Omitting types in infinitary $[0, 1]$-valued logic

Chris Eagle

March 8, 2013

1. Background - the first-order case
2. Infinitary logic for metric structures and omitting types
3. Applications

Background

L - countable first-order language, T an L-theory. We want to study countable models of T.

Background

L - countable first-order language, T an L-theory. We want to study countable models of T.

Example

$\langle \overline{\mathbb{Q}}, +, -, \cdot, 0, 1 \rangle \preceq \langle \overline{\mathbb{Q}(\pi)}, +, -, \cdot, 0, 1 \rangle$, but
$\langle \overline{\mathbb{Q}}, +, -, \cdot, 0, 1 \rangle \ncong \langle \overline{\mathbb{Q}(\pi)}, +, -, \cdot, 0, 1 \rangle$

Background

L - countable first-order language, T an L-theory. We want to study countable models of T.

Example

$\langle \overline{\mathbb{Q}}, +, -, \cdot, 0, 1 \rangle \preceq \langle \overline{\mathbb{Q}(\pi)}, +, -, \cdot, 0, 1 \rangle$, but
$\langle \overline{\mathbb{Q}}, +, -, \cdot, 0, 1 \rangle \not\cong \langle \overline{\mathbb{Q}(\pi)}, +, -, \cdot, 0, 1 \rangle$
No formula, even with parameters from $\overline{\mathbb{Q}}$, can distinguish these structures.

Definition

Let $\Sigma(x_1, \ldots, x_n)$ be a set of L-formulas. If $\mathcal{M}$ is an L-structure such that there are $a_1, \ldots, a_n \in \mathcal{M}$ such that $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ for all $\phi(x_1, \ldots, x_n) \in \Sigma$, we say $\mathcal{M}$ **realizes** Σ. Otherwise $\mathcal{M}$ **omits** Σ.

Definition

Let $\Sigma(x_1, \ldots, x_n)$ be a set of L-formulas. If $\mathcal{M}$ is an L-structure such that there are $a_1, \ldots, a_n \in \mathcal{M}$ such that $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ for all $\phi(x_1, \ldots, x_n) \in \Sigma$, we say $\mathcal{M}$ **realizes** Σ. Otherwise $\mathcal{M}$ **omits** Σ.

If T is an L-theory and there is $\mathcal{M} \models T$ such that $\mathcal{M}$ realizes Σ, then we say Σ is a **type of** T.

Example

Let $\Sigma(x) = \{a_0 + a_1 x + \cdots + a_n x^n \neq 0 : n \in \omega, a_i \in \mathbb{Z}\}$.

Then π realizes Σ in $\overline{\mathbb{Q}(\pi)}$, but $\overline{\mathbb{Q}}$ omits Σ.

Example

Let $\Sigma(x) = \{a_0 + a_1 x + \cdots + a_n x^n \neq 0 : n \in \omega, a_i \in \mathbb{Z}\}$.

Then π realizes Σ in $\overline{\mathbb{Q}(\pi)}$, but $\overline{\mathbb{Q}}$ omits Σ.

Types increase expressive power.

Example

Let T be the theory of abelian groups, and

$$\Sigma(x) = \{x \neq 0, x + x \neq 0, x + x + x \neq 0, \ldots\} .$$

Then $\mathcal{M} \models T$ omits Σ if and only if $\mathcal{M}$ is a torsion group.

Example

Let T be the theory of abelian groups, and

$$\Sigma(x) = \{x \neq 0, x + x \neq 0, x + x + x \neq 0, \ldots\}.$$

Then $\mathcal{M} \models T$ omits Σ if and only if $\mathcal{M}$ is a torsion group.

Example

Let $T = PA$, and

$$\Sigma(x) = \{x \neq 0, x \neq S0, x \neq SS0, \ldots\}.$$

Then $\mathcal{M} \models T$ omits Σ if and only if $\mathcal{M}$ is standard.

Definition

A type $\Sigma(\overline{x})$ is **principal** over T if there is $\phi(\overline{x})$ such that $T \cup \{\phi(\overline{x})\} \models \Sigma(\overline{x})$.

Definition

A type $\Sigma(\overline{x})$ is **principal** over T if there is $\phi(\overline{x})$ such that $T \cup \{\phi(\overline{x})\} \models \Sigma(\overline{x})$.

Theorem (Henkin-Orey 1957)

For each n, let Σ_n be a type of T which is non-principal. Then there is a (countable) $\mathcal{M} \models T$ which omits every Σ_n.

Corollary

Prime models realize only principal types.

Corollary

Prime models realize only principal types.

Corollary

Suppose $\mathcal{M}$ *is an ordered field, and that for every* ϕ *if* $\mathcal{M} \models \exists x \phi(x)$ *then there is a finite* $a \in \mathcal{M}$ *such that* $\mathcal{M} \models \phi(a)$. *Then* $\mathcal{M}$ *is elementarily equivalent to an Archimedean field.*

Corollary

Prime models realize only principal types.

Corollary

Suppose $\mathcal{M}$ is an ordered field, and that for every ϕ if $\mathcal{M} \models \exists x \phi(x)$ then there is a finite $a \in \mathcal{M}$ such that $\mathcal{M} \models \phi(a)$. Then $\mathcal{M}$ is elementarily equivalent to an Archimedean field.

Corollary

If $\mathcal{M} \models PA$ (or $\mathcal{M} \models ZF$) is countable then there exists a countable end-extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{M} \preceq \mathcal{N}$.

Theorem (Keisler 1973)

Let L be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and let T be an L-theory. For each n, let Σ_n be a type of T which is non-principal. Then there is a (countable) $\mathcal{M} \models T$ which omits every Σ_n.

$[0, 1]$-valued Model Theory

Signatures: Function symbols, predicate symbols each have moduli of uniform continuity.

$[0, 1]$-valued Model Theory

Signatures: Function symbols, predicate symbols each have moduli of uniform continuity.

Structures: Metric spaces of diameter ≤ 1. Interpretations of symbols respect the moduli of continuity from the signature.

