APPLICATIONS OF LOGIC TO OPERATOR ALGEBRAS

KEVIN CARLSON, ENOCH CHEUNG, ALEXANDER GERHARDT-BOURKE, LEANNE MEZUMAN, AND ALEXANDER SHERMAN

Abstract. This work is the report produced by students participating in the Fields-Mitacs Undergraduate Summer Research Program 2012. Under the supervision of Bradd Hart and Ilijas Farah, we aimed to explore interactions between mathematical logic and operator algebras. This report introduces the concepts important to some interesting ideas that we studied. The paper finishes with using model theory to study the structure of some specific C^*-algebras, in particular, we characterise both UHF and AF algebras.

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1. Continuous Model Theory

1.1. Metric Structures.

We begin the paper with a brief introduction to model theory of metric structures, a relatively new development in the field. For a more in depth introduction, we direct the reader to I. Ben Yaacov et al. We begin with the definition of metric space and then go into metric structures in particular.

Definition 1.1. An ordered pair, (M, d), is called a *metric space* if M is a collection of elements and $d : M \times M \to \mathbb{R}$ is a metric on M, i.e. a function satisfying the following axioms:

(1) $d(x,y) \geq 0 \quad \forall x, y \in M$, and in particular $d(x,y) = 0 \iff x = y$
(2) $d(x,y) = d(y,x) \quad \forall x, y \in M$, and
(3) $d(x,y) = d(x,z) + d(z,y) \quad \forall x, y, z \in M$

We say (M, d) is complete if every Cauchy sequence in M under the metric d converges in M. We now define the different categories of functions that will be used repeatedly throughout this paper.

Definition 1.2. Let (M, d) be a complete, bounded metric space. Then we have the following types of functions on M:

(1) A *predicate* $P : M^n \to \mathbb{R}$ is a uniformly continuous function from an n-tuple in the metric space into a bounded interval in $\mathbb{R}$. For the rest of this paper, we generally assume $P : M^n \to [0,1]$.
(2) A *function* or *operation* $f : M^n \to M$ is a uniformly continuous function from an n-tuple in the metric space back into the metrix space.

In both cases, we call the arity of a function or predicate n.

The next definition is one that only holds true in the case of metric structures, which is fine for our purposes.

Definition 1.3. A *signature* $\mathcal{L}$ is a set of predicate symbols, $(P_i : i \in I)$, functions symbols, $(f_j : j \in J)$, and "distinguished elements", or constant symbols $(a_k : k \in K)$. For each predicate and function the signature also consists of the arity of the function or predicate, as well as moduli of uniform continuity for each.

Signatures contain purely syntatical objects that require "interpretation" from an interpretation function. Such a function takes in the symbol of a predicate, function, or constant, and outputs an actual predicate, function, or constant. This may sound confusing, but we will now define what a metric structure is, give a few examples, and hopefully clear up any confusion.

Definition 1.4. A metric structure $\mathcal{M}$ is an ordered triple

$$\mathcal{M} = (M, \mathcal{L}, I)$$

where (M, d) is a complete, bounded metric space, $\mathcal{L}$ is a signature, and I is an interpretation function taking

$$f \to f^{\mathcal{M}}, \quad P \to P^{\mathcal{M}}, \quad c \to c^{\mathcal{M}}$$

for function symbols, predicate symbols, and constant symbols in $\mathcal{L}$ respectively. A metric structure is often denoted as

$$\mathcal{M} = (M, P_i, f_j, a_k : i \in I, j \in J, k \in K)$$

where each P_i, f_j, a_k each refer to the interpreted predicates, functions, and constants by $\mathcal{M}$. We will often call any metric stucture based on a particular signature $\mathcal{L}$ an $\mathcal{L}$-stucture.

So a signature can be viewed as a set of symbols distinct of any metric structure, and a metric structure is something that takes those symbols and intreprets them as it likes, maintaining the arity and modulus of uniform continuity for each predicate and function.

Example 1.5. The simplest example of a metric structure is index sets I, J, K are all empty. This forms a metric space with no structure where our metric space is complete and bounded.

Example 1.6. A discrete metric M where the distance between any two objects in the set of interest is either zero or one also forms a metric structure. The predicate in this case maps to the set 0,1. In this case, distinct elements $a, b \in M$, $d(a, b) = 1$.

