The surface signature of internal waves

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Workshop on Ocean Wave Dynamics (Fields Institute 2013)

Collaborators

Joint work with

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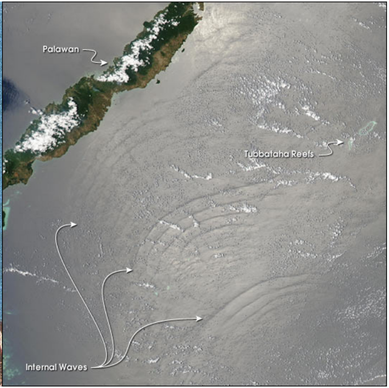
Surface signature of internal waves

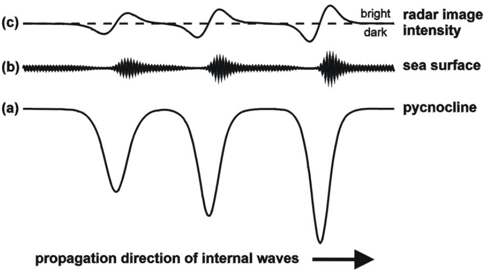
Strait of Gibraltar - ESA



Sulu Sea - NASA

- Internal waves are not directly visible to the observer, however their surface signature appears as strips of rough water (the ‘rip’), producing changes in reflectance on satellite images

Rip phenomenon



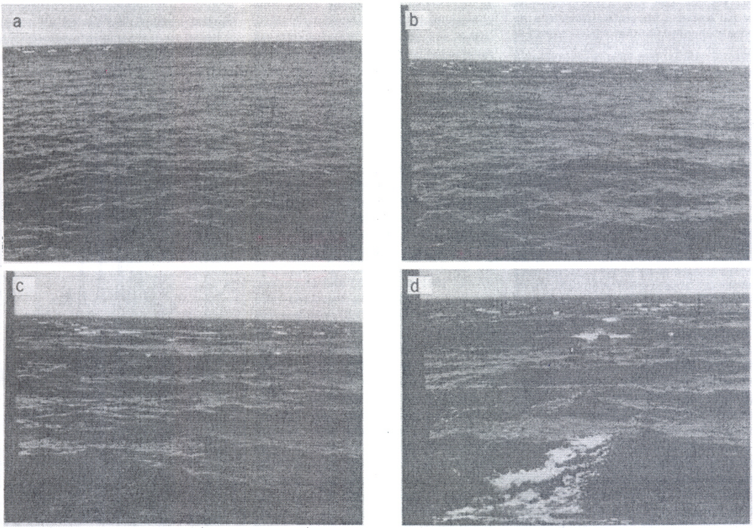
Schematic of internal-surface wave interaction (www.ifm.zmaw.de/ ers-sar)

Surface roughness induced by an internal wave (Forgot where?)

Rip phenomenon

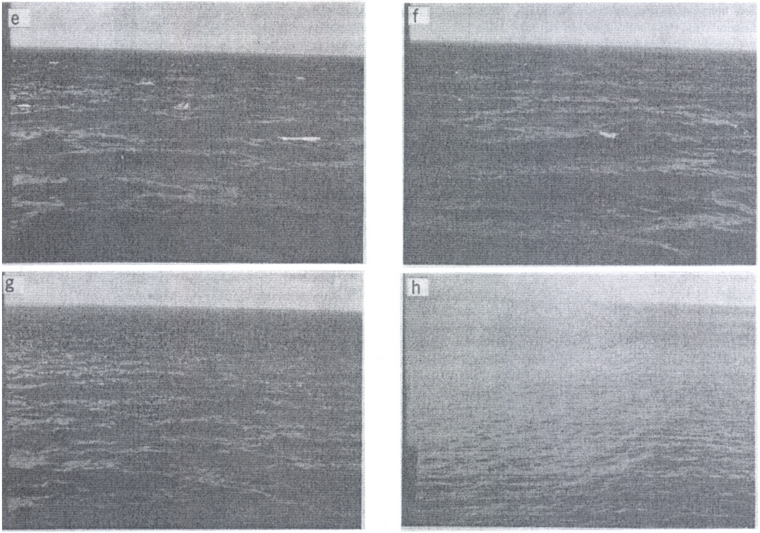
http://myweb.dal.ca/kelley/SLEIWEX/gallery (Ile Aux Lievres, Quebec)