$[0,1]$-valued $\mathcal{L}_{\omega_1,\omega}$

Fix a signature S. The formulas of $\mathcal{L}_{\omega_1,\omega}(S)$ are:
Atomic Formulas: $d(x,y)$, $R(x_1,\dots,x_n)$, constants for each $r \in \mathbb{Q} \cap (0,1)$.

$[0,1]$-valued $\mathcal{L}_{\omega_1,\omega}$

Fix a signature S. The formulas of $\mathcal{L}_{\omega_1,\omega}(S)$ are:
Atomic Formulas: $d(x,y)$, $R(x_1,\dots,x_n)$, constants for each $r \in \mathbb{Q} \cap (0,1)$.

Formulas:

- $\phi \to \psi \quad (= \min\{1-\phi+\psi, 1\})$

$[0,1]$-valued $\mathcal{L}_{\omega_1,\omega}$

Fix a signature S. The formulas of $\mathcal{L}_{\omega_1,\omega}(S)$ are:
Atomic Formulas: $d(x,y)$, $R(x_1,\ldots,x_n)$, constants for each $r \in \mathbb{Q} \cap (0,1)$.

Formulas:

- $\phi \to \psi \quad (= \min\{1 - \phi + \psi, 1\})$
- $\neg\phi \qquad (= 1 - \phi)$

$[0,1]$-valued $\mathcal{L}_{\omega_1,\omega}$

Fix a signature S. The formulas of $\mathcal{L}_{\omega_1,\omega}(S)$ are:
Atomic Formulas: $d(x,y)$, $R(x_1,\ldots,x_n)$, constants for each $r \in \mathbb{Q} \cap (0,1)$.

Formulas:

- $\phi \to \psi \quad (= \min\{1-\phi+\psi, 1\})$
- $\neg\phi \qquad (= 1-\phi)$
- $\phi \wedge \psi \quad (= \min\{\phi,\psi\})$
- $\phi \vee \psi \quad (= \max\{\phi,\psi\})$

$[0,1]$-valued $\mathcal{L}_{\omega_1,\omega}$

Fix a signature S. The formulas of $\mathcal{L}_{\omega_1,\omega}(S)$ are:
Atomic Formulas: $d(x,y)$, $R(x_1,\ldots,x_n)$, constants for each $r \in \mathbb{Q} \cap (0,1)$.

Formulas:

- $\phi \to \psi \quad (= \min\{1-\phi+\psi, 1\})$
- $\neg\phi \qquad (= 1-\phi)$
- $\phi \wedge \psi \quad (= \min\{\phi,\psi\})$
- $\phi \vee \psi \quad (= \max\{\phi,\psi\})$
- $\sup_x \phi$

$[0,1]$-valued $\mathcal{L}_{\omega_1,\omega}$

Fix a signature S. The formulas of $\mathcal{L}_{\omega_1,\omega}(S)$ are:
Atomic Formulas: $d(x,y)$, $R(x_1,\dots,x_n)$, constants for each $r \in \mathbb{Q} \cap (0,1)$.

Formulas:

- $\phi \to \psi \quad (= \min\{1-\phi+\psi, 1\})$
- $\neg\phi \quad (= 1-\phi)$
- $\phi \wedge \psi \quad (= \min\{\phi,\psi\})$
- $\phi \vee \psi \quad (= \max\{\phi,\psi\})$
- $\sup_x \phi$
- $\sup_{n\in\omega} \phi_n$

We can recover other connectives as limits. Starting with:

$$\frac{1}{2}x = \lim_{n\to\infty} \bigvee_{i=1}^{n} \left(\frac{i}{n} \wedge \neg(x \to \frac{i}{n}) \right).$$

By Stone-Weierstrass, combinations of our connectives uniformly approximate any continuous $F : [0,1]^n \to [0,1]$.

Truth values: $\phi^{\mathcal{M}} \in [0,1]$. $\mathcal{M} \models \phi$ means $\phi^{\mathcal{M}} = 1$.

Truth values: $\phi^{\mathcal{M}} \in [0,1]$. $\mathcal{M} \models \phi$ means $\phi^{\mathcal{M}} = 1$.

Example

In the signature of Banach algebras, let $\phi(x)$ be the formula $1 - \|x \cdot x\|$. Then $\mathcal{M} \models \phi(a)$ if and only if $a^2 = 0$ in $\mathcal{M}$.

Truth values: $\phi^{\mathcal{M}} \in [0,1]$. $\mathcal{M} \models \phi$ means $\phi^{\mathcal{M}} = 1$.

Example

In the signature of Banach algebras, let $\phi(x)$ be the formula $1 - \|x \cdot x\|$. Then $\mathcal{M} \models \phi(a)$ if and only if $a^2 = 0$ in $\mathcal{M}$.

Note: $\mathcal{M} \models \phi \to \psi$ if and only if $\phi^{\mathcal{M}} \leq \psi^{\mathcal{M}}$.

Examples of classes of Banach spaces axiomatizable in $\mathcal{L}_{\omega_1,\omega}$:

- Classes axiomatizable in finitary continuous logic ($L^p(\mu)$ spaces, $C(K)$ spaces, some classes of Nakano spaces, ...),

Examples of classes of Banach spaces axiomatizable in $\mathcal{L}_{\omega_1,\omega}$:

- Classes axiomatizable in finitary continuous logic ($L^p(\mu)$ spaces, $C(K)$ spaces, some classes of Nakano spaces, ...),
- uniformly convex spaces,

Examples of classes of Banach spaces axiomatizable in $\mathcal{L}_{\omega_1,\omega}$:

- Classes axiomatizable in finitary continuous logic ($L^p(\mu)$ spaces, $C(K)$ spaces, some classes of Nakano spaces, ...),
- uniformly convex spaces,
- spaces which are not super-reflexive,

Examples of classes of Banach spaces axiomatizable in $\mathcal{L}_{\omega_1,\omega}$:

- Classes axiomatizable in finitary continuous logic ($L^p(\mu)$ spaces, $C(K)$ spaces, some classes of Nakano spaces, ...),
- uniformly convex spaces,
- spaces which are not super-reflexive,
- spaces which are not hereditarily indecomposible,

Examples of classes of Banach spaces axiomatizable in $\mathcal{L}_{\omega_1,\omega}$:

- Classes axiomatizable in finitary continuous logic ($L^p(\mu)$ spaces, $C(K)$ spaces, some classes of Nakano spaces, ...),
- uniformly convex spaces,
- spaces which are not super-reflexive,
- spaces which are not hereditarily indecomposible,
- spaces which are unstable (in the sense of Krivine-Maurey).

