N}{\partial k}, \\ q &= \frac{25}{16} \pi^3 g^{3/2} E = \frac{25}{8} \pi^3 g^{3/2} \int_0^\infty k^{3/2} N_k \, dk. \end{aligned} \tag{3.17}$$

Here $E$ is the total energy. Thus in the isotropic case $x_1 = 5/2$ and we get for function $F(x)$ the following estimate:

$$F = \frac{5}{2} \frac{25}{8} \pi^3 \frac{1}{5/2 - x} = \frac{241.86}{5/2 - x}. \tag{3.18}$$

On Figure 1 is presented the plot of function $F(x)$ for the isotropic case that we calculated numerically. One can see that in the interval $x_1 < x < x_2$ function

$F(x)$ has exactly two zeros at

$$x = y_1 = 4, \quad x = y_2 = \frac{23}{6}. \tag{3.19}$$

To prove this result, let us consider that spectra are isotropic and present conservation laws of energy and wave action in the differential form:

$$\frac{\partial I_k}{\partial t} = 2\pi k \omega_k \frac{\partial N_k}{\partial t} = -\frac{\partial P}{\partial k}, \tag{3.20}$$

$$P = 2\pi \int_0^k k\omega_k \, S_{nl} \, dk, \tag{3.21}$$

$$2\pi k \frac{\partial N_k}{\partial t} = \frac{\partial Q}{\partial k}, \tag{3.22}$$

$$Q = 2\pi \int_0^k k \, S_{nl} \, dk. \tag{3.23}$$

Here $P$ is the flux of energy directed to high wave numbers, while $Q$ is the flux of wave action directed to small wave numbers. Equations

$$P = P_0 = const, \quad Q = Q_0 = const \tag{3.24}$$

apparently are solutions of stationary equation $S_{nl} = 0$. We will look for the solution in the powerlike form $N = \lambda \, k^{-x}$; then equations (3.24) read

$$P_0 = 2\pi g^2 \, \lambda^3 \frac{F(x)}{3(x-4)} \, k^{-3(x-4)} \tag{3.25}$$

$$Q_0 = -2\pi g^{3/2} \, \lambda^3 \, \frac{F(x)}{3(x-26/3)} \, k^{-3(x-26/3)} \tag{3.26}$$

One can see that $P_0$ and $Q_0$ are finite only if $F(4) = 0$ and $F(26/3) = 0$, moreover, if $F'(4) > 0$ and $F'(26/3) < 0$. We conclude that equation $S_{nl} = 0$ has the following solutions:

$$N_k^{(1)} = c_p \left(\frac{P_0}{g^2}\right)^{1/3} \frac{1}{k^4}, \tag{3.27}$$

$$N_k^{(2)} = c_q \left(\frac{Q_0}{g^{3/2}}\right)^{1/3} \frac{1}{k^{23/6}}. \tag{3.28}$$

Here $c_p, c_q$ are dimensionless Kolmogorov constants

$$c_p = \left(\frac{3}{2\pi \, F'(4)}\right)^{1/3}, \quad c_q = \left(\frac{3}{2\pi \, |F'(23/6)|}\right)^{1/3}.$$

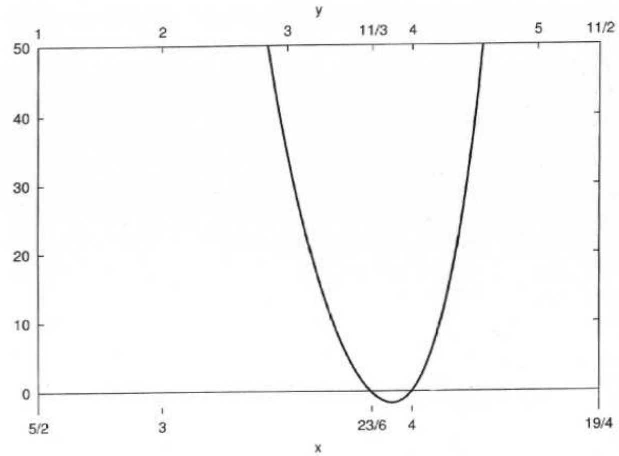


Figure 2: Plot of function $F(x)$; zoom in the vertical direction.

On Figure 2 is presented the zoom of function $F(x)$ in vertical coordinate. The numerics gives $F'(4) = 45.2$ and $F'(23/6) = -40.4$. In the area of zeros $F(x)$ can be approximated by parabola,

$$F(x) \simeq 256.8(x - 23/6)(x - 4). \tag{3.29}$$

To estimate the Kolmogorov constants, we need the value of $F(x)$ at $x = 9/2$; that is:

$$F(9/2) = 85.6. \tag{3.30}$$

Thus we get

$$c_p = 0.219, \quad c_q = 0.227, \tag{3.31}$$

and see that the both Kolmogorov constants are numerically small. In the "parabolic approximation" (3.29)

In the isotropic case, the energy spectrum $F(\omega)$ defined by (1.8) can be expressed through $N_k$,

$$F(\omega)d\omega = 2\pi\omega_k \, N_k \, k \, dk, \tag{3.32}$$

and the energy spectrum corresponding to solution (3.27) has the following form, called Zakharov-Filonenko spectrum:

$$F^{(1)}(\omega) = 4\pi c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^2}{\omega^4}. \tag{3.33}$$

This spectrum was found in 1966 as a solution of equation $S_{nl} = 0$ [10]. For the spatial spectrum

$$I_k \, dk = 2\pi\omega_k \, N(k) \, k \, dk, \tag{3.34}$$

solution (3.27) transforms to

$$I_k^{(1)} = 2\pi\, c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^{1/2}}{k^{5/2}} \simeq k^{-2.5}. \qquad (3.35)$$

Spectra (3.27), (3.33), (3.25) are realized if we have a source of energy that is concentrated at small wave number and generates the amount of energy $P$ in a unit of time. For the spectrum (3.28), first reported by Zakharov in 1966 [34],

$$I_k^{(2)} = 2\pi\, c_q\, Q^{1/3}\, k^{-7/3} \simeq 2\pi\, c_q\, Q^{1/3}\, k^{2.33}, \qquad (3.36)$$

$$F^{(2)}(\omega) = 4\pi\, c_q\, Q^{1/3} \frac{g^{4/3}}{\omega^{11/3}}. \qquad (3.37)$$

Spectra (3.30) and (3.36) can be realized in the case of a small source of wave action in the high wave numbers area.

The described spectra exhaust all powerlike isotropic solutions of the stationary kinetic equation $S_{nl} = 0$. It is important to stress that thermodynamical solutions $N = const$ and $N = c/k^{1/2}$ are not the solutions of this equation, because their exponents $x = 0$ and $x = 1/2$ are far below the lower end of the "window of possibility" $x_1 = 5/2$. This fact means that thermodynamics has nothing in common with the theory of wind-driven sea.

Solutions (3.29) and (3.30) are not the unique stationary solutions of $S_{nl} = 0$. The general isotropic solution describes the situation when both the energy source at small wave numbers and the wave action source exist simultaneously and have the following form:

$$N_k^{(3)} = c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{1}{k^4}\, L\left(\frac{g^{1/2}\, Q\, k^{1/2}}{P}\right). \qquad (3.38)$$

Here $L$ is an unknown function of one variable,

$$L \to 1 \quad \text{at} \quad k \to 0, \qquad L(\xi) \to \frac{c_q}{c_p}\, \xi^{1/3} \quad \text{at} \quad k \to \infty. \qquad (3.39)$$

In Chapter 5 we will present the approximate but rather accurate form of (3.38). Moreover, we will show that equation (1.21), (3.1) has a rich family of anisotropic solutions parameterized by an arbitrary function of one variable: the angular dependence of the spectrum at a certain value of frequency. In particular we will present anisotropic KZ solutions depending on three fluxes: wave action $Q$, energy $P$, and momentum $R_x$.