Millpond effect



Rip approaching the vessel (Osborne & Burch 1980)

Millpond effect

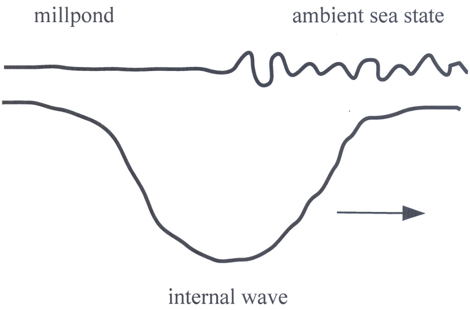


During and after the rip (Osborne & Burch 1980)

Millpond effect

From Osborne & Burch (1980):

*"Observable in the distance is a long band of breaking waves about 1.8 m high approaching from due west ... the rip continues to approach ... the rip band reaches the survey vessel with 1.8-m waves. This condition persisted for several minutes until the trailing edge of the rip passed by ... and the wave heights quickly dropped to less than 0.1 m. The surface of the Andaman Sea had the appearance of a **millpond**".*



Ray approach

Phase-averaged model for surface waves

$$\partial_t N + \nabla_k N \cdot \dot{\boldsymbol{k}} + \nabla_x N \cdot \dot{\boldsymbol{x}} = S$$

combined with ray equations

$$\dot{\boldsymbol{x}} = \nabla_k \omega \ , \quad \dot{\boldsymbol{k}} = -\nabla_x \omega$$

and

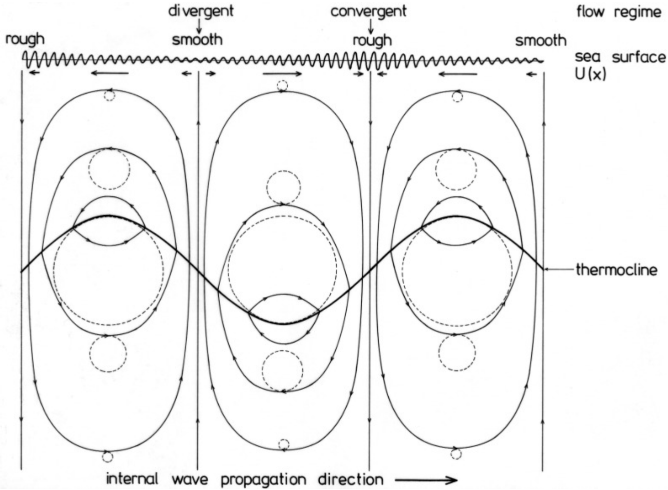
$$\omega = \sqrt{gk} + \boldsymbol{k} \cdot \boldsymbol{U}$$

where

- N : spectral density (wave action) for surface waves
- ω : linear dispersion relation
- $\boldsymbol{U}$: (prescribed) near-surface current induced by internal waves

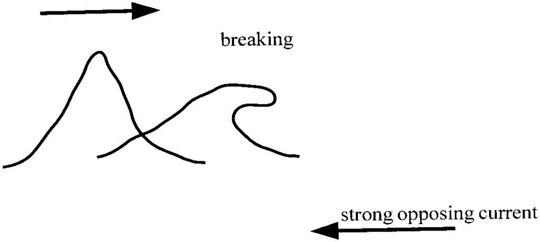
See e.g. Gargett & Hughes (1972), Basovich & Talanov (1977), Alpers (1985), Gasparovic et al. (1988), Bakhanov & Ostrovsky (2002), etc.