A **fragment** of $\mathcal{L}_{\omega_1,\omega}(S)$ is a set of formulas L such that:

- L contains every atomic formula
- L is closed under $\rightarrow, \neg, \wedge, \vee, \sup_x$
- L is closed under substituting terms for free variables
- L is closed under subformulas

A **fragment** of $\mathcal{L}_{\omega_1,\omega}(S)$ is a set of formulas L such that:

- L contains every atomic formula
- L is closed under $\rightarrow, \neg, \wedge, \vee, \sup_x$
- L is closed under substituting terms for free variables
- L is closed under subformulas

From now on, L is a fixed countable fragment of $\mathcal{L}_{\omega_1,\omega}(S)$, S has no function symbols. For C a set of new constant symbols, L_C is the least fragment of $\mathcal{L}_{\omega_1,\omega}(S \cup C)$ containing L.

Definition

Let T be an L-theory. A type $\Sigma(\overline{x})$ is **principal** over T if there is a formula $\phi(\overline{x})$ consistent with T such that for some $r \in \mathbb{Q} \cap (0,1)$, $T \cup \{\phi(\overline{x}) \geq r\} \models \Sigma(\overline{x})$.

Definition

Let T be an L-theory. A type $\Sigma(\overline{x})$ is **principal** over T if there is a formula $\phi(\overline{x})$ consistent with T such that for some $r \in \mathbb{Q} \cap (0, 1)$, $T \cup \{\phi(\overline{x}) \geq r\} \models \Sigma(\overline{x})$.

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}_n) : n \in \omega\}$ be a collection of non-principal types of T. Then there is a (countable) $\mathcal{M} \models T$ omitting each Σ_n.

Topology

$Str_L(S)$ is the class of all S-structures.

Topology

$Str_L(S)$ is the class of all S-structures.

Closed classes: $Mod(T) = \{\mathcal{M} \in Str(S) : \mathcal{M} \models T\}$, T is an L-theory.

Topology

$Str_L(S)$ is the class of all S-structures.

Closed classes: $Mod(T) = \{\mathcal{M} \in Str(S) : \mathcal{M} \models T\}$, T is an L-theory.

Note $\mathcal{M}, \mathcal{N}$ are indistinguishable if and only if $\mathcal{M} \equiv_L \mathcal{N}$, so the space is not T_0.

Proposition

The space $Str_L(S)$ is completely regular.

Proposition

The space $Str_L(S)$ is completely regular.

Proof.

For any ϕ, ψ, $\mathcal{M} \models \phi \to \psi$ if and only if $\phi^M \leq \psi^M$.

Proposition

The space $Str_L(S)$ is completely regular.

Proof.

For any ϕ, ψ, $\mathcal{M} \models \phi \to \psi$ if and only if $\phi^M \leq \psi^M$.

Say $\mathcal{M} \notin Mod(T)$, so some ϕ is such that $\phi^M < 1$.

Proposition

The space $Str_L(S)$ is completely regular.

Proof.

For any ϕ, ψ, $\mathcal{M} \models \phi \to \psi$ if and only if $\phi^M \leq \psi^M$.

Say $\mathcal{M} \notin Mod(T)$, so some ϕ is such that $\phi^M < 1$.

Note $\phi^{-1}([a, b]) = \{\mathcal{N} : a \leq \phi^{\mathcal{N}} \leq b\} = Mod(a \leq \phi \wedge \phi \leq b)$ is closed, so ϕ is continuous. □

Proposition

$\Sigma(\overline{x})$ is principal if and only if $Mod_{L_{\overline{x}}}(T \cup \Sigma(\overline{x}))$ has nonempty interior in $Mod_{L_{\overline{x}}}(T)$.

Proposition

$\Sigma(\overline{x})$ is principal if and only if $Mod_{L_{\overline{x}}}(T \cup \Sigma(\overline{x}))$ has nonempty interior in $Mod_{L_{\overline{x}}}(T)$.

Proof.

If $\Sigma(\overline{x})$ is principal, there is ϕ, r such that $T \cup \{\phi(\overline{x}) \geq r\} \models \Sigma(\overline{x})$.

Proposition

$\Sigma(\overline{x})$ is principal if and only if $Mod_{L_{\overline{x}}}(T \cup \Sigma(\overline{x}))$ has nonempty interior in $Mod_{L_{\overline{x}}}(T)$.

Proof.

If $\Sigma(\overline{x})$ is principal, there is ϕ, r such that $T \cup \{\phi(\overline{x}) \geq r\} \models \Sigma(\overline{x})$. Then for any $r' \in (r, 1)$ we have

$$\underset{L_{\overline{x}}}{Mod}(T) \cap \underset{L_{\overline{x}}}{Mod}(\phi(\overline{x}) > r')$$

is a non-empty open subset of $Mod_{L_{\overline{x}}}(T \cup \Sigma(\overline{x}))$.

Proof (con't).

Conversely, there is $\phi(\overline{x})$ such that

$$\underset{L_{\overline{x}}}{Mod}(T) \cap \underset{L_{\overline{x}}}{Mod}(\phi(\overline{x}) > 0)$$

is a non-empty open subset of $Mod_{L_{\overline{x}}}(T \cup \Sigma(x))$.

Proof (con't).

Conversely, there is $\phi(\overline{x})$ such that

$$\underset{L_{\overline{x}}}{Mod}(T) \cap \underset{L_{\overline{x}}}{Mod}(\phi(\overline{x}) > 0)$$

is a non-empty open subset of $Mod_{L_{\overline{x}}}(T \cup \Sigma(x))$.