# 4 Damping due to nonlinear interaction

In this Chapter we show that $S_{nl}$ is the leading term in the balance equation (1.24). In fact, the forcing terms $S_{in}$ and $S_{diss}$ are not known well enough, thus it is reasonable to accept the most simple models of both terms assuming that they are proportional to the action spectrum:

$$S_{in} = \gamma_{in}(k)\, N(k), \tag{4.1}$$
$$S_{diss} = -\gamma_{diss}(k)\, N(k). \tag{4.2}$$

Hence

$$\gamma(k) = \gamma_{in}(k) - \gamma_{diss}(k). \tag{4.3}$$

In reality $\gamma_{diss}(k)$ depends dramatically on the overall steepness $\mu$. We will discuss this point in Chapter 8. So far let us notice that the balance kinetic equation (1.24) can be written in the form

$$S_{nl} + \gamma(k)\, N_k = 0, \tag{4.4}$$

and present the $S_{nl}$ term as follow:

$$\begin{aligned} S_{nl} &= F_k - \Gamma_k\, N_k, && (4.5)\\ F_k &= \pi g^2 \int |T_{kk_1k_2k_3}|^2\, \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \\ &\qquad \times N_{k_1} N_{k_2} N_{k_3}\, dk_1 dk_2 dk_3, && (4.6)\\ \Gamma_k &= \pi g^2 \int |T_{kk_1k_2k_3}|^2\, \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \\ &\qquad \times (N_{k_1} N_{k_2} + N_{k_1} N_{k_3} - N_{k_2} N_{k_3})\, dk_1 dk_2 dk_3. && (4.7) \end{aligned}$$

The solution of stationary equation (4.4) is the following:

$$N_k = \frac{F_k}{\Gamma_k - \gamma_k}. \tag{4.8}$$

The positive solution exists if $\Gamma_k > \gamma_k$. The term $\Gamma_k$ can be treated as the nonlinear damping that appear due to four-wave interaction. This damping has a very powerful effect. A "naive" dimensional consideration gives

$$\Gamma_k \simeq \frac{4\pi g^2}{\omega_k}\, k^{10}\, N_k^2, \tag{4.9}$$

however, this estimate works only if $k \simeq k_p$; $k_p$ being the wave number of the spectral maximum.

Let $k \gg k_p$. Now for $\Gamma_k$ one gets

$$\Gamma_k = 2\pi g^2 \int |T_{kk_1,kk_3}|^2 \, \delta(\omega_{k_1} - \omega_{k_3}) \, N_{k_1} N_{k_3} \, dk_1 dk_2. \tag{4.10}$$

The main source of $\Gamma_k$ is the interaction of long and short waves. To estimate integral (4.7) more accurately, we assume that the spectrum of long waves is narrow in angle, $N(k_1, \theta_1) = \tilde{N}(k_1)\,\delta(\theta_1)$. Long waves propagate along the axis $x$ and $\vec{k}$ is the wave vector of short wave propagating in direction $\theta$. For the coupling coefficient we must put $T_{kk_1,k_2,k_3} \simeq 2k_1^2 k \cos\theta$. Then

$$\Gamma_k = 8\pi g^{3/2}\, k^2 \, \cos^2\theta \int_0^\infty k_1^{13/2}\, \tilde{N}^2(k_1)\, dk_1. \tag{4.11}$$

Even for the most mildly decaying KZ spectrum, $N_k \simeq k^{-23/6}$, the integrand in (4.8) behaves like $k_1^{-7/6}$ and the integral diverges. For more steep KZ spectra the divergence is stronger.

Let us estimate $\Gamma_k$ for the case of "mature sea", when the spectrum can be taken in the form

$$N_k \simeq \frac{3}{2} \frac{E}{\sqrt{g}} \frac{k_p^{3/2}}{k^4}\, \theta(k - k_p). \tag{4.12}$$

Here $E$ is the total energy. By plugging (4.12) to (4.11) one gets the equation

$$\Gamma_\omega = 36\,\pi\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \, \cos^2\theta, \tag{4.13}$$

that includes a huge enhancing factor: $36\pi \simeq 113.04$. For the very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \cdot 10^{-4} \omega \left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta. \tag{4.14}$$

In the isotropic case, to find $\Gamma_k$ for $\omega/\omega_p \gg 1$ we need to perform simple integration over angles that yields:

$$\int_0^{2\pi}\int_0^{2\pi} T^2_{\theta_1,\theta_2}\, d\theta_1\, d\theta_2 = \frac{5}{2}(2\pi)^2.$$

Now, instead of (4.11) we get:

$$\Gamma_k = 5\pi g^{3/2} k^2 \int_0^\infty k_1^{13/2} \tilde{N}(k_1)^2 \, dk_1 \tag{4.15}$$

or

$$\Gamma_\omega = \frac{45\pi}{2} g^{3/2}\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4. \tag{4.16}$$

Finally, assuming that

$$N_{k_p} \simeq \frac{3}{2}\frac{E}{\sqrt{g}k_p^{5/2}},$$

we get from (4.9) the following estimate for $\Gamma_p = \Gamma|_{k=kp}$:

$$\Gamma_p \simeq 9\pi\omega_p\mu_p^4 \tag{4.17}$$

Even in this case we have a pretty high enhancing factor: $9\pi \simeq 28.26$. In Chapter 6 we will show that in all known models $\Gamma_k$ surpasses $\gamma_k$ at least in order of magnitude even for these very smooth waves.

In the presence of peakedness

$$\Gamma_p \simeq \Lambda\,\omega_p\mu_p^4. \tag{4.18}$$

Here $\Lambda \simeq 4\pi\omega_p/\delta\omega$ is the enhancing factor due to peakedness. If $\Lambda\,\mu_p^2 \sim 1$, then $\Gamma_p$ is associated with the maximal growth of modulational instability for monochromatic wave: $\Gamma_p \simeq \gamma_{mod} \sim \omega_p\mu_p^2$. If $\Lambda \sim 1/\mu_p^2$, the nonlinearity becomes so strong that the weak-turbulent statistical approach is not applicable. This is quite realistic situation. Suppose that $\mu_p \simeq 0.11$ and $\omega_p/\delta\omega \simeq 5$. Then $\Lambda\,\mu_p^2 \sim 0.76$ and the weak turbulent description is hardly correct. In the situation of strong nonlinearity the wind-driven sea generates freak waves (see [44, 45]). The very fact of their existence as a common phenomenon is an implicit proof of $S_{nl}$ domination in the energy balance.

Notice that $\Gamma_k$ diverges for KZ spectra. However, it does not hurt the spectra existence because in the full kinetic equation the divergence in $\Gamma_k$ is cancelled by divergence in $F_k$. Indeed, if we consider the contribution of small wave-numbers in integral (4.1), we end up with the following expression:

$$F_k = 2\pi g^2\, N_k \int |T_{kk_1,kk_3}|^2\,\delta(\omega_{k_1}-\omega_{k_3})\,N_{k_1}N_{k_3}\,dk_1dk_3 \simeq N_k\,\Gamma_k. \tag{4.19}$$

In negligence of $\gamma_k$, equation (4.5) is satisfied automatically.