Ray approach



Schematic of surface modulation by internal waves (www.ifm.zmaw.de/ ers-sar)

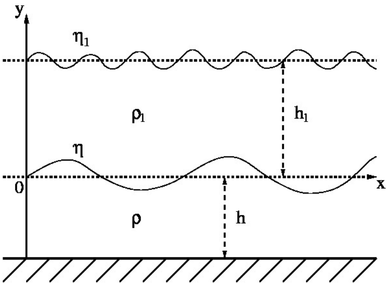
Wave blocking



- In the ray approach, the **millpond effect** is explained by focusing and breaking of surface waves due to opposing currents, leaving calmer water after the internal wave passage
- See e.g. Peregrine (1976), Phillips (1977), Chawla & Kirby (2002), etc.

(2D) Two-layer formulation for internal-surface waves

See e.g. Choi & Camassa (1999)



- g : acceleration due to gravity
- ρ, ρ_1 : densities of the lower and upper fluids respectively
  ($\rho > \rho_1$ in stable stratification, $\rho_1/\rho \sim 1$ in oceanic conditions)
- h, h_1 : mean depths of the lower and upper fluids
- $\eta(x,t)$: interface displacement, $x \in \mathbb{R}$, $t \geq 0$
- $\eta_1(x,t)$: surface elevation

Governing equations

Each fluid is incompressible, inviscid and the flow is irrotational

$$
\begin{aligned}
\Delta\varphi &= 0\,, && \text{for} \quad -h < y < \eta \\
\Delta\varphi_1 &= 0\,, && \text{for} \quad \eta < y < h_1 + \eta_1 \\
\partial_y\varphi &= 0\,, && \text{on} \quad y = -h \\
\partial_t\eta + \partial_x\eta\,\partial_x\varphi - \partial_y\varphi &= 0\,, && \text{on} \quad y = \eta \\
\partial_t\eta + \partial_x\eta\,\partial_x\varphi_1 - \partial_y\varphi_1 &= 0\,, && \text{on} \quad y = \eta \\
\rho\Big(\partial_t\varphi + \tfrac{1}{2}|\nabla\varphi|^2 + g\eta\Big) - \rho_1\Big(\partial_t\varphi_1 + \tfrac{1}{2}|\nabla\varphi_1|^2 + g\eta\Big) &= 0\,, && \text{on} \quad y = \eta \\
\partial_t\eta_1 + \partial_x\eta_1\,\partial_x\varphi_1 - \partial_y\varphi_1 &= 0\,, && \text{on} \quad y = h_1 + \eta_1 \\
\partial_t\varphi_1 + \tfrac{1}{2}|\nabla\varphi_1|^2 + g\eta_1 &= 0\,, && \text{on} \quad y = h_1 + \eta_1
\end{aligned}
$$

where $\varphi(x, y, t)$ and $\varphi_1(x, y, t)$ are the velocity potentials for the lower and upper fluids respectively

Hamiltonian formulation

See e.g. Benjamin & Bridges 1997, Zakharov & Kuznetsov 1997

- **Hamiltonian** (total energy)

$$H = \frac{1}{2}\int \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix}^{\top} \begin{pmatrix} G_{11}B^{-1}G(\eta) & -G(\eta)B^{-1}G_{12} \\ -G_{21}B^{-1}G(\eta) & \frac{1}{\rho_1}G_{22} - \frac{\rho}{\rho_1}G_{21}B^{-1}G_{12} \end{pmatrix} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} dx + \frac{1}{2}\int g(\rho-\rho_1)\eta^2\,dx + \frac{1}{2}\int g\rho_1(h_1+\eta_1)^2\,dx$$

with $\xi_1(x,t) = \rho_1\,\varphi_1|_{y=h_1+\eta_1}$ and $\xi(x,t) = (\rho\,\varphi - \rho_1\,\varphi_1)|_{y=\eta}$

