Six-waves scattering matrix for water wave equation

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Abstract

We study amplitudes of six-wave interactions for compact 1-D Zakharov equation. It was found that six-wave amplitude is not canceled for this equation. Thus, 1-D Zakharov equation is not integrable.

Key words: free surface, gravity waves, integrable equation
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1 Introduction

The work described here is motivated by remarkable fact regarding two-dimensional free surface hydrodynamics - four-wave interaction coefficient vanishes on the resonant manifold

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}. \end{aligned}$$

This cancellation was derived in [1] and brought the hypothesis of integrability of 2-D free surface hydrodynamics. Also the cancellation allows to consider

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surface waves moving in the same direction only. Namely, if initial state consists of such waves, evolution equation keeps this property. In this article we study the problem of integrability in more details.

So, we consider two-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid which is described by the following set of equations:

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 \qquad (\phi_z \to 0, z \to -\infty), \\ \eta_t + \eta_x \phi_x &= \phi_z \Big|_{z=\eta} \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta &= 0 \Big|_{z=\eta} ; \end{aligned}$$

here $\eta(x,t)$ - is the shape of a surface, $\phi(x,z,t)$ - is a potential function of the flow and g - is a gravitational acceleration. As was shown in[2] these equations are Hamiltonian with respect to variables $\eta(x,t)$ and $\psi(x,t) = \phi(x,z,t)\Big|_{z=\eta}$

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta} \qquad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}.$$

Here $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} v^2 dz \qquad U = \frac{g}{2}\int \eta^2 dx$$

Hamiltonian can be expanded in infinite series of characteristic wave steepness $k\eta_k << 1$ and we consider this series up to the fourth order:

$$\begin{aligned} H = \; & \frac{1}{2}\int g\eta^2 + \psi\hat{k}\psi dx - \frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \\ & + \frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx + \dots \end{aligned} \tag{1.1}$$

Applying canonical transformation along with introducing normal complex variable $b(x,t)$ Hamiltonian (1.1) transforms to the equivalent compact form :

$$H = \int b^* \hat{\omega}_k b dx + \frac{1}{2}\int |b'|^2 \left[\frac{i}{2}(bb'^* - b^* b') - \hat{k}|b|^2\right] dx. \tag{1.2}$$

Here $b' = \frac{\partial b}{\partial x}$, $\omega_k = \sqrt{gk}$ and $\hat{k}$ - modulus k operator. All the details of this transformation can be found in [3].

For Fourier harmonics Hamiltonian can be written as following

$$H = \int \omega_k |b_k|^2 + \\ + \frac{1}{2}\int T_{k_1 k_2}^{k_3 k_4} b_{k_1}^* b_{k_2}^* b_{k_3} b_{k_4} \delta_{k_1+k_2-k_3-k_4} dk_1 dk_2 dk_3 dk_4 \tag{1.3}$$

Here

$$T_{k_2 k_3}^{k k_1} = \frac{\theta(k)\theta(k_1)\theta(k_2)\theta(k_3)}{8\pi}\left[(kk_1(k+k_1) + k_2k_3(k_2+k_3)) - \\ -(kk_2|k-k_2| + kk_3|k-k_3| + k_1k_2|k_1-k_2| + k_1k_3|k_1-k_3|)\right]. \tag{1.4}$$

θ - functions in (1.4) is the manifestation of waves moving in the same direction. Corresponding evolution equation is the following:

$$i\frac{\partial b_k}{\partial t} = \omega_k b_k + \int T_{kk_1}^{k_2 k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3. \tag{1.5}$$

Below we will analyze this equation from the point of view of its integrability.

2 Scattering matrix for compact equation

For further consideration we introduce $c_k(t)$ in a following way:

$$b_k(t) = c_k(t)\, e^{-i\omega_k t}.$$

Then equation (1.5) can be rewritten as

$$i\dot{c}_k(t) = \int T_{k_2 k_3}^{k k_1} c_{k_1}^*(t) c_{k_2}(t) c_{k_3}(t) e^{i(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})t} \delta(k+k_1-k_2-k_3) dk_1 dk_2 dk_3 \tag{2.6}$$

and rewrite it in the Picard form ($\epsilon > 0$)

$$c_k(t) = c_k^- - i \lim_{\epsilon->0} \int_{-\infty}^{t} dt \int T_{k_2 k_3}^{k k_1} c_{k_1}^*(t) c_{k_2}(t) c_{k_3}(t) e^{i(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})t+\epsilon t} \delta(k+k_1-k_2-k_3) dk_1 dk_2 dk_3 \tag{2.7}$$