Let us return to equation (4.1) and assume that the total energy is finite, $E < \infty$, and the flux of all motion constants to and from infinity is zero. This is possible, if the following conditions are satisfied:

$$\int \gamma(k)\,N_k\,dk \;=\; 0, \tag{4.20}$$

$$\int \omega_k\,\gamma(k)\,N_k\,dk \;=\; 0, \tag{4.21}$$

$$\int \vec{k}\,\gamma(k)\,N_k\,dk \;=\; 0. \tag{4.22}$$

To satisfy these conditions, we must assume that $\gamma(k)$ is positive in some area bounded from both ends: $\gamma(k) > 0$ if $k_{min} < k < k_{max}$. We have to demand that $\gamma_k < 0$ for $k > k_{max}$ and $k < k_{min}$. Dissipation in high wave numbers is not a problem: either the white capping mechanism or the transformation to capillary waves are efficient enough to absorb the flux of energy headed to $k \to \infty$. The dissipation of long waves is a matter of discussion. Does such thing as "mature sea" really exist? Does the arrest of frequency downshift come from finite time of any duration and finite size of any fetch? This is a question directed to experimentalists.

In equation (4.8) we can put $\tilde{\Gamma}_k = \Gamma_k - \gamma_k$. If we know $\tilde{\Gamma}_k$, we can restore the spatial frequency spectrum $N_{k,\omega}$ as follow:

$$N_{k,\omega} = \frac{1}{\pi} \frac{\tilde{\Gamma}_k}{(\omega - \tilde{\omega}_k)^2 + \tilde{\Gamma}_k^2}. \tag{4.23}$$

Here

$$\tilde{\omega}_k = \omega_k + 2\pi g^2 \int |T_{kk_1}|^2 N_{k_1} \, dk_1 \tag{4.24}$$

is the dispersion law of surface waves renormalized due to nonlinear wave interaction.

# 5 Differential form of Hasselmann equation

In this Chapter we exploit the fact that Hasselmann equation conserves wave action, energy, and momentum at least on the formal level. We start with the equation in polar coordinates:

$$\frac{\partial N(\omega,\phi)}{\partial t} + \frac{g}{2\omega} \cos\phi \, \frac{\partial(\omega,\phi)}{\partial x} = S_{nl} + \gamma(\omega,\phi)\, N(\omega,\phi), \tag{5.1}$$

$$\begin{aligned} S_{nl}(\omega,\phi) \quad = \quad & 2\pi g^2 \int |T_{\omega,\omega_1,\omega_2,\omega_3}|^2 \, \delta(\omega + \omega_1 - \omega_2 - \omega_3) \times \\ & \times \delta(\omega^2 \cos\phi + \omega_1^2 \cos\phi_1 - \omega_2^2 \cos\phi_2 - \omega_3^2 \cos\phi_3) \times \\ & \times \delta(\omega^2 \sin\phi + \omega_1^2 \sin\phi_1 - \omega_2^2 \sin\phi_2 - \omega_3^2 \sin\phi_2) \times \\ & \times \Big\{ \omega^3 \, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3) + \omega_1^3 \, N(\omega,\phi)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3) - \\ & - \omega_2^2 \, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_3,\phi_3) - \omega_3^2 \, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2) \Big\} \\ & \qquad d\omega_1 \, d\omega_2 \, d\omega_3 \, d\phi_1 \, d\phi_2 \, d\phi_3. \end{aligned} \tag{5.2}$$

Exactly this form of $S_{nl}$ is used for numerical simulation of Hasselmann equation. Suppose that $N(\omega,\phi)=\omega^{-z}$ is isotropic spectrum. Then

$$S_{nl}=\frac{\omega^{-3z+13}}{4g^4}\,F\left(\frac{z+3}{2}\right)=\frac{G(z)}{g^4}\,\omega^{-3z+13}, \tag{5.3}$$

where $F(x)$ is presented on Figures 1, 2. Now the "window of opportunity" is: $2<z<13/2$. Zeros of $G(z)$ are posed at $z_1=5$ and $z_2=14/3$ and near these zeros $G(z)$ can be presented as parabola,

$$G(z)\simeq\,16.05(z-5)(z-14/3). \tag{5.4}$$

To make the motion constants more conspicuous, we introduce the elliptic differential operator

$$L\,f(\omega,\phi)=\left(\frac{\partial^2}{\partial\omega^2}+\frac{2}{\omega^2}\,\frac{\partial^2}{\partial\phi^2}\right)\,f(\omega,\phi) \tag{5.5}$$

with following parameters: $0<\omega<\infty,\;\;0<\phi<2\pi$. Equation

$$L\,G=\delta(\omega-\omega')\,\delta(\phi-\phi') \tag{5.6}$$

with boundary conditions

$$G|_{\omega\to 0}=0,\qquad G_{\omega\to\infty}<\infty,\qquad G(2\pi)=G(0),$$

can be resolved as

$$G(\omega,\omega',\phi-\phi')=\frac{1}{4\pi}\sqrt{\omega\omega'}\sum_{n=-\infty}^{\infty}e^{in(\phi-\phi')}\times$$
$$\times\left[\left(\frac{\omega}{\omega'}\right)^{\Delta_n}\,\Theta(\omega'-\omega)+\left(\frac{\omega'}{\omega}\right)^{\Delta_n}\,\Theta(\omega-\omega')\right], \tag{5.7}$$

where $\Delta_n=1/2\sqrt{1+8n^2}$. Now we present $S_{nl}$ in the form:

$$A(\omega,\phi)=\int_0^\infty d\omega'\int_0^{2\pi}d\phi'\,G(\omega,\omega',\phi-\phi')\,S_{nl}(\omega',\phi'). \tag{5.8}$$

Notice that $A(\omega,\phi)$ is a regular integral operator and suppose that $N(\omega,\phi)=\omega^{-z}$. Then

$$A[\omega^{-z}]=\frac{\omega^{-3z+15}}{g^4}\,H(z),$$
$$H(z)=\frac{G(z)}{9(z-5)(z-14/3)}. \tag{5.9}$$

Function $H(z)$ is positive and has no zeros. If $G(z)$ is presented by parabola (5.4), $H(z)$ is just a constant:

$$H(z) = H_0 = 16.05/9 = 1.83. \tag{5.10}$$

This fact leads to a bold idea. If we assume that

$$A = \frac{H_0}{g^4}\,\omega^{15}\,N^3, \tag{5.11}$$

the nonlinear term $S_{nl}$ turns to the elliptic operator:

$$S_{nl} = \frac{H_0}{g^4}\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\omega^{15}\,N^3. \tag{5.12}$$

This is the so-called "diffusion approximation", introduced in article [33]. Being very simple, it grasps the basic features of wind-driven sea theory. We will refer mostly to this model, having in mind that the real case (5.9) does not differ much from it, at least qualitatively.

Let us integrate equation (5.1) by angles. We get:

$$\frac{\partial N(\omega,t)}{\partial t} + \frac{\partial B(\omega,t)}{\partial x} = \frac{\partial Q}{\partial\omega} + S(\omega,t). \tag{5.13}$$

As before, here $N(\omega,t) = \int_0^{2\pi} N(\omega,\phi)\,d\phi$. Then

$$B(\omega,t) = \frac{g}{2\omega}\int_0^{2\pi}\cos\phi\,N(\omega,\phi)\,d\phi, \qquad S(\omega,t) = \int\gamma(\omega,\phi)\,N(\omega,\phi)\,d\phi, \tag{5.14}$$

and the flux of wave action is:

$$Q = \frac{\partial K}{\partial\omega}, \qquad K = \int_0^{2\pi} A(\omega,\phi)\,d\phi. \tag{5.15}$$

After multiplication of equation (5.9) by $\omega$ one obtains equation

$$\frac{\partial F(\omega,t)}{\partial t} + \frac{\partial}{\partial x}\,\omega\,B(\omega,t) + \frac{\partial P}{\partial\omega} = \omega\,S(\omega,t), \tag{5.16}$$

where $P = K - \omega\,\partial K/\partial\omega$ is the flux of energy.