- **Canonical equations of motion**

$$\partial_t \begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{\eta_1}H \\ \delta_{\xi_1}H \end{pmatrix}, \quad \partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{\eta}H \\ \delta_{\xi}H \end{pmatrix}$$

- Convenient for **perturbation calculations**

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots + \varepsilon^m H_m\,, \quad \varepsilon \ll 1$$

Craig et al. 2005, 2012

Dirichlet–Neumann operators

Following Craig & Sulem (1993)

- For the lower layer

$$G(\eta)\Phi(x) = \nabla\varphi \cdot N(1 + |\partial_x\eta|^2)^{1/2}$$

- For the upper layer

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} = \begin{pmatrix} -(\nabla\varphi_1 \cdot N)(x, \eta(x))(1 + (\partial_x\eta(x))^2)^{1/2} \\ (\nabla\varphi_1 \cdot N_1)(x, h_1 + \eta_1(x))(1 + (\partial_x\eta_1(x))^2)^{1/2} \end{pmatrix}$$

- We can use their Taylor series expansions in η and η_1

$$G(\eta) = \sum_{j=0}^{\infty} G^{(j)}(\eta)$$

$$\begin{pmatrix} G_{11}(\eta, \eta_1) & G_{12}(\eta, \eta_1) \\ G_{21}(\eta, \eta_1) & G_{22}(\eta, \eta_1) \end{pmatrix} = \sum_{m_1, m_2 = 0}^{\infty} \begin{pmatrix} G_{11}^{(m_0, m_1)}(\eta, \eta_1) & G_{12}^{(m_0, m_1)}(\eta, \eta_1) \\ G_{21}^{(m_0, m_1)}(\eta, \eta_1) & G_{22}^{(m_0, m_1)}(\eta, \eta_1) \end{pmatrix} .$$

Dirichlet–Neumann operators

For example, the first terms in these series are given by

$$G^{(0)} = D\tanh(hD)$$

$$\begin{pmatrix} G_{11}^{(0)} & G_{12}^{(0)} \\ G_{21}^{(0)} & G_{22}^{(0)} \end{pmatrix} = \begin{pmatrix} D\coth(h_1 D) & -D\mathrm{csch}(h_1 D) \\ -D\mathrm{csch}(h_1 D) & D\coth(h_1 D) \end{pmatrix}$$

$$\begin{pmatrix} G_{11}^{(10)}(\eta,\eta_1) & G_{12}^{(10)}(\eta,\eta_1) \\ G_{21}^{(10)}(\eta,\eta_1) & G_{22}^{(10)}(\eta,\eta_1) \end{pmatrix} =$$

$$\begin{pmatrix} D\coth(h_1 D)\eta(x)D\coth(h_1 D) - D\eta(x)D & -D\coth(h_1 D)\eta(x)D\mathrm{csch}(h_1 D) \\ -D\mathrm{csch}(h_1 D)\eta(x)D\coth(h_1 D) & D\mathrm{csch}(h_1 D)\eta(x)D\mathrm{csch}(h_1 D) \end{pmatrix}$$

$$\begin{pmatrix} G_{11}^{(01)}(\eta,\eta_1) & G_{12}^{(01)}(\eta,\eta_1) \\ G_{21}^{(01)}(\eta,\eta_1) & G_{22}^{(01)}(\eta,\eta_1) \end{pmatrix} =$$

$$\begin{pmatrix} -D\mathrm{csch}(h_1 D)\eta_1(x)D\mathrm{csch}(h_1 D) & D\mathrm{csch}(h_1 D)\eta_1(x)D\coth(h_1 D) \\ D\coth(h_1 D)\eta_1(x)D\mathrm{csch}(h_1 D) & -D\coth(h_1 D)\eta_1(x)D\coth(h_1 D) + D\eta_1(x)D \end{pmatrix}$$

where $D = -i\partial_x$

Coupled long-wave/modulational regime

- We consider **long waves** at the interface

$$\eta \sim \varepsilon^2 r(X,\tau), \quad X = \varepsilon x, \quad \tau = \varepsilon^3 t$$

where

$$\frac{a}{h} = \left(\frac{h}{l}\right)^2 = \varepsilon^2 \ll 1$$

(a: internal wave amplitude, l: internal wavelength)