So there is $r \in \mathbb{Q} \cap (0,1)$ such that $T \cup \{\phi(\overline{x}) \geq r\}$ is satisfiable.

Proof (con't).

Conversely, there is $\phi(\overline{x})$ such that

$$\underset{L_{\overline{x}}}{Mod}(T) \cap \underset{L_{\overline{x}}}{Mod}(\phi(\overline{x}) > 0)$$

is a non-empty open subset of $Mod_{L_{\overline{x}}}(T \cup \Sigma(x))$.

So there is $r \in \mathbb{Q} \cap (0,1)$ such that $T \cup \{\phi(\overline{x}) \geq r\}$ is satisfiable.

Let $\psi(\overline{x})$ be the formula $\phi(\overline{x}) \geq r$. Then ψ, $1 - s$ witness principality for any $s \in \mathbb{Q} \cap (0, r)$. □

We add countably many new constants $C = \{c_n : n \in \omega\}$ to S. We work in $\mathcal{W} \subseteq Str(L_C)$ where satisfaction of $\sup_x \phi$ is witnessed in the constant symbols.

We add countably many new constants $C = \{c_n : n \in \omega\}$ to S.
We work in $\mathcal{W} \subseteq Str(L_C)$ where satisfaction of $\sup_x \phi$ is witnessed in the constant symbols.

Lemma

For any $\mathbf{i} = i_0, \ldots, i_{n-1}$*, the natural map* $R_{\mathbf{i}} : \mathcal{W} \cap Mod(T) \to Mod_{L_{c_{\mathbf{i}}}}(T)$ *is open, continuous, and surjective.*

We add countably many new constants $C = \{c_n : n \in \omega\}$ to S. We work in $\mathcal{W} \subseteq Str(L_C)$ where satisfaction of $\sup_x \phi$ is witnessed in the constant symbols.

Lemma

For any $\mathbf{i} = i_0, \ldots, i_{n-1}$*, the natural map* $R_{\mathbf{i}} : \mathcal{W} \cap Mod(T) \to Mod_{L_{c_{\mathbf{i}}}}(T)$ *is open, continuous, and surjective.*

Proposition

Let $\Sigma(x_0, \ldots, x_{n-1})$ *be a nonprincipal type of* T*. Then for each* $\mathbf{i} \in \omega^n$, $R_{\mathbf{i}}^{-1}(Mod(T \cup \Sigma(c_{\mathbf{i}}))$ *is closed nowhere dense in* $\mathcal{W} \cap Mod(T)$.

Suppose there is
$\langle \mathcal{M}, \overline{a} \rangle \in \mathcal{W} \cap Mod(T) \setminus \bigcup_{n \in \omega} \bigcup_{\mathbf{i} \in \omega^n} R_{\mathbf{i}}^{-1}(T \cup \Sigma_n(c_{\mathbf{i}}))$.

Suppose there is
$\langle \mathcal{M}, \overline{a} \rangle \in \mathcal{W} \cap Mod(T) \setminus \bigcup_{n\in\omega} \bigcup_{\mathbf{i}\in\omega^n} R_{\mathbf{i}}^{-1}(T \cup \Sigma_n(c_{\mathbf{i}}))$.

Since $\langle \mathcal{M}, \overline{a} \rangle \in \mathcal{W}$, $\overline{a} \preceq \mathcal{M}$, and our choice ensures no subset of $\overline{a}$ realizes any Σ_n, so $\overline{a} \models T$ and omits each Σ_n.

Suppose there is
$\langle \mathcal{M}, \overline{a} \rangle \in \mathcal{W} \cap Mod(T) \setminus \bigcup_{n \in \omega} \bigcup_{\mathbf{i} \in \omega^n} R_{\mathbf{i}}^{-1}(T \cup \Sigma_n(c_{\mathbf{i}}))$.

Since $\langle \mathcal{M}, \overline{a} \rangle \in \mathcal{W}$, $\overline{a} \preceq \mathcal{M}$, and our choice ensures no subset of $\overline{a}$ realizes any Σ_n, so $\overline{a} \models T$ and omits each Σ_n.

It therefore suffices to show that $\mathcal{W} \cap Mod(T)$ is **Baire**, i.e., the countable union of closed nowhere dense sets is codense.

Definition

Let X be a completely regular space. A **complete sequence of open covers** of X is a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers such that if $\mathcal{F}$ is a centred family of closed sets such that for each $n \in \omega$ there is $F_n \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ such that $F_n \subseteq U_n$, then $\bigcap \mathcal{F} \neq \emptyset$.

Definition

Let X be a completely regular space. A **complete sequence of open covers** of X is a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers such that if $\mathcal{F}$ is a centred family of closed sets such that for each $n \in \omega$ there is $F_n \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ such that $F_n \subseteq U_n$, then $\bigcap \mathcal{F} \neq \emptyset$.

Definition

A completely regular space X is **Čech-complete** if it has a complete sequence of open covers.

Fact

If X is $T_{3\frac{1}{2}}$ then X is Čech-complete if and only if X is a G_δ in βX.

Fact

If X is $T_{3\frac{1}{2}}$ then X is Čech-complete if and only if X is a G_δ in βX.

Fact

If X is completely metrizable or locally compact Hausdorff, then X is Čech-complete.

Proposition

Let X be completely regular and Čech-complete. Then:

- *X is Baire.*
- *Every closed subspace of X is Čech-complete.*

Theorem

$\mathcal{W}$ is Čech-complete.

Proof: Recursively build a sequence of open covers such that:

- The sets in $\mathcal{U}_n$ prescribe ranges for the first n sentences.

Proof: Recursively build a sequence of open covers such that:

- The sets in $\mathcal{U}_n$ prescribe ranges for the first n sentences.
  - i.e., if $U \in \mathcal{U}_n$ then for each $m \leq n$ there is $I_m \subseteq [0,1]$ such that for all $\mathcal{M} \in U$, $\sigma_m^{\mathcal{M}} \in I_m$.