Let us introduce now the following definitions: the integrated by angle spectral density of momentum

$$M_x(\omega,t) = \frac{\omega^2}{g}\int_0^{2\pi}\cos\phi\,B(\omega,\phi)\,d\phi, \tag{5.17}$$

the quantity

$$C_x(\omega, t) = \frac{\omega}{2g} \int_0^{2\pi} \cos^2\phi\, N(\omega, \phi)\, d\phi, \tag{5.18}$$

the flux of momentum

$$R_x = \int_0^{2\pi} \cos\phi(\omega\, A - \frac{\omega^2}{2}\frac{\partial A}{\partial \omega})\, d\phi, \tag{5.19}$$

and the spectral density of horizontal flux

$$\tau_x = \frac{\omega^2}{g} \int_0^{2\pi} \gamma(\omega, \phi)\, N(\omega, \phi)\, \cos\phi\, d\phi. \tag{5.20}$$

All these quantities are connected by equation

$$\frac{\partial M_x}{\partial t} + \frac{\partial C_x}{\partial x} + \frac{\partial R_x}{\partial \omega} = \tau_x. \tag{5.21}$$

Equations (5.9), (5.12) and (5.17) are averaged by angle balance equations for the basic conservative quantities.

Now we can return to the question formulated above. How many solutions has the stationary kinetic equation (1.23), (3.1)? Notice that we simplified it to the linear equation

$$\left( \frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial \phi^2} \right) A = 0. \tag{5.22}$$

In particulary, kinetic equation has the KZ solution

$$A = \frac{1}{2\pi} \left\{ P + \omega\, Q + \frac{R_x}{\omega}\, \cos\phi \right\}, \tag{5.23}$$

where $P$ and $R_x$ are fluxes of energy and momentum at $\omega \to \infty$ and $Q$ is the flux of wave action directed to small wave numbers. In a general case, (5.23) is a nonlinear integral equation, however in the diffusion approximation the KZ solution can be found in the explicit form:

$$N(\omega, \phi) = \frac{1}{(2\pi\, H_0)^{1/3}} \frac{g^{4/3}}{\omega^5} \left( P + \omega\, Q + \frac{R_x}{\omega}\, \cos\phi \right)^{1/3}. \tag{5.24}$$

By comparison with (3.33), (3.37) we easily find that in this case

$$c_p = c_q = \frac{1}{2(2\pi\, H_0)^{1/3}} = 0.223, \qquad H_0 = 1.83.$$

This is exactly the arithmetic mean between the values of Kolmogorov constants given by (3.31).

By multiplication of (5.24) to $2\pi\omega$ we get the general KZ spectrum in the diffusion approximation:

$$F(\omega) = 2.78\, \frac{g^{4/3}}{\omega^4} \left( P + \omega\, Q + \frac{R_x}{\omega}\, \cos\phi \right)^{1/3}. \tag{5.25}$$

We must be sure that in the isotropic case $R_x = 0$, expression

$$F(\omega) = 2.78\, \frac{g^{4/3}}{\omega^4}\, (P + \omega\, Q)^{1/3} \tag{5.26}$$

approximates the generic KZ spectrum with accuracy up to few percents. In the general anisotropic spectrum (5.25) we must be more cautions.

If somehow we know the value of $A(\omega, \phi)$ on the circle $\omega = \omega_0$, then we can solve the external and internal Dirichlet boundary problem for equation (5.22). Suppose that

$$A(\omega, \phi) = A_0(\phi) - A_0 + A_1\, \cos\phi + \sum_{n=2}^{\infty} A_n \left( \frac{\omega_0}{\omega} \right)^{-1/2+\sqrt{1/4+4n^2}} \cos\phi. \tag{5.27}$$

First two terms in (5.27) present the KZ spectrum with $Q = 0$, $P = 2\pi A_n$, $R_x = 2\pi\omega_0\, A_1$. The next terms describe the fast stabilization of any arbitrary solution to the KZ spectrum at $\omega/\omega_0 \to \infty$. The first additional term in (5.27) decays as $(\omega_0/\omega)^{3.53}\, \cos 2\phi$.

This stabilization to KZ spectrum is actually the "angular spreading" on wind-driven wave spectra that is usually observed in field experiments (see, for instance [12]). If $Q = 0$, the general KZ solution (5.25) at $\omega \to 0$ is the following spectrum:

$$F(\omega) \to \frac{2.78}{\omega^4}\, g^{4/3}\, p^{1/3} \left( 1 + \frac{1}{3}\, \frac{R_x}{P\, \omega}\, \cos\phi + \cdots \right). \tag{5.28}$$

Similar results were predicted by Kontorovich and Kats [47] and Balk [48]. Inside the circle $\omega = \omega_0$, the solution of equation (5.22) is presented by series

$$A = A_0\, \frac{\omega}{\omega_0} + \sum_{n=1}^{\infty} A_n\, \cos n\phi \left( \frac{\omega}{\omega_0} \right)^{1/2+\sqrt{1/4+4n^2}}. \tag{5.29}$$

To get the finite value of $N$ at $\omega \to 0$ one must demand that $A(\omega) < \omega^{15}$. It presumes that first eight terms in (5.29) must be zero ($A_0 = 0, \ldots A_7 = 0$). Thus any finite in $\omega \to 0$ solution must badly oscillate in angle. It means that the pure conservative equation (5.22) is not applicable in the small frequency area. It must be augmented by nonstationary, nonuniformity or additional damping. Usually spectra in the area of small wave numbers are almost one-dimensional. Theory of these spectra started with the work of Zakharov and Smilga [58] and is pretty well developed now [59]. We will not discuss this theory in the presented article.

# 6 Direct and inverse cascades

In this Chapter we study the forced stationary Hasselmann equation

$$\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right) A + \gamma(\omega,\phi)\, N(\omega,\phi) = 0. \tag{6.1}$$

After integration by angle we get:

$$\frac{\partial^2}{\partial\omega^2} K + S(\omega) = 0. \tag{6.2}$$

Equations(6.2) are simple, nevertheless, very instructive. We have mentioned already that equation (6.1) is resolvable if only conditions (4.20–4.22) are satisfied. It presumes that we have some dissipation in the small wave vectors area. As far as this dissipation is an absolutely unexplored land, at the moment the study of regular solutions of this equation is a pure academic question. Instead we study the singular solutions of equation (6.2). For these solutions the flux of wave action is given by expression

$$Q = \frac{\partial K}{\partial\omega} = \int_\omega^\infty S(\omega)\, d\omega. \tag{6.3}$$

Then at $\omega \to 0$,

$$Q \to Q_0 = \int_0^\infty S(\omega)\, d\omega, \tag{6.4}$$

in the same way

$$P = K - \omega\frac{\partial K}{\partial\omega} = \int_0^\omega \omega\, S(\omega)\, d\omega, \tag{6.5}$$

$$P_0 = P(\infty) = \int_0^\infty \omega\, S(\omega)\, d\omega. \tag{6.6}$$

Function $K(\omega)$ is given by equation

$$K(\omega) = \int_0^\omega dp \int_p^\infty S(q)\, dq, \tag{6.7}$$

that is the solution of (6.2) under boundary conditions

$$K(0) = 0, \qquad \frac{\partial K}{\partial\omega} = 0 \quad \text{at} \quad \omega \to \infty. \tag{6.8}$$