Example 1.7. Let $\mathcal{L} = \{\hat{+}, \hat{\times}, \widehat{|\cdot|}, \hat{1}, \hat{0}\}$, where $\hat{+}$ and $\hat{\times}$ are 2-ary functions, $\widehat{||}$ is a 1-ary predicate, and $\hat{1}$ and $\hat{0}\}$ are constants. Then we can take our metric structure $\mathcal{M}$ to be based on the metric space $([-1, 1], d)$ where $d(x, y) = |x - y|$, and have that:

$$\hat{+} \to +, \quad \hat{\times} \to \times, \quad \widehat{|\cdot|} \to |\cdot|, \quad \hat{1} \to 1, \quad \hat{0} \to 0$$

Then $\mathcal{M}$ is just the space $[-1, 1]$ with the usual operations of addition and multiplication as $+$ and $\times$ along with the constants 0 and 1 and a norm function $\widehat{|\cdot|}$. We could have instead taken $\hat{+}$ to be interpreted as subtraction, or even multiplication, or we could have let $\hat{0}$ be interpreted as $1/2$.

1.2. **Terms and Formulas.**

The elements of a signature are refered to as "non-logical" symbols. We also have many other syntactical objects which we will be using, and all of which are called "logical" symbols. The explanation for the distinction will be made shortly.

With any signature $\mathcal{L}$ in the context of metric structures, we include the following symbols:

(1) A usually countable infinite set of variables $V_{\mathcal{L}}$
(2) The metric d for the underlying space
(3) Uniformly continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ for $n \in \mathbb{N}$
(4) The symbols sup and inf

The idea of logical symbols is that for any signature $\mathcal{L}$, the interpretation of a logical symbol is the same for any $\mathcal{L}$-structure. Although the metric d may not be the same between two $\mathcal{L}$-structures, the idea is that a structure must interpret d as its metric, and has no flexibility for interpreting it as some other 2-ary predicate.

We now define terms and formulas, purely syntactical objects generated from a signature $\mathcal{L}$, and are very important to understand.

Definition 1.8. *Terms* formed from a signature $\mathcal{L}$, called $\mathcal{L}$-terms, are formed inductively as follows: variable and constant symobls are $\mathcal{L}$-terms, and if f is an n-ary function in $\mathcal{L}$, and $t_1, \dots, t_n$ are $\mathcal{L}$-terms, then $f(t_1, \dots, t_n)$ is also an $\mathcal{L}$-term. Any possible $\mathcal{L}$-term is constructed in such a manner.

Example 1.9. Let our language be as in example 1.7. Then $\hat{1}$ and $\hat{0}$ are constant symbols, and v_1 and v_2 are variables, so all are $\mathcal{L}$-terms. Furthermore, so are $(\hat{1}\hat{+}\hat{1})\hat{\times}\hat{1}$ and $(v_1\hat{\times}v_2)\hat{+}\hat{1}$.

Definition 1.10. $\mathcal{L}$-*formulas*, which are formulas formed from a signature $\mathcal{L}$, are also formed inductively as follows:

(1) *Atomic Formulas* of $\mathcal{L}$, which are expressions of the form $P(t_1, \dots, t_n)$ where P is an n-ary predicate symbol of $\mathcal{L}$, and $t_1, \dots, t_n$ are $\mathcal{L}$-terms, along with $d(t_1, t_2)$, where t_1, t_2 are $\mathcal{L}$-terms, are all $\mathcal{L}$-formulas.
(2) If $u : \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous, and $\varphi_1, \dots, \varphi_n$ are $\mathcal{L}$-formulas, then $u(\varphi_1, \dots, \varphi_n)$ is an $\mathcal{L}$-formula. Such functions are called the *connectives* of $\mathcal{L}$.
(3) If φ is an $\mathcal{L}$-formula, and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are $\mathcal{L}$-formulas. Here, sup and inf are called the *quantifiers* of $\mathcal{L}$.

We note that for a variable x which occurs in an $\mathcal{L}$-formula is said to be *bound* if it lies within a "subformula" of the form $\sup_x \varphi$, and otherwise it is said to be *free*.

We note that if we have an $\mathcal{L}$-term t with free variables $x_1, \dots, x_n$ occuring in it, then we write t as $t(x_1, \dots, x_n)$. Likewise, if we have an $\mathcal{L}$-formula φ with free variables $x_1, \dots, x_n$ occuring in it, then we write it as $\varphi(x_1, \dots, x_n)$.

An $\mathcal{L}$-formula with no free variables occuring in it is referred to as a *sentence*.