This equation can be solved by iterations:

$$c_k = c_k^{(0)} + c_k^{(1)}(t) + c_k^{(2)}(t) + ... \qquad c_k^{(0)} = c_k(-\infty)$$

Following [4] we introduce a so-called formal scattering matrix for the equation (2.7) for

$$c_k^- = \lim_{t->-\infty} c_k(t), \qquad c_k^+ = \lim_{t->=+\infty} c_k(t).$$

$$c_k^+ = S[c_k^-] \tag{2.8}$$

So far as (2.6) has only four-wave vortex, scattering matrix has the form:

$$S[c_k^-] = c_k^- + S_{22}[c_k^-] + S_{33}[c_k^- + \dots \tag{2.9}$$

Element S_{22} has already been calculated in [6]. In spite of it has logarithmic divergence (it is why scattering matrix is formal) , it does not produce "new" wave vectors as should be in the integrable systems. Below we will calculate S_{33}. Performing two iteration of (2.7) one can get for $c_k^{(1)}(t)$:

$$c_k^{(1)}(t) = -i \lim_{\epsilon\to 0} \int\limits_{-\infty}^{t} dt \int T_{k_2k_3}^{kk_1} c_{k_1}^{*(0)} c_{k_2}^{(0)} c_{k_3}^{(0)} e^{i(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})t+\epsilon t} \delta(k+k_1-k_2-k_3) dk_1 dk_2 dk_3$$

or

$$c_k^{(1)}(t) = -\int \frac{T_{k_2k_3}^{kk_1} c_{k_1}^{*(0)} c_{k_2}^{(0)} c_{k_3}^{(0)}}{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} e^{i(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})t} \delta(k+k_1-k_2-k_3) dk_1 dk_2 dk_3.$$

Performing second iteration one can get:

$$c_{p_1}^{(2)}(+\infty) = -i \int\limits_{-\infty}^{+\infty} dt \int dr dp_2 dp_3 dq_1 dq_2 dq_3 c_{p_2}^{*(0)} c_{p_3}^{*(0)} c_{q_1}^{(0)} c_{q_2}^{(0)} c_{q_3}^{(0)} e^{i(\omega_{p_1}+\omega_{p_2}+\omega_{p_3}-\omega_{q_1}-\omega_{q_2}-\omega_{q_3})t} \times$$

$$\times \Bigg[\frac{T_{q_1q_2}^{p_1r} T_{rq_3}^{p_2p_3} \delta(r+p_1-q_1-q_2)\delta(r+q_3-p_2-p_3)}{\omega_r+\omega_{q_3}-\omega_{p_2}-\omega_{p_3}} + \frac{T_{q_3q_2}^{p_1r} T_{rq_1}^{p_2p_3} \delta(r+p_1-q_2-q_3)\delta(r+q_1-p_2-p}{\omega_r+\omega_{q_1}-\omega_{p_2}-\omega_{p_3}}$$

$$+ \frac{T_{q_1q_3}^{p_1r} T_{rq_2}^{p_2p_3} \delta(r+p_1-q_1-q_3)\delta(r+q_2-p_2-p_3)}{\omega_r+\omega_{q_2}-\omega_{p_2}-\omega_{p_3}} + \frac{T_{rq_1}^{p_1p_2} T_{q_2q_3}^{rp_3} \delta(r+q_1-p_1-p_2)\delta(r+p_3-q_2-q_3)}{\omega_r+\omega_{p_3}-\omega_{q_2}-\omega_{q_3}}$$

$$+ \frac{T_{rq_2}^{p_1p_2} T_{q_1q_3}^{rp_3} \delta(r+q_2-p_1-p_2)\delta(r+p_3-q_1-q_3)}{\omega_r+\omega_{p_3}-\omega_{q_1}-\omega_{q_3}} + \frac{T_{rq_3}^{p_1p_2} T_{q_2q_1}^{rp_3} \delta(r+q_3-p_1-p_2)\delta(r+p_3-q_1-q_2)}{\omega_r+\omega_{p_3}-\omega_{q_1}-\omega_{q_2}}$$