These conditions mean that neither energy nor the wave action come to the system from outside. The corresponding energy spectra are singular:

$$F(\omega) \to 4\pi\, c_q \frac{g^{4/3}\, Q^{1/3}}{\omega^{11/3}}, \qquad \omega \to 0 \tag{6.9}$$

$$F(\omega) \to 4\pi\, c_p\, \frac{g^{4/3}\, P_0^{1/3}}{\omega^4}, \qquad \omega \to \infty \tag{6.10}$$

Suppose that the source term $S(\omega)$ is concentrated at certain frequency $\omega = \omega_0$:

$$S(\omega) = S_0\,\delta(\omega - \omega_0). \tag{6.11}$$

Then

$$Q(\omega) = S_0 = Q_0 \quad \text{if} \quad \omega < \omega_0, \qquad Q(\omega) = 0 \qquad \text{if} \quad \omega > \omega_0. \tag{6.12}$$

On the contrary,

$$P(\omega) = 0 \quad \text{if} \quad \omega < \omega_0, \qquad P(\omega) = P_0 = \omega_0\,S_0 \quad \text{if} \quad \omega > \omega_0. \tag{6.13}$$

The spectral range $\omega > \omega_0$ can be called the "area of direct cascade", while the spectral range $\omega < \omega_0$ is the "area of inverse cascade". In the framework of diffusion model the spectra in both areas are exactly spectra (6.9) and (6.10). In the real sea the transition between two areas is smooth, however, as our numerical experiments show, in the transition zone from inverse to direct cascade they are rather narrow as is seen on Figure 2.

In the real sea $S(\omega)$ does not have a clear maximum in a certain spectral band and separation of direct and inverse cascades is less obvious. One can consider that if $P < \omega\,Q$, the inverse cascade prevails, in the opposite case the direct cascade dominates. In the marginal case $S(\omega) \simeq \omega^{-3/2}$, for all frequencies we have $P = \omega\,Q$. In this case $N_\omega \simeq \omega^{-5+1/6} = \omega^{-29/6}$ and $F(\omega) \simeq \omega^{-23/6}$. This regime is realized if $\gamma(\omega) \simeq \omega^{10/3}$. In all existing models $\gamma(\omega)$ grows more slowly and separation of inverse and direct cascades is possible. It is important to notice that if the spectrum is approximated by powerlike function $F(\omega) \simeq \omega^{-\nu(\omega)}$, exponent $\nu$ varies in the very narrow interval $11/3 < \nu(\omega) < 4$. In real sea all the energy transported to high wave frequency region must be absorbed by dissipation. Hence $P(\omega) \to 0$ at $\omega \to \infty$.

By differentiation of equation (6.5) we get an interesting identity:

$$\frac{\partial P}{\partial \omega} = -\omega\,\frac{\partial^2 K}{\partial \omega^2} = -\omega\,\frac{\partial Q}{\partial \omega}. \tag{6.14}$$

It denotes that in the area $\partial P/\partial \omega < 0$, the energy is absorbed, while in the area, where $\partial Q/\partial \omega > 0$, the flux of wave action grows. As far as $Q \to 0$ at $\omega \to \infty$, it means that for large wave numbers $Q < 0$ and at certain $\omega = \omega_{max}$ the flux of wave action is zero, $Q = 0$. One can say that waves born in the spectral range $\omega > \omega_x$ move to high wave numbers, while waves born in the range $\omega < \omega_{max}$ move to small wave numbers, loosing their energy during this way. Apparently $K > 0$ reaches its maximum at $\omega = \omega_{max}$.

In reality energy and wave number spectra are regular at small wave numbers. The inverse cascade is matched with nonstationary or nonhomogeneous downshift.

At certain point $\omega = \omega_{min}$, the energy flux $P$ changes its sign. This is a real lower border of equilibrium area.

Typical distribution of fluxes during time evolution are presented on Figure 3. In experimentally observed spectra the inverse and the direct cascades are clearly distinguishable. An example of such spectrum is presented on Figure 4.

# 7 Interaction with wind

No doubts that ocean waves are generated by wind, however a reasonable theory of this phenomenon is not developed yet, in spite of enormous invested efforts. The difficulties are caused not only by the turbulence of atmospheric boundary layer over the sea surface, they are caused also by necessity to take into account the inverse influence of waves to the atmosphere. Let us put together the reliable information about this process.

Let $u, v, w$ be $x, y, z$ components of the air velocity. The boundary layer is characterized by the following measurable quantities: the averaged horizontal velocity

$$U(z) = \langle u \rangle \tag{7.1}$$

and the downward Reynolds stress

$$-\langle u\, w \rangle = u_*^2. \tag{7.2}$$

Velocity $U(z)$ is a slowly growing function on $z$, while according to the theory $\partial/\partial z \langle u\, w \rangle = 0$. Thus, at least theoretically $u_*$ does not depend on height. For this reason $u_*$ together with $g$ are often used for the scaling of all quantities connected with the boundary layer. As for $U(z)$, this function is usually evaluated on the "height of standard anemometer", $z = 10\ m$, is denoted as $U_{10} = U|_{z=10}$ and is also used for scaling.

Numerous measurements show that in the open sea the following relation holds with a fair accuracy:

$$U_{10} \simeq 28\, u_*. \tag{7.3}$$

The Reynolds stress $u_*^2$ is directly connected with the momentum flux from air to water,

$$\tau = \rho_{air}\, u_*^2 = \epsilon\, u_*^2, \tag{7.4}$$

and with the drag coefficient

$$c_d = \frac{u_*^2}{U_{10}^2}. \tag{7.5}$$

Relation (7.3) is the first approximation only; in fact the drag coefficient slowly depends on $U_{10}$. In 1977, Garratt offered the following law for this dependance [49]:

$$c_d = (0.75 + 0.067\ U_{10}) \cdot 10^{-3} \tag{7.6}$$

Value (7.3) is realized for $U_{10} = 11.17\ \ m/sec$.

Both relations (7.4) and (7.6) are purely empiric. Nevertheless, with a good accuracy,

$$\frac{1}{28} = \frac{1}{\sqrt{\epsilon}}. \tag{7.7}$$

So far, this interesting fact hasn't any theoretical interpretation. Theory of atmospheric boundary layer over sea is not developed yet in a proper degree. Traditionally oceanographers refer to Von-Karman - Prandtl theory of turbulent boundary layer over the rigid flat surface. According to this theory (see, for instance [50]), the profile of $u(z)$ is a logarithmic function,

$$u(z) = 2.5\ u_* \ln \frac{z}{z_0} \tag{7.8}$$

Here $z_0$ is the characteristic length associated either with thickness of viscous sublayer

$$z_v \simeq \frac{0.13\ \nu}{u_*} \tag{7.9}$$

or with characteristic roughness of the surface

$$z_0 \simeq \sqrt{(\nabla\,\eta)^2}. \tag{7.10}$$

Theory of turbulent boundary layer over the rigid flat surface is question of great practical importance and subject of numerous theoretical and experimental studies. Relation (7.9) can be considered as very well established. In the wind-driven sea the viscous length $z_\nu$ is very small: with the kinematic viscosity of fluid $\nu \simeq 0.150\ cm^2/sec$ and a typical value of $u_* = 50\ cm/sec$, we get

$$z_\nu \simeq 4 \cdot 10^{-4}\ cm. \tag{7.11}$$

Hence we conclude that $z_0$ must be the roughness of fluid. We can find $z_0$ from equation (7.8) assuming that at $z = 10^3$, $U(z) = 28\ u_*$. We get:

$$\ln \frac{10^3}{z_0} = 11.2 \qquad z_0 = 0.0136\ cm\ \ \simeq 0.1\ mm \tag{7.12}$$

This is much more than estimate (7.11). Thus, viscosity does not play an important role in the momentum transport.