1.3. **Semantics.** We now come to an important part of our development of continuous model theory for metric structures, which is semantics, with respect to sentences. For a sentence σ, we will now the *value* of σ in $\mathcal{M}$, and it will be a real value and is denoted $\sigma^{\mathcal{M}}$, defined inductively on formulas as follows:

Definition 1.11. Let $\mathcal{M}$ be an $\mathcal{L}$-structure for some signature $\mathcal{L}$. Then we have the following definitions:

(1) For the metric, we have that $(d(t_1, t_2))^{\mathcal{M}} = d(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ where t_1 and t_2 are any $\mathcal{L}$-terms
(2) For any n-ary predicate P in $\mathcal{L}$, and $t_1, \dots, t_n$ $\mathcal{L}$-terms,
$$(P(t_1, \dots, t_n))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})$$
(3) For any connective n-ary u in $\mathcal{L}$, and senctences $\sigma_1, \dots, \sigma_n$ all in $\mathcal{L}$,
$$(u(\sigma_1, \dots, \sigma_n))^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}}, \dots, \sigma_n^{\mathcal{M}})$$
(4) For any $\mathcal{L}$-formula $\varphi(x)$,
$$(\sup_x \varphi(x))^{\mathcal{M}} = \sup\{\varphi(x) : x \in M\}$$
and similarly
$$(\inf_x \varphi(x))^{\mathcal{M}} = \inf\{\varphi(x) : x \in M\}$$

We note now that if we have an $\mathcal{L}$-formula $\varphi(x_1,\ldots,x_n)$, we define the function $\varphi^{\mathcal{M}}$ for simplicity as:

$$\varphi^{\mathcal{M}}(x_1,\ldots,x_n) = (\varphi(x_1,\ldots,x_n))^{\mathcal{M}}$$

Remark 1.12. We make a note here that sup and inf are the quantifiers in continuous logic, and are very analogous to $\forall$ and $\exists$ from discrete logic, which the reader is probably very familiar with. If we let $\mathcal{L}$ be as in example 1.7, and we have an $\mathcal{L}$-formula $\varphi(x) = |2a-1|$, where $a \in [-1,1]$. Then consider the sentence

$$\inf_x \varphi(x) = \inf_x |2a-1|$$

It's clear that $inf_x\varphi(x) = 0$ if there exists an element $a^{-1} \in [-1,1]$ such that $aa^{-1} = 1$. In other words if a is invertible, the value of the sentence is 0.

It is not too harmful to think of such a sentence as saying, $\inf_x \varphi(x) = 0$ if and only if a is invertible, but the reader must be aware that it is often not the case that such a thing is true! Because the infimum only reports the least upper bound, we are not guaranteed that we can find an element such that, in our case, $\varphi(x) = 0$. This is a major difference of continuous logic in contrast to discrete logic, and the difference will consume a large amount of theory and significance in our paper.

1.4. Conditions of $\mathcal{L}$.

Definition 1.13. An $\mathcal{L}$-*condition* E is a formal expression of the form $\varphi = 0$, where φ is an $\mathcal{L}$-formula, and can be a sentence or have free variables.

If E is the $\mathcal{L}$-condition $\varphi(x_1,\ldots,x_n) = 0$, and $a_1,\ldots,a_n \in M$, where $\mathcal{M}$ is an $\mathcal{L}$-structure, we say $a_1,\ldots,a_n$ *satisfies*, or is *true* of E, and write that $\mathcal{M} \vDash \varphi(a_1,\ldots,a_n) = 0$, or $\mathcal{M} \vDash E$, if $\varphi^{\mathcal{M}}(a_1,\ldots,a_n) = 0$.

For a non-negative real number r, we define $\varphi \leq r$ to mean $\varphi \dot{-} r = 0$, where $\dot{-}$ is the connective in $\mathcal{L}$ such that

$$\varphi \dot{-} r := \dot{-}(\varphi, r) = \max(\varphi - r, 0)$$

We note that from here on out we will call simply call $\mathcal{L}$-conditions just, conditions.

Definition 1.14. A *theory* in $\mathcal{L}$ is a set of conditions $\sigma = 0$, where σ is a sentence from $\mathcal{L}$.

If T is a theory in $\mathcal{L}$, and $\mathcal{M}$ is an $\mathcal{L}$-structure, we say that $\mathcal{M}$ models T, and write $\mathcal{M} \vDash T$, if $\mathcal{M} \vDash \sigma$ for every $\sigma = 0$ in T.

We also define the *theory of* $\mathcal{M}$, denote $Th(\mathcal{M})$, to be the set of conditions $\sigma = 0$ such that $\sigma^{\mathcal{M}} = 0$ and we say that T is complete if it is of such a form.