Proof: Recursively build a sequence of open covers such that:

- The sets in $\mathcal{U}_n$ prescribe ranges for the first n sentences.
  - i.e., if $U \in \mathcal{U}_n$ then for each $m \leq n$ there is $I_m \subseteq [0, 1]$ such that for all $\mathcal{M} \in U$, $\sigma_m^{\mathcal{M}} \in I_m$.
- The sets in $\mathcal{U}_{n+1}$ prescribe ranges for the first n sentences that refine the ranges from sets in $\mathcal{U}_n$.

Proof: Recursively build a sequence of open covers such that:

- The sets in $\mathcal{U}_n$ prescribe ranges for the first n sentences.
    - i.e., if $U \in \mathcal{U}_n$ then for each $m \leq n$ there is $I_m \subseteq [0, 1]$ such that for all $\mathcal{M} \in U$, $\sigma_m^{\mathcal{M}} \in I_m$.
- The sets in $\mathcal{U}_{n+1}$ prescribe ranges for the first n sentences that refine the ranges from sets in $\mathcal{U}_n$.
- When a set prescribes a range for $\sup_n \phi_n$ it also picks an n and specifies that ϕ_n be in the same range.

Proof: Recursively build a sequence of open covers such that:

- The sets in $\mathcal{U}_n$ prescribe ranges for the first n sentences.
  - i.e., if $U \in \mathcal{U}_n$ then for each $m \leq n$ there is $I_m \subseteq [0,1]$ such that for all $\mathcal{M} \in U$, $\sigma_m^{\mathcal{M}} \in I_m$.
- The sets in $\mathcal{U}_{n+1}$ prescribe ranges for the first n sentences that refine the ranges from sets in $\mathcal{U}_n$.
- When a set prescribes a range for $\sup_n \phi_n$ it also picks an n and specifies that ϕ_n be in the same range.
- When a set prescribes a range for $\sup_x \phi$ it also picks an n and specifies that $\phi(c_n)$ be in the same range.

Let $\mathcal{F}$ be a centred family of closed sets. For each n, pick $F_n \in \mathcal{F}$, $U_n \in \mathcal{U}$ such that $F_n \subseteq U_n$. For $F \in \mathcal{F}$, let T_F be a theory such that $F = Mod(T_F)$.

Let $\mathcal{F}$ be a centred family of closed sets. For each n, pick $F_n \in \mathcal{F}$, $U_n \in \mathcal{U}$ such that $F_n \subseteq U_n$. For $F \in \mathcal{F}$, let T_F be a theory such that $F = Mod(T_F)$.

Lemma

$\bigcap \mathcal{F} = \bigcap_{n \in \omega} F_n$.

Let $\mathcal{F}$ be a centred family of closed sets. For each n, pick $F_n \in \mathcal{F}$, $U_n \in \mathcal{U}$ such that $F_n \subseteq U_n$. For $F \in \mathcal{F}$, let T_F be a theory such that $F = Mod(T_F)$.

Lemma

$\bigcap \mathcal{F} = \bigcap_{n \in \omega} F_n$.

Proof.

If $\mathcal{M} \notin \bigcap \mathcal{F}$ then there is some $F \in \mathcal{F}$ and some $\sigma \in T_F$ such that $\mathcal{M} \not\models \sigma$, i.e., $\sigma^{\mathcal{M}} < 1$.

Let $\mathcal{F}$ be a centred family of closed sets. For each n, pick $F_n \in \mathcal{F}$, $U_n \in \mathcal{U}$ such that $F_n \subseteq U_n$. For $F \in \mathcal{F}$, let T_F be a theory such that $F = Mod(T_F)$.

Lemma

$\bigcap \mathcal{F} = \bigcap_{n \in \omega} F_n$.

Proof.

If $\mathcal{M} \notin \bigcap \mathcal{F}$ then there is some $F \in \mathcal{F}$ and some $\sigma \in T_F$ such that $\mathcal{M} \not\models \sigma$, i.e., $\sigma^{\mathcal{M}} < 1$.

If $\mathcal{M} \in \bigcap_{n \in \omega} F_n$ then there is $n \in \omega$ large enough for U_n to specify that $\sigma < 1$. So $F_n \models \sigma < 1$.

Let $\mathcal{F}$ be a centred family of closed sets. For each n, pick $F_n \in \mathcal{F}$, $U_n \in \mathcal{U}$ such that $F_n \subseteq U_n$. For $F \in \mathcal{F}$, let T_F be a theory such that $F = Mod(T_F)$.

Lemma

$\bigcap \mathcal{F} = \bigcap_{n \in \omega} F_n$.

Proof.

If $\mathcal{M} \notin \bigcap \mathcal{F}$ then there is some $F \in \mathcal{F}$ and some $\sigma \in T_F$ such that $\mathcal{M} \not\models \sigma$, i.e., $\sigma^{\mathcal{M}} < 1$.

If $\mathcal{M} \in \bigcap_{n \in \omega} F_n$ then there is $n \in \omega$ large enough for U_n to specify that $\sigma < 1$. So $F_n \models \sigma < 1$.

So $F \cap F_n = \emptyset$, contradiction. □

For each n, pick $\mathcal{M}_n \in F_0 \cap \cdots \cap F_n$. Pick an ultrafilter $\mathcal{D}$ on ω. Let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_n$.

For each n, pick $\mathcal{M}_n \in F_0 \cap \cdots \cap F_n$. Pick an ultrafilter $\mathcal{D}$ on ω. Let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_n$.

Lemma

For each L_C-sentence σ, $\sigma^{\mathcal{M}} = \lim_{n \to \mathcal{D}} \sigma^{\mathcal{M}_n}$.

Proof.

By induction on the complexity of σ. Clear from the definition for σ atomic, and continuous connectives by definition of limits.

For each n, pick $\mathcal{M}_n \in F_0 \cap \cdots \cap F_n$. Pick an ultrafilter $\mathcal{D}$ on ω. Let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_n$.