If we believe in logarithmic profile (7.8), we should offer some analytic expression for $z_0$. Charnock [51] offered the following formula for $z_0$:

$$z_0 = c_{Ch} \frac{u_*^2}{g}. \tag{7.13}$$

To obtain $z_0 \simeq 0.0136$ at $u_* = 50$ we must choose the Charnock constant very small,

$$c_{Ch} \simeq 5.3 \cdot 10^{-3}. \tag{7.14}$$

Charnock formula (7.13) is widely used but is in fact very vulnerable for criticism. For the scales of 0.1 $mm$ the gravitational effects are completely suppressed by capillarity. Moreover, the smallness of $c_{Ch}$ is a new riddle that has no reasonable solution.

It is more naturally to assume that $z_0$ is defined by surface tension:

$$z_0 = c_z \frac{\sigma}{u_*^2}. \tag{7.15}$$

Now $c_z \simeq 0.46$ is a constant of order of unit. Implicitly formula (7.15) presumes that all transport of momentum is realized by capillary waves. These waves must be very much nonlinear, so it is not astonishing that constant $c_z$, which is just a characteristic steepness of those capillary waves, is of order of unit. Theoretically speaking, this is an acceptable scenario but a more detailed study shows that it is consistent neither with experiment nor with a common sense.

Logarithmic law (7.8) means that all variations of velocity from zero to characteristic wing velocity are going on in a very thin layer over the sea surface. Let $z = 100\ cm\, z_0 = 1.36\ cm$. According to (7.8), at this height

$$U|_{z=1.36\ cm} = 0.41\ U_{10}. \tag{7.16}$$

This is unrealistically high level of velocity. The see surface is not a polished steel plate and wind velocity grows much more slower with the height, otherwise neither swimming no sailing would be possible in the windy sea. Moreover, visual observations as well as optical and radio experiments show that in the range of scales $10\ mm < l < 0.1\ mm$, the sea surface is relatively smooth. Visually observed maximum of steepness is $z_0 \simeq 1 \div 2\ cm$. This corresponds exactly to the transition between gravity and capillary dominated waves. Theoretical justification of this view-point is published in the article by Newell and Zakharov [52].

There are another arguments about logarithmic profile of $U(z)$. It was mentioned in the monograph of Young [53] that logarithmic profile predicts too low values of $\gamma_k$, even if we neglect the turbulence and perform the calculation in the quasilaminar approach. In this approximation the coefficient of interaction with the wind is given by Miles formula

$$\gamma_k = \pi\,\epsilon\,\omega_k\,\frac{u''}{|u'|}|\chi_k|^2\Bigg|_{z=z_{cr}} \tag{7.17}$$

Here $U(z_{cr}) = \omega/k\,\cos\phi$ and $\chi_k(z)$ is a solution of Taylor-Goldstein equation

$$\frac{d^2\chi}{dz^2} = \left[k^2 + \frac{w_0''}{w_0}\right]\,\chi \tag{7.18}$$

with boundary conditions

$$\chi(0) = 1, \qquad \chi \to 0 \quad \text{at} \quad z \to \infty$$

and

$$w_0(z) = U_0(z) - \frac{\omega_k}{k\,\cos\phi} \tag{7.19}$$

Let us add unit to (7.21) to avoid singularity at $z = 0$:

$$U_0(z) = \frac{u_*}{\kappa}\,\ln\left(\frac{z}{z_0} + 1\right)$$

By introducing of new variable $y = z/z_0$, we transform (7.20) to

$$\chi'' = \left(\xi^2 + \frac{w''}{w}\right)\,\chi = \left(\xi^2 - \frac{1}{(1+y)^2}\,\frac{1}{\ln(1+y) - \lambda}\right)\,\chi. \tag{7.20}$$

Here $w = \ln(1+y) - \lambda$, $\xi = k\,z_0$, and

$$\lambda = \frac{\omega\kappa}{k\,\cos\phi\,u_*} = \frac{\omega_p}{\omega}\,\frac{\kappa\,U_{10}}{u_*\,\cos\phi} = 11.2\,\frac{\omega_p}{\omega\,\cos\phi}. \tag{7.21}$$

Thereafter we will assume $\cos\phi = 1$. Notice, that $\lambda \to 0$ if $\omega/\omega_p \to \infty$.

In the area

$$\lambda \ll \ln\frac{1}{\xi} \tag{7.22}$$

the boundary problem (7.20), (7.21) can be solved analytically. For $y \ll 1/\xi$, equation (7.20) can be simplified up to the form

$$\chi'' = \frac{w''}{w}\,\chi \tag{7.23}$$

This equation can be solved explicitly:

$$\chi = 1 + \frac{1}{\lambda}\,\ln(1+y) + c\,[\ln(1+y) - \lambda]\int_0^y \frac{dz}{[\ln(1+z) - \lambda]^2} \tag{7.24}$$

Here $c$ is indefinite, so far is a constant. The condition $\chi(0) = 1$ is satisfied. Another form of the same solution is the following:

$$\chi = 1 - \frac{1+c}{\lambda} \ln(1+y) + c\left(-y + [\ln(1+y) - \lambda] \int_0^y \frac{dz}{\ln(1+z) - \lambda}\right) \qquad (7.25)$$

From (7.24) one can see that

$$\chi\Big|_{z=z_{cr}} = -c(1+y) = -\frac{c\, z_{cr}}{z_0} \qquad (7.26)$$

Solution (7.24) should be matched with the far asymptotic solution $\chi \simeq e^{-\xi y}$ in the area $y \sim 1/\xi$. In accordance with (7.22), in this area by we can find asymptotic behavior for $\chi$:

$$\chi \simeq -\frac{1}{\lambda} \ln y + \frac{cy}{\ln y} \qquad (7.27)$$

From condition $\chi'/\chi \simeq -\xi$ at $y \simeq 1/\xi$, we get

$$c = \frac{\xi}{2\lambda} \ln^2 \frac{1}{\xi}. \qquad (7.28)$$

Combining (7.19), (7.26) and (7.27) we get:

$$\frac{\gamma_k}{\omega_k} \simeq \epsilon \frac{\pi}{4} \frac{k\, z_{cr}}{11.22} \ln^4 \frac{1}{k\, z_0^2} \left(\frac{\omega}{\omega_p}\right)^2. \qquad (7.29)$$

If we consider the short wave limit

$$z_{cr} \to z_0, \quad k \simeq 10\ cm^{-1}, \quad \ln \frac{1}{kz_0} \simeq 6.6, \quad k \simeq 0.4,$$

we get

$$\frac{\gamma}{\omega} = 0.012\ \epsilon \left(\frac{\omega}{\omega_p}\right)^2. \qquad (7.30)$$

Let us compare this formula with the "less aggressive" empiric formula for $\gamma/\omega$, offered by Hsiao and Shemdin [2] and Plant and Wright [3]:

$$\frac{\gamma}{\omega} \simeq 0.04\, \epsilon \left(\frac{\omega}{\omega_p}\right)^2.$$

We see that Miles formula underestimates the wind input term at least in factor three. The main reason of discrepancy is the hypothesis of logarithmic shape on the boundary layer. In fact there is no serious arguments in support of this approximation. On the hard polished plate, where indeed the logarithmic profile takes place, all momentum transport happens exactly on the surface of plate at $z = 0$. In the sea, the extraction of momentum from air is distributed in height. Each wave with wave factor $\vec{k}$ interacts with the critical layer with horizontal velocity $U_{cr} = \omega/k \cos\phi$. Withholding of energy and momentum from this critical layer is called usually "radiation stress". The total value of this stress is given by integral

$$\tau_{rad} = \int_0^{2\pi} d\phi \int_0^{\infty} \omega^2 \, \gamma(\omega, \phi) \, N(\omega, \phi) \, d\omega. \tag{7.31}$$

Apparently,

$$\tau_{rad} \leq \epsilon \, u_*^2 \tag{7.32}$$

Inequality (7.32) can be used for checking of validity of different models for $\gamma_k$.