In set builder notation this can be written as:

$$Th(\mathcal{M}) = \{\sigma = 0 : \sigma^{\mathcal{M}} = 0\}$$

1.5. Embeddings.

We now present a section on embeddings between metric structures.

Definition 1.15. Let $\mathcal{L}$ be a signature and $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{L}$-structures. An *embedding* from $\mathcal{M}$ into $\mathcal{N}$ is a metric space isometry

$$\phi : (M, d_1) \to (N, d_2)$$

such that:

- For any n-ary function f of $\mathcal{L}$, and $a_1, \ldots, a_n \in M$,
$$f^{\mathcal{N}}(\phi(a_1), \ldots, \phi(a_n)) = \phi(f^{\mathcal{M}}(a_1, \ldots, a_n))$$
- For any n-ary predicate P of $\mathcal{L}$ and $a_1, \ldots, a_n \in M$,
$$P^{\mathcal{N}}(\phi(a_1), \ldots, \phi(a_n)) = P^{\mathcal{M}}(a_1, \ldots, a_n)$$
- For any constant symbol c of $\mathcal{L}$,
$$c^{\mathcal{N}} = \phi(c^{\mathcal{M}})$$

If the embedding is surjective, then it is an *isomorphism*, and we say that $\mathcal{M}$ is isomorphic to $\mathcal{N}$ and write $\mathcal{M} \cong \mathcal{N}$

Remark 1.16. It is clear that if there exists an embedding from $\mathcal{M}$ into $\mathcal{N}$, then for any quantier free condition E $\varphi = 0$, $\mathcal{M} \models E \iff \mathcal{N} \models E$. For statements with quantifiers this does not always hold though. The Tarski-Vaught test does characterize when all sentences are true in both structures, but we will omit it as it does not relate to our central task.

Definition 1.17. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures, then we have the following definitions:

(1) We say that $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$, if for all sentences σ of $\mathcal{L}$, $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$.

(2) We say that $\mathcal{M}$ is an *elementary extension* of $\mathcal{N}$, and write $\mathcal{M} \preceq \mathcal{N}$, if $M \subset N$ and for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in M$, we have that
$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(a_1, \ldots, a_n)$$
In this case we see that necessarily $\mathcal{M} = \mathcal{N}$.

If we elementarily equivalent $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, then it follows that $Th(\mathcal{M}) = Th(\mathcal{N})$.

We note that we can have two structures with the same cardinalities in the underlying set which are elementarily equivalent, but not isomorphic. This weaker relationship between two structures does sometimes give an isomorphism, and we will study a few cases in which such a thing happens, both abstractly with general structures and concretely with a certain class of structures.

Definition 1.18. For any function $f : X \to Y$, the zero set, D, of f such that $\forall x \in D, f(x) = 0$, where $D \subseteq X$.

1.6. Ultraproducts and the Compactness Theorem.

1.7. Types.

Definition 1.19. Consider a metric structure $\mathcal{M}$ for a language $\mathcal{L}$. Let M be the underlying space. For any $A \subseteq M$, let $\mathcal{L}(A)$ be the language obtained by adding constants c_a for each $a \in A$ to the language $\mathcal{L}$.

Definition 1.20. A *n-type* (of $\mathcal{M}$) over A is a set of $\mathcal{L}(A)$-conditions $\phi(x_1, \ldots, x_n) \leq r$.

Definition 1.21. A type t is *realized* by $\bar{b}$ if every condition in t is satisfied by $\bar{b}$.

Definition 1.22. A type t is *consistent* if for each finite subset $\{\phi_i(\bar{x}) \leq r_i \mid i \leq k\} \subseteq t$ there exists $\bar{b}$ such that $\forall i \leq k \quad \phi_i(\bar{b}) \leq r_i$.

Clearly if a type t is realized, then it must be consistent. However the converse is not generally true. However, by the compactness theorem, t is consistent in $\mathcal{M}$ implies that t is realized in some ultrapower $\mathcal{M}^{\mathcal{U}}$.

Definition 1.23. A model $\mathcal{M}$ is *countably saturated (ω-saturated)* if for every countable $A \subseteq M$, every consistent type over A is realized in M.

Definition 1.24. A model $\mathcal{M}$ *omits* a type t if there is no element in M that realizes t.

Definition 1.25. A consistent type t is *complete* if it is a maximal (with respect to inclusion) consistent type.