- Smaller **modulated monochromatic waves** at the surface

$$\eta_1 \sim \varepsilon_1 v(X,\tau_1) e^{ik_0 x - i\omega(k_0)t} + c.c., \quad \tau_1 = \varepsilon^2 t$$

where

$$k_0 a_1 = \varepsilon_1 = \varepsilon^{2+\alpha}, \quad \alpha > 0$$

(a_1: surface wave amplitude, k_0: surface wave number)

KdV-linear Schrödinger system

See also Kawahara et al. 1975, Ma 1983, Lee et al. 2007

- At lowest order in ε, we find

$$\begin{aligned} \partial_\tau r &= -a_1 r \partial_X r - a_2 \partial_X^3 r && \text{(KdV)} \\ -i\partial_{\tau_1} v &= -\delta^2 \partial_X^2 v + r(X)\, v && \text{(Schrödinger)} \end{aligned}$$

  with the resonance condition

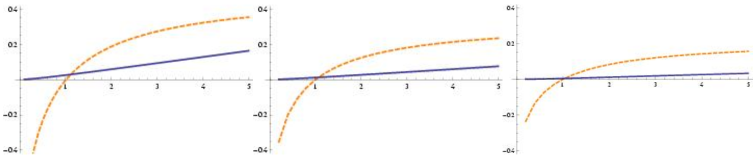
  (surface group velocity) $\quad \partial_k \omega(k_0) = c \quad$ (internal phase velocity)

- The coefficients a_1, a_2 and δ depend on the physical parameters ρ, ρ_1, h and h_1
- An internal wave solution is given by the soliton

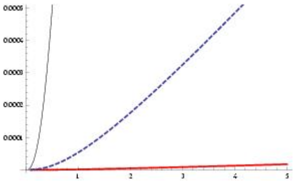
$$r(X) = \frac{3a_2 r_0}{a_1} \operatorname{sech}^2\left(\frac{\sqrt{r_0}}{2} X\right)$$

  which plays the role of a potential in the linear Schrödinger equation for surface waves

Coefficients



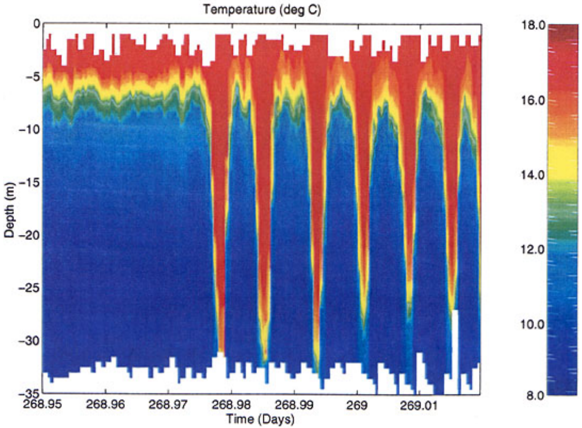
a_1 (dashed line) and a_2 (solid line) as functions of h_1/h for $\rho_1/\rho = 0.95$ (left), 0.99 (center) and 0.998 (right)



δ^2 as a function of h_1/h for $\rho_1/\rho = 0.95$ (thin line), 0.99 (thick line) and 0.998 (dashed line)

For $\rho_1/\rho \sim 1$ and $h_1/h < 1$, the internal soliton is of depression (i.e. potential well) and $\delta^2 \ll 1$