$$+ \frac{T_{rq_1}^{p_1p_3} T_{q_2q_3}^{rp_2} \delta(r+q_1-p_1-p_3)\delta(r+p_2-q_2-q_3)}{\omega_r+\omega_{p_2}-\omega_{q_2}-\omega_{q_3}} + \frac{T_{rq_2}^{p_1p_3} T_{q_1q_3}^{rp_2} \delta(r+q_2-p_1-p_3)\delta(r+p_2-q_1-q_3)}{\omega_r+\omega_{p_2}-\omega_{q_1}-\omega_{q_3}}$$

$$+ \frac{T_{rq_3}^{p_1p_3} T_{q_2q_1}^{rp_2} \delta(r+q_3-p_1-p_3)\delta(r+p_2-q_1-q_2)}{\omega_r+\omega_{p_2}-\omega_{q_1}-\omega_{q_2}} \Bigg]$$

or

$$c_{p_1}^{(2)}(+\infty) = -2\pi i \int T_{q_1q_2q_3}^{p_1p_2p_3} c_{p_2}^{*(0)} c_{p_3}^{*(0)} c_{q_1}^{(0)} c_{q_2}^{(0)} c_{q_3}^{(0)} \delta(\omega_{p_1}+\omega_{p_2}+\omega_{p_3}-\omega_{q_1}-\omega_{q_2}-\omega_{q_3}) \delta(p_1+p_2+p_3-q_1-q_2-q_3$$

Here $T_{q_1q_2q_3}^{p_1p_2p_3}$ is equal to:

$$T_{q_1q_2q_3}^{p_1p_2p_3} = \Bigg[\frac{T_{p_2+p_3-q_3q_3}^{p_2p_3} T_{q_1q_2}^{q_1+q_2-p_1p_1}}{\omega_{q_1+q_2-p_1}+\omega_{p_1}-\omega_{q_1}-\omega_{q_2}} + \frac{T_{p_2+p_3-q_1q_1}^{p_2p_3} T_{q_2q_3}^{q_2+q_3-p_1p_1}}{\omega_{q_2+q_3-p_1}+\omega_{p_1}-\omega_{q_2}-\omega_{q_3}} + \frac{T_{p_2+p_3-q_2q_2}^{p_2p_3} T_{q_1q_3}^{q_1+q_3-p_1p_1}}{\omega_{q_1+q_3-p_1}+\omega_{p_1}-\omega_{q_1}-\omega_{q_3}}$$

$$+\frac{T^{p_1p_2}_{p_1+p_2-q_1q_1}T^{q_2+q_3-p_3p_3}_{q_2q_3}}{\omega_{q_2+q_3-p_3}+\omega_{p_3}-\omega_{q_2}-\omega_{q_3}}+\frac{T^{p_1p_2}_{p_1+p_2-q_2q_2}T^{q_1+q_3-p_3p_3}_{q_1q_3}}{\omega_{q_1+q_3-p_3}+\omega_{p_3}-\omega_{q_1}-\omega_{q_3}}+\frac{T^{p_1p_2}_{p_1+p_2-q_3q_3}T^{q_1+q_2-p_3p_3}_{q_1q_2}}{\omega_{q_1+q_2-p_3}+\omega_{p_3}-\omega_{q_1}-\omega_{q_2}}+$$
$$+\frac{T^{p_1p_3}_{p_1+p_3-q_1q_1}T^{q_2+q_3-p_2p_2}_{q_2q_3}}{\omega_{q_2+q_3-p_2}+\omega_{p_2}-\omega_{q_2}-\omega_{q_3}}+\frac{T^{p_1p_3}_{p_1+p_3-q_2q_2}T^{q_1+q_3-p_2p_2}_{q_1q_3}}{\omega_{q_1+q_3-p_2}+\omega_{p_2}-\omega_{q_1}-\omega_{q_3}}+\frac{T^{p_1p_3}_{p_1+p_3-q_3q_3}T^{q_1+q_2-p_2p_2}_{q_1q_2}}{\omega_{q_1+q_2-p_2}+\omega_{p_2}-\omega_{q_1}-\omega_{q_2}}\Bigg]$$

Element $T^{p_1p_2p_3}_{q_1q_2q_3}$ is the kernel of six waves element of scattering matrix S_{33}. Symbolically in can be represented as 9 diagrams, given in the Appendix. Now we calculate 6-waves element $T^{p_1p_2p_3}_{q_1q_2q_3}$ on the resonant manifold

$$\begin{cases} p_1+p_2+p_3=q_1+q_2+q_3 \\ \omega_{p_1}+\omega_{p_2}+\omega_{p_3}=\omega_{q_1}+\omega_{q_1}+\omega_{q_3} \end{cases}$$