Definition 1.26. We say that t is a *complete type of* $\bar{b}$ for some $\bar{b} \in M$ if t is the set of all $\mathcal{L}(A)$-conditions $E(\bar{x})$ such that $\mathcal{M} \vDash E(\bar{b})$. We denote $t = \mathrm{tp}_{\mathcal{M}}(\bar{b})$.

1.8. **Definability.**

Definition 1.27. A predicate P is *definable* over A if there are $\mathcal{L}(A)$-formulas $\phi_n(\bar{x})$ such that uniformly converges to P (over bounded sets)

$$P(\bar{x}) = \lim_{n\to\infty} \phi_n(\bar{x}) \quad \text{uniformly.}$$

Definition 1.28. A set D is *definable* if the predicate

$$P(x) = d(x, D) := \inf_{y\in D} \|x - y\|$$

is a definable predicate.

1.9. **Atomic models.**

Definition 1.29. A complete type p is *principal* (or *definable*) if $\{x \mid x \text{ realizes } p\}$ is a definable set.

Definition 1.30. A model $\mathcal{M}$ is *atomic* if every realized complete type is principal.

Lemma 1.31. *Every model $\mathcal{M}$ has a countably saturated elementary extension $\mathcal{N}$.*

Proof. Let F be the collection of finite subsets of J where J is a set with cardinality $\geq$ the cardinality of the set of all conditions in $\mathcal{M}$. Let $\mathcal{U}$ be the ultrafilter on F generated by the sets $S_j = \{i \in F \mid j \in i\}$ for each $j \in J$. Notice that it does indeed generate an ultrafilter, since it satisfies the finite intersection property ($\{j,k\} \in S_j \cap S_k$, thus finite intersections are nonempty). Let $\mathcal{N} := \mathcal{M}^{\mathcal{U}}$. Clearly $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, since $\mathcal{M}$ embeds into $\mathcal{N}$ via the diagonal embedding ($a \in M \mapsto (a, \dots, a) \in M^{\mathcal{U}}$), and by Łoś's theorem, $\mathcal{N} \equiv \mathcal{M}$.

To show that $\mathcal{N}$ is countably saturated, consider any countable subset $A \subseteq M$. Let $t(\bar{x})$ be any consistent type over A. Since t is consistent, every finite subset $u \subseteq t$ is realized by some $a_u \in M^n$ (i.e. for all $\mathcal{L}(A)$-condition $E(\bar{x}) \in u$, $\mathcal{M} \vDash E(a_u)$).

To show that type t will be realized in $\mathcal{M}^{\mathcal{U}}$, let α be an surjective function from J to t. Such a function exists because J has cardinality $\geq$ the cardinality of the set of all conditions, and a type is a set of conditions. Given any finite subset i of J, $i = \{j_1, \dots, j_k\} \in F$. Let $\bar{a}_i \in M^n$ be the element that realizes finite subset $\{\alpha(j_1), \dots \alpha(j_k)\} \subseteq$

t. Such an element exists for every finite subset due to consistency of t. Then $((\bar{a}_i)_{i\in F})_{\mathcal{U}} \in M^{\mathcal{U}}$ realizes type t in $\mathcal{M}^{\mathcal{U}}$, since by Łoś's theorem, given any condition $\phi(\bar{x}) \le r$ in t,

$$\phi^{\mathcal{M}^{\mathcal{U}}}(((\bar{a}_i)_{i\in F})_{\mathcal{U}}) = \lim_{i,\mathcal{U}} \phi^{\mathcal{M}}(\bar{a}_i) \le r$$

since $\phi^{\mathcal{M}}(\bar{a}_i) \le r$ for every $i \in F$ since $\bar{a}_i$ was chosen to realize the finite subset picked by i via α.

□

Lemma 1.32. *Let $\mathcal{M}$ be a model and let principal type $p(x_1,\dots,x_m) = \mathrm{tp}_{\mathcal{M}}(a_1,\dots,a_n)$ for $a_1,\dots,a_n \in \mathcal{M}$. Let $n > m$ and $q(x_1,\dots,x_n)$ be an extension of p (i.e. any condition in type p is also in type q). Let q be a principal (and complete) type.*

For each $\epsilon > 0$, there exists $(b_1,\dots,b_n)$ that realizes q in $\mathcal{M}$ and satisfies $d(a_j,b_j) \le \epsilon$ for $j = 1,\dots,m$.