Internal solitary waves



Off the Oregon coast - NOAA

See e.g. Helfrich & Melville 2006, Grimshaw, Pelinovsky & Talipova 2007

Schrödinger equation

- Looking for solutions

$$v(X,\tau_1) = \int e^{i\lambda\tau_1} u(X,\lambda) d\mu(\lambda)$$

the linear Schrödinger equation reduces to the eigenvalue problem

$$-\delta^2 u'' + r\,u = \lambda\,u$$

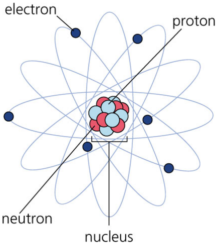
- The spectrum consists of a finite number of negative eigenvalues

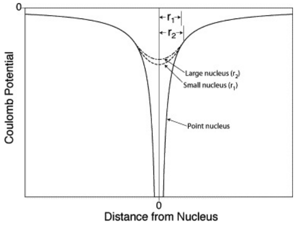
$$\lambda_0 < \lambda_1 < \cdots < \lambda_J < 0$$

along with the continuous part $\lambda > 0$

- For $\rho_1/\rho \sim 1$ and $h_1/h < 1$, recall $\delta^2 \ll 1$ which implies a large number of negative eigenvalues λ_j with corresponding localized bound states $u_j(X)$ characterized by their number of zeros

Analogy with quantum mechanics





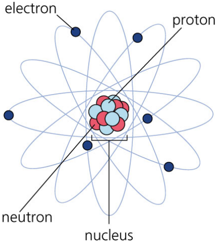
Consider the 1D Schrödinger equation for electrons

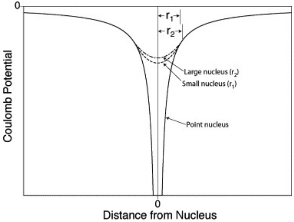
$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \Big[\lambda - V(x)\Big]\psi = 0$$

where

- ψ : wave function (probability density) for electrons
- V : attractive potential due to the nucleus
- $\hbar = 6.62 \times 10^{-34}$ m^2 kg s^{-1} : Planck's constant

Analogy with quantum mechanics

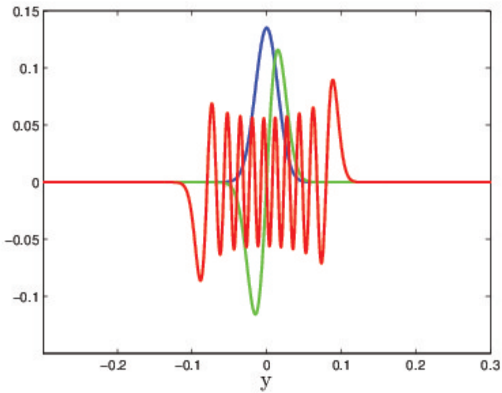




Scattering theory for the Schrödinger equation says

- $\lambda < V(\pm\infty)$: discrete/quantized spectrum (bound states)
  Electrons that are 'trapped' near the nucleus
- $\lambda > V(\pm\infty)$: continuous spectrum (scattering states)
  The 'other guys' who wander far away ...

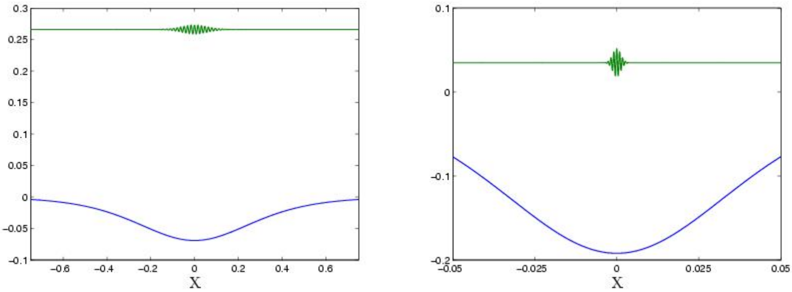
Bound states



Bound states corresponding to λ_0 (ground state, blue), λ_2 (green) and λ_{20} (red)