Lemma

For each L_C-sentence σ, $\sigma^{\mathcal{M}} = \lim_{n \to \mathcal{D}} \sigma^{\mathcal{M}_n}$.

Proof.

By induction on the complexity of σ. Clear from the definition for σ atomic, and continuous connectives by definition of limits.

Say $\sigma = \sup_{n \in \omega} \theta_n$.

For each n, pick $\mathcal{M}_n \in F_0 \cap \cdots \cap F_n$. Pick an ultrafilter $\mathcal{D}$ on ω. Let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_n$.

Lemma

For each L_C-sentence σ, $\sigma^{\mathcal{M}} = \lim_{n \to \mathcal{D}} \sigma^{\mathcal{M}_n}$.

Proof.

By induction on the complexity of σ. Clear from the definition for σ atomic, and continuous connectives by definition of limits.

Say $\sigma = \sup_{n \in \omega} \theta_n$.

If $a < \sigma^{\mathcal{M}}$ then there is $m^* \in \omega$, $a < \theta_{m^*}^{\mathcal{M}}$.

For each n, pick $\mathcal{M}_n \in F_0 \cap \cdots \cap F_n$. Pick an ultrafilter $\mathcal{D}$ on ω. Let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_n$.

Lemma

For each L_C-sentence σ, $\sigma^{\mathcal{M}} = \lim_{n \to \mathcal{D}} \sigma^{\mathcal{M}_n}$.

Proof.

By induction on the complexity of σ. Clear from the definition for σ atomic, and continuous connectives by definition of limits.

Say $\sigma = \sup_{n \in \omega} \theta_n$.

If $a < \sigma^{\mathcal{M}}$ then there is $m^* \in \omega$, $a < \theta_{m^*}^{\mathcal{M}}$. By induction hypothesis, $\{n \in \omega : \theta_{m^*} > a\} \in \mathcal{D}$.

For each n, pick $\mathcal{M}_n \in F_0 \cap \cdots \cap F_n$. Pick an ultrafilter $\mathcal{D}$ on ω. Let $\mathcal{M} = \prod_{\mathcal{D}} \mathcal{M}_n$.

Lemma

For each L_C-sentence σ, $\sigma^{\mathcal{M}} = \lim_{n \to \mathcal{D}} \sigma^{\mathcal{M}_n}$.

Proof.

By induction on the complexity of σ. Clear from the definition for σ atomic, and continuous connectives by definition of limits.

Say $\sigma = \sup_{n \in \omega} \theta_n$.

If $a < \sigma^{\mathcal{M}}$ then there is $m^* \in \omega$, $a < \theta_{m^*}^{\mathcal{M}}$. By induction hypothesis, $\{n \in \omega : \theta_{m^*} > a\} \in \mathcal{D}$. Hence

$$\{n \in \omega : \sigma^{\mathcal{M}_n} > a\} \in \mathcal{D}.$$

If $\left\{n \in \omega : \sigma^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$, by induction it suffices to find $m^* \in \omega$ such that $\left\{n \in \omega : \theta_{m^*}^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$.

If $\left\{n \in \omega : \sigma^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$, by induction it suffices to find $m^* \in \omega$ such that $\left\{n \in \omega : \theta_{m^*}^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$.

Find k large enough to specify that $\sigma > a$. Then there is m^* such that k also specifies $\theta_{m^*} > a$.

If $\left\{n \in \omega : \sigma^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$, by induction it suffices to find $m^* \in \omega$ such that $\left\{n \in \omega : \theta_{m^*}^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$.

Find k large enough to specify that $\sigma > a$. Then there is m^* such that k also specifies $\theta_{m^*} > a$.

So cofinitely many $\mathcal{M}_n \models \theta_{m^*} > a$, so $\left\{n \in \omega : \theta_{m^*}^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$.

If $\left\{n \in \omega : \sigma^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$, by induction it suffices to find $m^* \in \omega$ such that $\left\{n \in \omega : \theta_{m^*}^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$.

Find k large enough to specify that $\sigma > a$. Then there is m^* such that k also specifies $\theta_{m^*} > a$.

So cofinitely many $\mathcal{M}_n \models \theta_{m^*} > a$, so $\left\{n \in \omega : \theta_{m^*}^{\mathcal{M}_n} > a\right\} \in \mathcal{D}$.

Similar proof for $\sigma = \sup_x \phi$. □

It follows that $\mathcal{M} \in \bigcap_{n\in\omega} F_n = \bigcap \mathcal{F} \neq \emptyset$. So $\mathcal{W}$ is Čech-complete.

$\square$

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of non-principal types of T. Then there is $\mathcal{M} \models T$ omitting each Σ_n.

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of non-principal types of T. Then there is $\mathcal{M} \models T$ omitting each Σ_n.

What if we wanted $\mathcal{M}$ to be based on a *complete* metric space?

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of non-principal types of T. Then there is $\mathcal{M} \models T$ omitting each Σ_n.

What if we wanted $\mathcal{M}$ to be based on a *complete* metric space?

Example

If $(c_i)_{i\in\omega}$ are constant symbols and T is a theory which implies that (c_i) is a Cauchy sequence, then the type $\Sigma(x)$ expressing that $x = \lim_{i\to\infty} c_i$ can be omitted in a metric structure, but not in a complete metric structure.

Definition

For $\Sigma(x_1, \dots, x_n)$ a type, and $\delta \in \mathbb{Q} \cap (0,1)$,

$$\Sigma^\delta = \left\{ \sup_{y_1} \dots \sup_{y_n} \left(\bigwedge_{k \leq n} d(x_k, y_k) \leq \delta \wedge \sigma(y_1, \dots, y_n) \right) : \sigma \in \Sigma \right\}.$$

Definition

For $\Sigma(x_1, \ldots, x_n)$ a type, and $\delta \in \mathbb{Q} \cap (0,1)$,

$$\Sigma^\delta = \left\{ \sup_{y_1} \ldots \sup_{y_n} \left(\bigwedge_{k \leq n} d(x_k, y_k) \leq \delta \wedge \sigma(y_1, \ldots, y_n) \right) : \sigma \in \Sigma \right\}.$$

Definition

A type $\Sigma(\overline{x})$ of T is **metrically principal** over T if for every $\delta > 0$ the type $\Sigma^\delta(\overline{x})$ is principal over T.