Proof. Let $D = \{\bar{x} \in M^m \mid \bar{x} \text{ realizes } p\}$ and $E = \{\bar{x} \in M^n \mid \bar{x} \text{ realizes } q\}$. Since p and q are principal, D and E are definable sets. Thus $d(\bar{x},D)$ and $d(\bar{x},E)$ are definable predicates. Define the predicate

$$F(\bar{x}) \coloneqq \inf_{\bar{y}\in M^{n-m}} |d(\bar{x},D) - d((\bar{x},\bar{y}),E)|$$

Note that F is a definable predicate because it is built from definable predicates. Note also that $\bar{x} \in M^m$ and $(\bar{x},\bar{y}) \in M^n$. Let $\mathcal{N}$ be a countably saturated elementary extension of $\mathcal{M}$ ($\mathcal{N}$ exists by Lemma 1.31). Since $(a_1,\dots,a_m)$ is a realization of p in $\mathcal{M}$, q must be realized in $\mathcal{N}$ by an extension of $(a_1,\dots,a_m)$ (since $\mathcal{N}$ is countably saturated means that q, which is consistent since it is complete, will be realized in $\mathcal{N}$). Now consider F interpreted by $\mathcal{N}$, since F is a definable predicate. Thus, $F^{\mathcal{N}}(a_1,\dots,a_m) = 0$. Since $\mathcal{M} \equiv \mathcal{N}$, any formula have the same value under both interpretations, thus the limit of any sequence of formulas have the same value under both interpretations, thus any definable predicate have the same value under both interpretations. Thus $F^{\mathcal{M}}(a_1,\dots,a_m) = 0$ also.

From $F(a_1,\dots,a_n) = 0$ in $\mathcal{M}$ and the definition of F

$$F(a_1,\dots,a_m) = \inf_{\bar{y}\in M^{n-m}} |d((a_1,\dots,a_m),D) - d((a_1,\dots,a_m,\bar{y}),E)| = 0$$

Recall that $(a_1,\dots,a_m)$ realizes p, thus $d((a_1,\dots,a_m),D) = 0$, therefore

$$F(a_1,\dots,a_m) = \inf_{\bar{y}\in M^{n-m}} d((a_1,\dots,a_m,\bar{y}),E) = 0$$

Therefore, given any $\epsilon > 0$, there exists $\bar{c} \in M^{n-m}$ such that $d((a_1,\dots,a_m,\bar{c}),E) \le \frac{\epsilon}{2}$. By definition of distance,

$$\inf_{\bar{b}\in E} d((a_1,\dots,a_m,\bar{c}),\bar{b}) = d((a_1,\dots,a_m,\bar{c}),E) \le \frac{\epsilon}{2}$$

which means that there exists $\bar{b} \in E$ such that $d((a_1,\dots,a_m,\bar{c}),\bar{b}) \le \epsilon$.

Therefore, this $\bar{b}$ realizes q, and $d(a_j,b_j) \le \epsilon$ for $j = 1,\dots,m$.

□

Theorem 1.33. *Given $\mathcal{M}$ and $\mathcal{N}$ separable atomic models, then*

$$\mathcal{M} \equiv \mathcal{N} \iff \mathcal{M} \cong \mathcal{N}$$

Proof. ($\Leftarrow$) If $\mathcal{M}$ is isomorphic to $\mathcal{N}$, then clearly they have the same theory, thus they are elementarily equivalent.

($\Rightarrow$) Assume $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent. We will use the "back and forth" method to show an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

Since M is separable, consider a countable set A dense subset of M. Since we will be taking elements from A one by one, we will let $\{u_1, u_2, \dots\} = A$ be an indexing of A by $\mathbb{N}$. However, when we visit elements from A, we will do it in the following order:

$$u_1, \quad u_1, u_2, \quad u_1, u_2, u_3, \quad u_1, u_2, u_3, u_4, \quad u_1, u_2 \dots$$

Note that this will still include every element of A, and by construction, we will in fact visit very element infinitely many times. Similarly, consider countable set C dense subset of N, and do the same with it. For any $\epsilon > 0$, consider $\delta_k = \frac{\epsilon}{2^{n+1}}$, so that $\sum_{k=0}^{\infty} \delta_k = \epsilon$. Using Lemma 1.32 we will generate by induction for each $n \geq 0$ sequences $a_n \in M^n$ and $c_n \in N^n$ such that

(1) $a_0 = c_0 = \varnothing$
(2) $\mathrm{tp}_{\mathcal{M}}(a_n) = \mathrm{tp}_{\mathcal{N}}(c_n)$
(3) $d((a_{n+1})_i, (a_n)_i) \leq \delta_n$ and $d((c_{n+1})_i, (c_n)_i) \leq \delta_n$ for every $i \leq n$

To follow this step by step, we first establish (1) as our basis step for induction. Certainly (2) holds for the empty sequence ($n = 0$), since $\mathcal{M} \equiv \mathcal{N}$, thus they agree on the level of sentences. We pick the first element in A using the method above and put it in a_1, then consider $p_0(x_1) = \mathrm{tp}_{\mathcal{M}}(a_1)$, which is a principal type since $\mathcal{M}$ is atomic. Thus, we can consider type p_0 in $\mathcal{N}$ and use Lemma 1.32 to show that it is realized by what we will put into c_1, since type p_0 is an extension of the type that holds all true sentences. Note (3) is automatic for $n = 0$.