$N = 8192$ collocation points are used

Numerical simulations

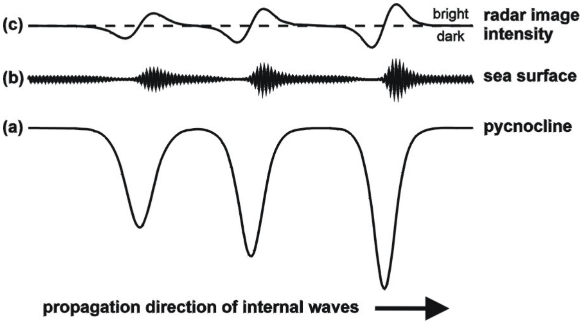


Left: internal wave with surface signature for $\rho_1/\rho = 0.997$, $h_1/h = 0.266$, $a/h = 0.069$ and $k_0h = 396.4$ (Andaman Sea)

Right: internal wave with surface signature for $\rho_1/\rho = 0.998$, $h_1/h = 0.035$, $a/h = 0.192$ and $k_0h = 3696.2$ (off the Oregon coast)

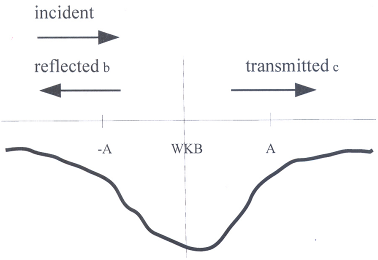
- Blue line: internal wave η
- Green line: surface signature η_1 (ground state + oscillations) $\Longrightarrow$ the rip

Looks like ...



Schematic of internal-surface wave interaction (www.ifm.zmaw.de/ ers-sar)

Continuous spectrum $\lambda > 0$



By successive approximations, consider the solution for $X < -A$

$$
\begin{aligned}
u(X) &= e^{-i\sqrt{\lambda}X/\delta} + be^{i\sqrt{\lambda}X/\delta} \\
&\quad + \frac{1}{\delta\sqrt{\lambda}} \int_{-\infty}^{X} \sin\left[\frac{\sqrt{\lambda}}{\delta}(X-Y)\right] r(Y) \left(e^{-i\sqrt{\lambda}Y/\delta} + be^{i\sqrt{\lambda}Y/\delta}\right) dY
\end{aligned}
$$

and, for $X > A$,

$$
u(X) = c\left(e^{-i\sqrt{\lambda}X/\delta} - \frac{1}{\delta\sqrt{\lambda}} \int_{-\infty}^{X} \sin\left[\frac{\sqrt{\lambda}}{\delta}(X-Y)\right] r(Y) e^{-i\sqrt{\lambda}Y/\delta} dY\right)
$$

Continuous spectrum $\lambda > 0$

After *veeery looooonnngg calculations*, we find the following

Proposition:

In the limit $\delta, \lambda \to 0$ with $\sqrt{\lambda}/\delta \to 0$, the coefficients of reflection and transmission have the asymptotic values

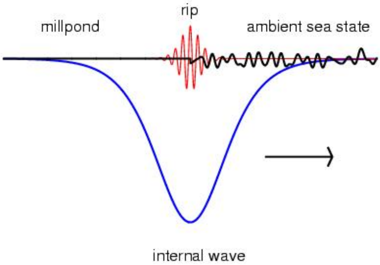
$$\begin{aligned} b &= -1 + O(\sqrt{\lambda}) \\ c &= 0 + O(\sqrt{\lambda}) \end{aligned}$$

respectively

In other words:

- Quasi-monochromatic surface waves tend to be completely reflected from the internal wave region
- The fact that very little is transmitted may contribute to the millpond effect

Summary



For an internal soliton moving into an ambient sea state:

- Absorption into bound states above the soliton (the rip)
- Reflection from and transmission through the soliton region
- Total reflection in the monochromatic limit $\sqrt{\lambda}/\delta \to 0$
- Wave absorption + reflection provide an explanation for the millpond effect (**no wave breaking needed**)

A ‘dirty’ analogy

A passing internal wave sweeps the sea surface of waves

The end ...

THANK YOU!