It is clear from all written above that the wind-wave interaction is a self-consistent process and a detailed study of the inverse influence of the waves on the shaping of the atmospheric boundary layer is an important and urgent problem. Some steps on this direction were done by Fabrikant [54] and Janssen [55, 56]. They developed the so-called "quasilinear theory of wind-wave interaction", similar to quasilinear theory of interaction of electrons and Langmuir waves in non-magnetized plasma (see, for instance [57]). This theory cannot be considered as accomplished, because the question about separation of roles between radiation $\tau_{rad}$ and turbulent $\tau_{turb}$ stresses is not yet clear, though it demonstrates much better approximation to reality than the "primitive" theory of the logarithmic boundary layer. Nevertheless, this theory did not became a basic model for description of air-water interaction. One of the reasons is the deficit of experimental data, both the field experiments and the laboratory ones. The number of such experiments is a few and their accuracy is rather poor. The scatter in measurements of $\gamma/\omega$ is of order of this quantity.

On Figure 5 is plotted the dimensionless growth rate $\gamma/f$ as a function of $u_*/C$ compiled from experimental measurements by Plant [??]. The solid line drawn amid the experimental dots is exactly the quasilaminar Miles theory evaluated by Janssen [??]. The dashed straight line drawn above corresponds to $\Gamma_\omega/f$, the dissipation due to nonlinear interaction. This term was calculated according to equation (4.14) at $\theta = 0$ and $u_* = \frac{1}{28}\frac{g}{\omega_p}$, that gives for dependance the following estimate: $\Gamma_f/f \simeq 97(u_*/c)^3$. It is clearly seen that the nonlinear damping surpasses the income from wind at least in order of magnitude.

A waste literature is devoted to description of different heuristic models of $S_{in}$. However all of them are easily surpassed by the nonlinear dissipation term $\Gamma_\omega$, hence the question about the optimal choice of $S_{in}$ is not a question of primary importance.

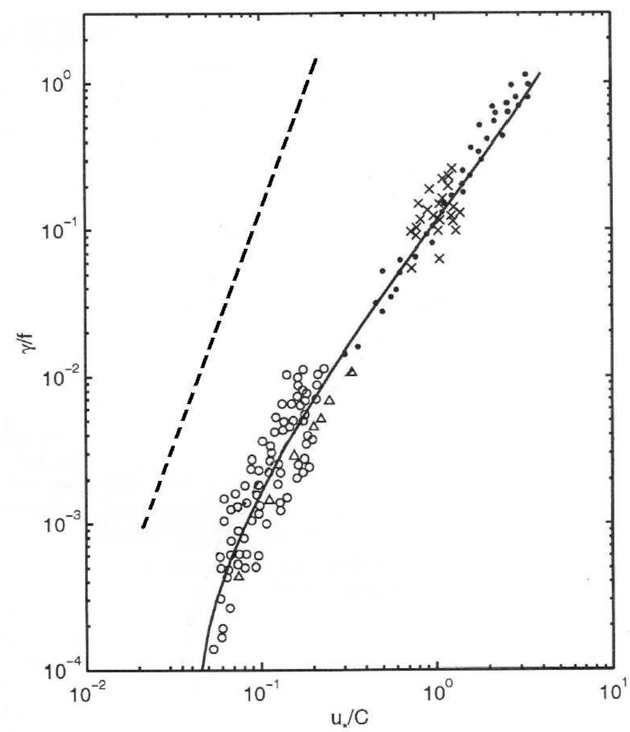


Figure 3: Dominance of nonlinear damping over input from wind. On the right: a comparison of quasilaminar Miles theory with observational wave growth vs frequency $\gamma/f$ as given in [??]. On the left: dissipation due to nonlinear interaction $\Gamma_f/f$ calculated according to (4.14).

# 8 Phillips sea

Numerous experiments show that that the energy spectrum $F(\omega)$ could not exceed some experimental value

$$F(\omega) < \frac{\alpha\, g^2}{\omega^5}. \tag{8.1}$$

In different experiments the dimensionless constant $\alpha$ varies between $0.7 < \alpha < 0.01$. Corresponding measurements of spatial correlation function give

$$I_k \simeq \frac{\alpha_c}{k^4}. \tag{8.2}$$

Here $\alpha_c \simeq 10^{-3}$.

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59.

Equation (1.15) also has a rich family of anisotropic solutions. For a real ocean case, they are not so far properly explored. To find these solutions numerically, we need to perform a lot of calculations; however their basic properties could be understood in a framework of a "toy" differential model. Instead of the real complicated function (2.18), let us consider another function that is homogeneous in $k_i$ and satisfies the symmetry conditions (2.17). Then, we assume that this function is "local", concentrated in in the region, where all $k_i$ are close to each other. In this case the kinetic Hasselmann equation can be replaced by the fourth-order differential equation. This idea was offered independently by Hasselmann et al [30] and by Iroshnikov, who performed a corresponding tedious calculation [31]. However, they did not manage to present their results in the compact form and analyze the solutions of obtained PDE's. The differential wave kinetic equation was derived in [33] on the base of a very elementary consideration; see also [46]. Moreover, in the same article was offered a drastic simplification of this equation that makes possible to reduce the fourth-order PDE kinetic equation to a nonlinear diffusion equation. This very simple model, which can be easily studied both analytically and numerically, inherits the basic properties of original Hasselmann equation.

To derive the PDE Hasselmann equation one has to exploit three facts only: this equation has a proper set of motion constants, Rayley-Jeans thermodynamic solutions, and the kernel that is a homogenous function of order 3. These conditions make possible to construct the kinetic equation by a unique way up to an arbitrary constant. It is easier to perform in the polar coordinates in $k$-space, replacing $|k|$ by $\omega^2/g$.

Now we replace $N_k \to n(\omega, \phi)$ and introduce the operator

$$L = \frac{1}{2}\frac{\partial^2}{\partial \omega^2} + \frac{1}{\omega^2}\frac{\partial^2}{\partial \phi^2}, \tag{8.3}$$

where $\phi$ is the angle. The PDE analog for the kinetic equation reads as

$$\frac{\partial n}{\partial t} = \frac{c}{g^8\,\omega^3}\, L\, n^4\, \omega^{26}\, L\, \frac{1}{n} = S_{nl}. \tag{8.4}$$

This is a unique equation, which has thermodynamic solutions (3.2) as well as conserving quantities (2.36-2.38), and gives for axially-symmetric solutions of the stationary equation the following expression:

$$S_{nl} = \frac{c}{g^8\,\omega^3}\frac{\partial^2}{\partial \omega^2}\, n^4\, \omega^{26}\, \frac{\partial^2}{\partial \omega^2}\frac{1}{n} = 0. \tag{8.5}$$

Looking for solution in the form $n = \lambda(\omega^2/g)^{-x}$ we end up with:

$$S_{nl} \;\; = \;\; c\,\lambda^3\, g^{3/2}\left(\frac{\omega^2}{g}\right)^{-3x+9/4} F(x), \tag{8.6}$$

$$F(x) \quad = \quad 72x(2x-1)(x-4)(x-23/6).$$

In this case "the window of opportunity" is the whole real axis. Function $F(x)$ has four zeros. Besides the KZ zeros at $x = 4$ and $x = 23/6$, it has "thermodynamic zeros" at $x = 0$ and $x = 1/2$. For the differential version of Hasselmann equation, the thermodynamic distribution of wave action $n = P/(\omega + \nu)$ is a real solution.