Now we assume that $\mathrm{tp}_{\mathcal{M}}(a_n) = \mathrm{tp}_{\mathcal{N}}(c_n)$ and call it type p. Pick the next element of A that is not already in a_n using the method described above and append it to a_n to make a_{n+1}. Consider the extension $q(x_1, \dots, x_{n+1}) = tp_{\mathcal{M}}(a_{n+1})$ of q. Note that since c_n realizes p, and q is an extension of p, and the types are principal since we are in an atomic model, by Lemma 1.32 q is realized in $\mathcal{N}$, and we will call the sequence that realizes it c_{n+1}. Note that Lemma 1.32 also gives us $d((c_{n+1})_i, (c_n)_i) \leq \delta_n$ for every $i \leq n$. Thus (2) holds: $\mathrm{tp}_{\mathcal{M}}(a_{n+1}) = \mathrm{tp}_{\mathcal{N}}(c_{n+1})$, and c_{n+1} is close to c_n in the style of (3). Note also that since we are only appending when creating a_{n+1}, $d((a_{n+1})_i, (a_n)_i) = 0 \leq \delta_n$, thus (3) holds.

Since this is a "back and forth" argument, we will do two steps after n. Now that we have $q(x_1, \dots, x_{n+1}) = \mathrm{tp}_{\mathcal{M}}(a_{n+1}) = \mathrm{tp}_{\mathcal{N}}(c_{n+1})$, we will extend it again but using C this time. Pick the next element using the above method from C that is not already in c_{n+1} and append it to the end of c_{n+1}, making c_{n+2}. Let $r(x_1, \dots, x_{n+2}) = \mathrm{tp}_{\mathcal{N}}(c_{n+2})$. Similar to above, r is an extension of q which was realized in $\mathcal{N}$ and both are principal types. By Lemma 1.32 r is also realized in $\mathcal{N}$, and we will call the realization a_{n+2}. Thus $\mathrm{tp}_{\mathcal{M}}(a_{n+2}) = \mathrm{tp}_{\mathcal{N}}(c_{n+2})$. From the lemma, for $i = 1, \dots, n+1$, $d((a_{n+2})_i, (a_{n+1})_i) \leq \delta_{n+1}$. Obviously $d((c_{n+2})_i, (c_{n+1})_i) = 0 \leq \delta_{n+1}$ as well.

Therefore, we have created sequences $a_n \in M^n$ and $c_n \in N^n$ for every n, and any element in dense subsets $A \subseteq M$ or $C \subseteq N$ will eventually be included in some a_n or c_n. However, notice that the a_n and c_n shifts slightly every two steps, i.e. earlier ones are not initial segments of later ones. This is how we fix it:

For each j, the consider the sequence of the j^{th} coordinate of a_n and c_n. In other words, consider sequences $(a_n)_j$ and $(c_n)_j$ as n goes from j to infinity (we start from j because anything lower will not have a j^{th} coordinate). Notice that the two sequences are Cauchy, because in each step we have bounded the difference by δ_k, and the sum of all $\sum \delta_k = \epsilon$. Therefore, given any $\varepsilon > 0$, there exists K such that $\sum_{k=0}^{K} \delta_k > \epsilon - \varepsilon$, thus the difference between two terms with indices larger than K will be at most $\sum_{k=K}^{\infty} \delta_k \leq \varepsilon$.

Because we have Cauchy sequences and the metric space is complete, we will call the limits

$$(j^{th} \text{ coordinate of } a_n) \longrightarrow b_j$$
$$(j^{th} \text{ coordinate of } c_n) \longrightarrow d_j$$

Therefore, our isomorphism will take b_j to d_j. To verify this, we need to check that it is an elementary map.