Definition

For $\Sigma(x_1, \dots, x_n)$ a type, and $\delta \in \mathbb{Q} \cap (0,1)$,

$$\Sigma^\delta = \left\{ \sup_{y_1} \dots \sup_{y_n} \left(\bigwedge_{k \le n} d(x_k, y_k) \le \delta \wedge \sigma(y_1, \dots, y_n) \right) : \sigma \in \Sigma \right\}.$$

Definition

A type $\Sigma(\overline{x})$ of T is **metrically principal** over T if for every $\delta > 0$ the type $\Sigma^\delta(\overline{x})$ is principal over T.

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is $\mathcal{M} \models T$ such that $\overline{\mathcal{M}}$ omits each Σ_n.

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is $\mathcal{M} \models T$ such that $\overline{\mathcal{M}}$ omits each Σ_n.

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is $\mathcal{M} \models T$ such that $\overline{\mathcal{M}}$ omits each Σ_n.

But is $\mathcal{M} \equiv_L \overline{\mathcal{M}}$?

Theorem

Let T be an L-theory, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is $\mathcal{M} \models T$ such that $\overline{\mathcal{M}}$ omits each Σ_n.

But is $\mathcal{M} \equiv_L \overline{\mathcal{M}}$?

Definition

A fragment L of $\mathcal{L}_{\omega_1,\omega}$ is **continuous** if every formula $\phi(x_1, \ldots, x_n) \in L$ defines a *continuous* function $\phi : \mathcal{M}^n \to [0,1]$ for every L-structure $\mathcal{M}$.

Theorem

Let T be an L-theory, where L is a countable continuous fragment, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is a complete $\mathcal{M} \models T$ omitting each Σ_n.

Theorem

Let T be an L-theory, where L is a countable continuous fragment, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is a complete $\mathcal{M} \models T$ omitting each Σ_n.

- If L is first-order continuous logic, this is Henson's Omitting Types Theorem.

Theorem

Let T be an L-theory, where L is a countable continuous fragment, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is a complete $\mathcal{M} \models T$ omitting each Σ_n.

- If L is first-order continuous logic, this is Henson's Omitting Types Theorem.
- Restricted to discrete structures, this is Keisler's Omitting Types Theorem.

Theorem

Let T be an L-theory, where L is a countable continuous fragment, and let $\{\Sigma_n(\overline{x}) : n \in \omega\}$ be a collection of types of T which are not metrically principal. Then there is a complete $\mathcal{M} \models T$ omitting each Σ_n.

- If L is first-order continuous logic, this is Henson's Omitting Types Theorem.
- Restricted to discrete structures, this is Keisler's Omitting Types Theorem.
- If L is first-order and we restrict to discrete structures, this is the classical Omitting Types Theorem.

Application - Separable Quotients

Question

Let X be a Banach space. Is there a subspace $Y \subseteq X$ such that X/Y is separable?

Application - Separable Quotients

Question

Let X be a Banach space. Is there a subspace $Y \subseteq X$ such that X/Y is separable?

Equivalently,

Question

Let X be a non-separable Banach space. Is there a separable Banach space Y and a surjective bounded linear operator $T : X \to Y$?

Theorem

Let X, Y be Banach spaces with density$(X) >$ density(Y), and let $T : X \to Y$ be a surjective bounded linear map.

Theorem

Let X, Y *be Banach spaces with density*$(X) >$ *density*(Y)*, and let* $T : X \to Y$ *be a surjective bounded linear map. Let* L *be a continuous countable fragment of* $\mathcal{L}_{\omega_1,\omega}$*. Then there are* X', Y', T' *such that:*

Theorem

Let X, Y be Banach spaces with density$(X) >$ density(Y), and let $T : X \to Y$ be a surjective bounded linear map. Let L be a continuous countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Then there are X', Y', T' such that:

- *X', Y' are Banach spaces, $T' : X' \to Y'$ is a surjective bounded linear map,*

Theorem

Let X, Y be Banach spaces with density$(X) >$ density(Y), and let $T : X \to Y$ be a surjective bounded linear map. Let L be a continuous countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Then there are X', Y', T' such that:

- *X', Y' are Banach spaces, $T' : X' \to Y'$ is a surjective bounded linear map,*
- *X' has density $\aleph_1$,*

Theorem

Let X, Y be Banach spaces with density$(X) >$ density(Y), and let $T : X \to Y$ be a surjective bounded linear map. Let L be a continuous countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Then there are X', Y', T' such that:

- *X', Y' are Banach spaces, $T' : X' \to Y'$ is a surjective bounded linear map,*
- *X' has density $\aleph_1$,*
- *Y' is separable,*

Theorem

Let X, Y be Banach spaces with $density(X) > density(Y)$, and let $T : X \to Y$ be a surjective bounded linear map. Let L be a continuous countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Then there are X', Y', T' such that:

- *X', Y' are Banach spaces, $T' : X' \to Y'$ is a surjective bounded linear map,*
- *X' has density $\aleph_1$,*
- *Y' is separable,*
- *$(X, Y, T) \equiv_L (X', Y', T')$.*

Proof (Outline).

By Downward Löwenheim-Skolem, we may assume $|X| = \kappa^+$, $|Y| = \kappa$.

Proof (Outline).

By Downward Löwenheim-Skolem, we may assume $|X| = \kappa^+$, $|Y| = \kappa$.

Attach a (discrete) copy of the order $(\kappa^+, \leq)$, a constant for κ, and a function f so that $f|\kappa$ enumerates Y and $f|[\kappa, \kappa^+)$ enumerates X, to (X, Y, T) to form $\mathcal{M}$.

Proof (Outline).