On this manifold expression for $T^{p_1p_2p_3}_{q_1q_2q_3}$ can be simplified:

$$T^{p_1p_2p_3}_{q_1q_2q_3}=\frac{1}{2\left(\omega_{p_1}\omega_{p_2}\omega_{p_3}-\omega_{q_1}\omega_{q_2}\omega_{q_3}\right)}\times$$
$$\begin{aligned}
\times\Big[\, &T^{p_2p_3}_{p_2+p_3-q_3q_3}T^{q_1+q_2-p_1p_1}_{q_1q_2}\left(\omega_{p_1}-\omega_{q_1}-\omega_{q_2}-\omega_{q_1+q_2-p_1}\right)\left(\omega_{p_1}-\omega_{q_3}\right)+\\
&+T^{p_2p_3}_{p_2+p_3-q_1q_1}T^{q_2+q_3-p_1p_1}_{q_2q_3}\left(\omega_{p_1}-\omega_{q_2}-\omega_{q_3}-\omega_{q_2+q_3-p_1}\right)\left(\omega_{p_1}-\omega_{q_1}\right)+\\
&+T^{p_2p_3}_{p_2+p_3-q_2q_2}T^{q_1+q_3-p_1p_1}_{q_1q_3}\left(\omega_{p_1}-\omega_{q_1}-\omega_{q_3}-\omega_{q_1+q_3-p_1}\right)\left(\omega_{p_1}-\omega_{q_2}\right)+\\
&+T^{p_1p_2}_{p_1+p_2-q_1q_1}T^{q_2+q_3-p_3p_3}_{q_2q_3}\left(\omega_{p_3}-\omega_{q_2}-\omega_{q_3}-\omega_{q_2+q_3-p_3}\right)\left(\omega_{p_3}-\omega_{q_1}\right)+\\
&+T^{p_1p_2}_{p_1+p_2-q_2q_2}T^{q_1+q_3-p_3p_3}_{q_1q_3}\left(\omega_{p_3}-\omega_{q_1}-\omega_{q_3}-\omega_{q_1+q_3-p_3}\right)\left(\omega_{p_3}-\omega_{q_2}\right)+\\
&+T^{p_1p_2}_{p_1+p_2-q_3q_3}T^{q_1+q_2-p_3p_3}_{q_1q_2}\left(\omega_{p_3}-\omega_{q_1}-\omega_{q_2}-\omega_{q_1+q_2-p_3}\right)\left(\omega_{p_3}-\omega_{q_3}\right)+\\
&+T^{p_1p_3}_{p_1+p_3-q_1q_1}T^{q_2+q_3-p_2p_2}_{q_2q_3}\left(\omega_{p_2}-\omega_{q_2}-\omega_{q_3}-\omega_{q_2+q_3-p_2}\right)\left(\omega_{p_2}-\omega_{q_1}\right)+\\
&+T^{p_1p_3}_{p_1+p_3-q_2q_2}T^{q_1+q_3-p_2p_2}_{q_1q_3}\left(\omega_{p_2}-\omega_{q_1}-\omega_{q_3}-\omega_{q_1+q_3-p_2}\right)\left(\omega_{p_2}-\omega_{q_2}\right)+\\
&+T^{p_1p_3}_{p_1+p_3-q_3q_3}T^{q_1+q_2-p_2p_2}_{q_1q_2}\left(\omega_{p_2}-\omega_{q_1}-\omega_{q_2}-\omega_{q_1+q_2-p_2}\right)\left(\omega_{p_2}-\omega_{q_3}\right)\Big]
\end{aligned} \tag{2.10}$$

Manifold can be parametrized using three parameters λ, α, and β as following

$$\begin{cases}
p_1=\dfrac{\omega^2_{p_1}}{g}=\left(1+\lambda\dfrac{1+\alpha}{1+\alpha+\alpha^2}\right)^2\\
p_2=\dfrac{\omega^2_{p_2}}{g}=\left(1+\lambda\dfrac{\alpha(1+\alpha)}{1+\alpha+\alpha^2}\right)^2\\
p_3=\dfrac{\omega^2_{p_3}}{g}=\left(1-\lambda\dfrac{\alpha}{1+\alpha+\alpha^2}\right)^2\\
q_1=\dfrac{\omega^2_{q_1}}{g}=\left(1+\lambda\dfrac{1+\beta}{1+\beta+\beta^2}\right)^2\\
q_2=\dfrac{\omega^2_{q_2}}{g}=\left(1+\lambda\dfrac{\beta(1+\beta)}{1+\beta+\beta^2}\right)^2\\
q_3=\dfrac{\omega^2_{q_3}}{g}=\left(1-\lambda\dfrac{\beta}{1+\beta+\beta^2}\right)^2
\end{cases}$$