Thermodynamic solutions are not the subject of our interest; we are concerned with the fast-decaying solutions only. For these solutions we can simplify the kinetic equation further and replace it with the nonlinear diffusion equation:

$$\frac{\partial N(\omega,\phi)}{\partial t} = \frac{a}{g^4}\, L\,\omega^{15}\, N^3(\omega,\phi) = \frac{a}{g^4}\left(\frac{1}{2}\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\omega^{15}\, N^3(\omega,\phi). \tag{8.7}$$

Here $a$ is some dimensionless constant and

$$N(\omega,\phi) = \frac{2\omega^3}{g^2}\, n(\omega,\phi). \tag{8.8}$$

Thereafter for any function $g(\phi)$ we will notate $\langle g\rangle = \int_0^{2\pi} g(\phi)\, d\phi$; then introduce

$$\tilde{N}(\omega) = \langle N(\omega,\phi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} N(\omega,\phi)\, d\phi. \tag{8.9}$$

Apparently,

$$F(\omega) = \omega\tilde{N}(\omega), \quad N = \int_0^{\infty} \hat{N}(\omega)\, d\omega, \quad \langle\sigma^2\rangle = E = \int_0^{\infty} F(\omega)\, d\omega.$$

For the components of spectral density momentum,

$$\begin{aligned} M_x &= \frac{1}{g}\langle\omega^2\, N(\omega,\phi)\,\cos\phi\rangle, \\ M_y &= \frac{1}{g}\langle\omega^2\, N(\omega,\phi)\,\sin\phi\rangle, \end{aligned} \tag{8.10}$$

total values of momentum are:

$$M_x = \int_0^{\infty} M_x(\omega)\, dx, \quad M_y = \int_0^{\infty} M_y(\omega)\, dy. \tag{8.11}$$

From equation (3.44) one gets

$$\frac{\partial\tilde{N}}{\partial t} = \frac{\partial^2}{\partial\omega^2} K, \quad K = \frac{a}{2g^4}\langle\omega^{15}\, N^3(\omega,\phi)\rangle. \tag{8.12}$$

In the similar way we obtain the conservation laws of wave action and energy in differential form:

$$\frac{\partial \tilde{N}}{\partial t} = \frac{\partial Q}{\partial \omega}, \quad Q = \frac{\partial K}{\partial \omega}, \tag{8.13}$$

$$\frac{\partial F(\omega)}{\partial t} = -\frac{\partial P}{\partial \omega}, \quad P = K - \omega \frac{\partial K}{\partial \omega}. \tag{8.14}$$

Conservation of momentum can also be written in differential form:

$$\frac{\partial \hat{M}_x}{\partial t} + \frac{\partial R_x}{\partial \omega} = 0, \quad \frac{\partial \hat{M}_y}{\partial t} + \frac{\partial R_y}{\partial \omega} = 0. \tag{8.15}$$

Here $R_x, R_y$ are components of momentum flux. For the diffusion model of the Hasselmann equation

$$R_x = \frac{a}{g^5}\langle \cos\phi \left(\omega - \frac{\omega^2}{2}\frac{\partial}{\partial \omega}\right) \omega^{15} N^3(\omega,\phi)\rangle$$

$$R_y = \frac{a}{g^5}\langle \sin\phi \left(\omega - \frac{\omega^2}{2}\frac{\partial}{\partial \omega}\right) \omega^{15} N^3(\omega,\phi)\rangle \tag{8.16}$$

Now we can find the general KZ (Kolmogorov-Zakharov) solution of the stationary equation:

$$\frac{a}{g^4} L\,\omega^{15} N^3(\omega,\phi) = 0, \tag{8.17}$$

$$N(\omega,\phi) = \left(\frac{2g^4}{a}\right)^{1/3} \frac{1}{\omega^5} \left(P + \omega Q + \frac{2R_x}{\omega}\cos\phi\right)^{1/3}. \tag{8.18}$$

Here $P, Q, R$ are fluxes of energy, wave action and $x$-component of momentum. In the isotropic case $R_x = 0$, and we get the general axially-symmetric solution

$$N(\omega,\phi) = \left(\frac{2g^4}{a}\right)^{1/3} \frac{1}{\omega^5} (P + \omega Q)^{1/3} \tag{8.19}$$

In the particular cases $Q = 0$ and $P = 0$ we get KZ spectra for energy and wave action. If $R_x = 0$, $P > 0$, $Q > 0$, solution (3.55) is strictly positive. If $R_x \neq 0$ in some area of small frequencies, $N$ changes sign. Negative $N$ has no physical meaning; it means that generic equation (3.54) can be realized only for large enough frequencies.

From equation (3.35) we get the energy and the wave action spectra for KZ-equation:

$$F^{(1)}(\omega) = \left(\frac{2}{a}g^4\right)^{1/3} \frac{P^{1/3}}{\omega^4}, \qquad F^{(2)}(\omega) = \left(\frac{2}{a}g^4\right)^{1/3} \frac{Q^{1/3}}{\omega^{11/3}}. \tag{8.20}$$

To reach coincidence with (3.33) and (3.37) we have to neglect the difference between $c_p$ and $c_q$ and put $a = 2/(4\pi c_p)^3$. For $c_p \simeq 0.22$, we get $a = 0.094$.

Returning to the general Hasselmann equation, we can again introduce coordinates $\omega, \phi$ and rewrite the equation as follow:

$$\frac{\partial N(\omega,\phi)}{\partial t} = \left( \frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) A = 2LA. \tag{8.21}$$

Here

$$A = \frac{1}{g^2} L^{-1} \omega^3 S_{nl}, \tag{8.22}$$

and $N(\omega, \phi)$ is defined by (3.45). The explicit expression for $L^{-1}$ is presented in Appendix 2. The solution of equation

$$LA = 0 \tag{8.23}$$

can be chosen as follow:

$$A(\omega,\phi) = P + \omega Q + \frac{2R_x\, g\, \cos\phi}{\omega}. \tag{8.24}$$

As before, here $P$ is flux of energy to high wave numbers, $Q$ is flux of wave action to small wave numbers, and $R_x$ is flux of $x$-component of momentum to high wave numbers. Thus, general solutions, symmetric with reflection $y \to -y$, depend on three arbitrary constants. At $\omega \to 0$, the energy spectrum behaves as (3.39). If $Q = 0$ we get ZF asymptotics (3.35).

The general anisotropic solution becomes negative for backward direction $\phi = \pi$ in the area of small frequency. If $P = 0, Q = 0$, one gets the third KZ spectrum,

$$F^3(\omega,\phi) = f(\phi) \left( \frac{R_x}{g^2} \right)^{1/3} \frac{g^2}{\omega^{13/3}}, \tag{8.25}$$

however from the symmetry consideration follows that $f(\pi - \phi) = -f(\phi)$. Thus this spectrum has no independent importance.

From (3.58) one can see that the general anisotropic spectrum (3.58????) has tendency for isotropisation at $\omega \to \infty$. Weakly anisotropic KZ spectra were studied by Kantorovich and Katz [46] and by Balk [47].