By Downward Löwenheim-Skolem, we may assume $|X| = \kappa^+$, $|Y| = \kappa$.

Attach a (discrete) copy of the order $(\kappa^+, \leq)$, a constant for κ, and a function f so that $f|\kappa$ enumerates Y and $f|[\kappa, \kappa^+)$ enumerates X, to (X, Y, T) to form $\mathcal{M}$.

Using Downward Löwenheim-Skolem, let $\langle X_0, Y_0, T_0, L_0, c, \trianglelefteq_0, f_0 \rangle = \mathcal{M}_0 \preceq \mathcal{M}$ be countable.

Add countably many constant symbols $(d_l)_{l \in L}$, and another constant d^*.

Add countably many constant symbols $(d_l)_{l \in L}$, and another constant d^*.

Let $T = diag_{el}(\mathcal{M}_0) \cup \{d_l \trianglelefteq_0 d^* : l \in L_0\}$.

Add countably many constant symbols $(d_l)_{l \in L}$, and another constant d^*.

Let $T = diag_{el}(\mathcal{M}_0) \cup \{d_l \trianglelefteq_0 d^* : l \in L_0\}$.

Define $\Sigma(x) = \{x \in L\} \cup \{f_0(x) \in Y_0\} \cup \{d(x, d_l) = 1 : l \trianglelefteq_0 c\}$.

Add countably many constant symbols $(d_l)_{l \in L}$, and another constant d^*.

Let $T = diag_{el}(\mathcal{M}_0) \cup \{d_l \trianglelefteq_0 d^* : l \in L_0\}$.

Define $\Sigma(x) = \{x \in L\} \cup \{f_0(x) \in Y_0\} \cup \{d(x, d_l) = 1 : l \trianglelefteq_0 c\}$.

Then Σ is non-principal over T.

By Omitting Types there is a countable $\mathcal{M}_1 \models T$ which omits Σ.

By Omitting Types there is a countable $\mathcal{M}_1 \models T$ which omits Σ.

Then $\mathcal{M}_0 \preceq \mathcal{M}_1$, and $\mathcal{M}_1$ has no new elements of L above c. So Y_0 is dense in Y_1.

By Omitting Types there is a countable $\mathcal{M}_1 \models T$ which omits Σ.

Then $\mathcal{M}_0 \preceq \mathcal{M}_1$, and $\mathcal{M}_1$ has no new elements of L above c. So Y_0 is dense in Y_1.

Repeat ω_1 times, and take the union of the elementary chain to get $\mathcal{M}_{\omega_1} = \langle X_{\omega_1}, Y_{\omega_1}, T_{\omega_1}, \ldots \rangle$.

By Omitting Types there is a countable $\mathcal{M}_1 \models T$ which omits Σ.

Then $\mathcal{M}_0 \preceq \mathcal{M}_1$, and $\mathcal{M}_1$ has no new elements of L above c. So Y_0 is dense in Y_1.

Repeat ω_1 times, and take the union of the elementary chain to get $\mathcal{M}_{\omega_1} = \langle X_{\omega_1}, Y_{\omega_1}, T_{\omega_1}, \ldots \rangle$.

Take $\langle X', Y', T' \rangle$ to be the completion of $\langle X_{\omega_1}, Y_{\omega_1}, T_{\omega_1} \rangle$

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where $density(X) > density(Y)$. Then there are X', Y' and $T' : X' \to Y'$, with $density(X') = \aleph_1$ and Y' separable, such that:

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where $\mathit{density}(X) > \mathit{density}(Y)$. Then there are X', Y' and $T' : X' \to Y'$, with $\mathit{density}(X') = \aleph_1$ and Y' separable, such that:

- X and X' have isometric ultrapowers,

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where $density(X) > density(Y)$. Then there are X', Y' and $T' : X' \to Y'$, with $density(X') = \aleph_1$ and Y' separable, such that:

- X and X' have isometric ultrapowers,
- Y and Y' have isometric ultrapowers,

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where $\mathit{density}(X) > \mathit{density}(Y)$. Then there are X', Y' and $T' : X' \to Y'$, with $\mathit{density}(X') = \aleph_1$ and Y' separable, such that:

- X and X' have isometric ultrapowers,
- Y and Y' have isometric ultrapowers,
- if X is uniformly convex then so is X',

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where $\mathit{density}(X) > \mathit{density}(Y)$. Then there are X', Y' and $T' : X' \to Y'$, with $\mathit{density}(X') = \aleph_1$ and Y' separable, such that:

- X and X' have isometric ultrapowers,
- Y and Y' have isometric ultrapowers,
- if X is uniformly convex then so is X',
- if X is not hereditarily indecomposible then neither is X',

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where *density*$(X) >$ *density*(Y). Then there are X', Y' and $T' : X' \to Y'$, with *density*$(X') = \aleph_1$ and Y' separable, such that:

- X and X' have isometric ultrapowers,
- Y and Y' have isometric ultrapowers,
- if X is uniformly convex then so is X',
- if X is not hereditarily indecomposible then neither is X',
- if X is not super-reflexive then neither is X',

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where $\mathit{density}(X) > \mathit{density}(Y)$. Then there are X', Y' and $T' : X' \to Y'$, with $\mathit{density}(X') = \aleph_1$ and Y' separable, such that:

- X and X' have isometric ultrapowers,
- Y and Y' have isometric ultrapowers,
- if X is uniformly convex then so is X',
- if X is not hereditarily indecomposible then neither is X',
- if X is not super-reflexive then neither is X',
- all the above, with Y instead of X,

Example

Suppose $T : X \to Y$ is bounded, linear, surjective, where *density*$(X) >$ *density*(Y). Then there are X', Y' and $T' : X' \to Y'$, with *density*$(X') = \aleph_1$ and Y' separable, such that:

- X and X' have isometric ultrapowers,
- Y and Y' have isometric ultrapowers,
- if X is uniformly convex then so is X',
- if X is not hereditarily indecomposible then neither is X',
- if X is not super-reflexive then neither is X',
- all the above, with Y instead of X,
- $\vdots$

Thank you!