Plugging ω_i, p_i and q_i for different values of λ, α, and β one can check that

$$T^{p_1p_2p_3}_{q_1q_2q_3} \neq 0.$$

This is the proof of nonintegrability of compact 1-D Zakharov equation.

Note, that if consider above theory for Nonlinear Schrodinger Equation for which

$$T^{k_1k_2}_{k_3k_4} = 1, \qquad \omega_k = k^2$$

simple calculations end up with $T^{p_1p_2p_3}_{q_1q_2q_3} = 0$, as it must be for integrable system.

3 Conclusions

Compact 1-D Zakharov equation (1.5), or equivalent system with Hamiltonian (1.1) is nonintegrable system. However the question about integrability of fully nonlinear system (1.1) is still unclear. Exact equation has his own six wave term which could make total six wave coefficient changed.

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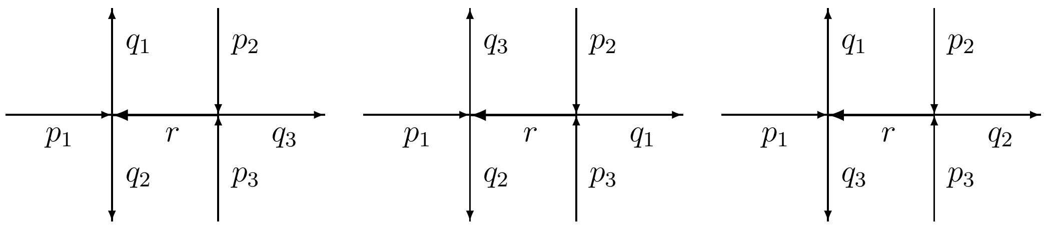
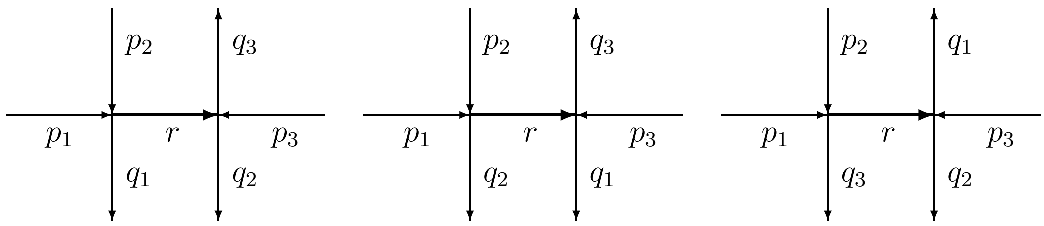
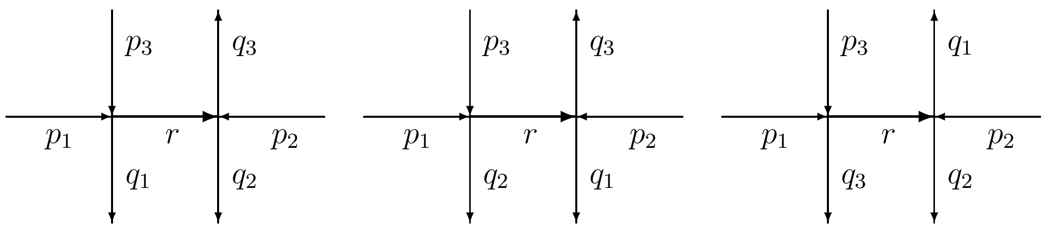
5 Appendix



$q_1 + q_2 - p_1 > 0$ $\qquad$ $q_2 + q_3 - p_1 > 0$ $\qquad$ $q_1 + q_3 - p_1 > 0$



$p_1 + p_2 - q_1 > 0$ $\qquad$ $p_1 + p_2 - q_2 > 0$ $\qquad$ $p_1 + p_2 - q_3 > 0$



$p_1 + p_3 - q_1 > 0$ $\qquad$ $p_1 + p_3 - q_2 > 0$ $\qquad$ $p_1 + p_3 - q_3 > 0$