Consider any formula $\phi(x_1, \dots, x_m)$. The value of ϕ on b_j's can be written as $\phi(b_{\sigma(1)}, \dots, b_{\sigma(m)})$. Let $\phi^{\mathcal{M}}(b_{\sigma(1)}, \dots, b_{\sigma(m)}) = R$. Since $\mathrm{tp}_{\mathcal{M}}(b_1, \dots, b_n) = \mathrm{tp}_{\mathcal{N}}(d_1, \dots, d_n)$ for all $n \geq 0$, let $n = \max\{\sigma(k) \mid 1 \leq k \leq m\}$. Thus

$$\forall r < R \quad (\phi(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \leq r) \notin \mathrm{tp}_{\mathcal{M}}(b_1, \dots, b_n)$$

and

$$\forall r \geq R \quad (\phi(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \leq r) \in \mathrm{tp}_{\mathcal{M}}(b_1, \dots, b_n)$$

Therefore

$$\forall r < R \quad (\phi(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \leq r) \notin \mathrm{tp}_{\mathcal{N}}(d_1, \dots, d_n)$$

and

$$\forall r \geq R \quad (\phi(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \leq r) \in \mathrm{tp}_{\mathcal{N}}(d_1, \dots, d_n)$$

Thus,

$$\phi(d_{\sigma(1)}, \dots, d_{\sigma(m)}) = R = \phi(b_{\sigma(1)}, \dots, b_{\sigma(m)})$$

as desired, showing that $b_j \mapsto d_j$ is an isomorphism.

To extend this isomorphism to $\mathcal{M}$ and $\mathcal{N}$, we now have to verify that the the sets $\{b_j \mid j \in \mathbb{N}\}$ is dense in M, and $\{d_j \mid j \in \mathbb{N}\}$ is dense in N.

Given any $\epsilon' > 0$ and any element $x \in M$, we recall that

$$d(b_j, (a_n)_j) \leq \sum_{k=n}^{\infty} \delta_k = \epsilon - \sum_{k=0}^{n-1} \delta_k$$

Thus we can pick K such that for all $n > K$,

$$d(b_j, (a_n)_j) \leq \frac{\epsilon'}{2}$$

Now since A is a dense subset of M, and every $a \in A$ is included in infinitely many a_n due to each element being visited infinitely many times, the set $\{a \mid \exists n \quad a \in a_n\}$ is also dense in M. In fact, due to elements being infinitely many times, every $a \in A$ is included in some a_n even when we consider n where $n > K$. Therefore, the following set is also dense in M.

$$S := \{a \mid \exists n > K \quad a \in a_n\}$$

Note that by definition of K, any $s \in S$ has $d(s, b_j) \leq \frac{\epsilon'}{2}$ for some j depending on which coordinate the s came from. By density of S in M, given any $x \in M$, there exists $s \in S$ such that $d(x, s) \leq \frac{\epsilon'}{2}$. By triangle inequality there exists some b_j such that

$$d(x,b_j) \le d(x,s) + d(s,b_j) \le \frac{\epsilon'}{2} + \frac{\epsilon'}{2} \le \epsilon'$$

Therefore, $\{b_j \mid j \in \mathbb{N}\}$ is dense in M, and by the same argument, $\{d_j \mid j \in \mathbb{N}\}$ is dense in N. Thus, from the isomorphism $b_j \mapsto d_j$, we have an isomorphism F between $\mathcal{M}$ and $\mathcal{N}$ by continuity of formulas using the following:

Given any $x \in M$, by density of $\{b_j \mid j \in \mathbb{N}\}$ there exists a sequence $(b_{\sigma(j)})$ of elements in $\{b_j \mid j \in \mathbb{N}\}$ that converges to x. Let $y \in N$ be the limit of the sequence $(d_{\sigma(j)})$. Define $F(x) := y$. By the continuity of formulas, given any formula ϕ,

$$\phi(x) = \phi(\lim b_{\sigma(j)}) = \lim \phi(b_{\sigma(j)}) = \lim \phi(d_{\sigma(j)}) = \phi(\lim d_{\sigma(j)}) = \phi(y)$$

Therefore, $F : M \to N$ is an isomorphism.

□

1.10. **Stability.**

Definition 1.34. A condition $\phi(x,\bar{b}) \le r$ is *stable* if for all $\epsilon \ge 0$ there is $\delta \ge 0$ such that

$$\phi(a,\bar{b}) \le r + \delta \implies \text{there exists } a' \text{ such that } \|a' - a\| \le \epsilon \text{ and } \phi(a,\bar{b}) \le r$$

Sometimes we will refer to formulas or definable predicates as being stable if it is obvious which r we are talking about (usually when $r = 0$)

Our primary motivation for considering stable conditions is that we wish to quantify over them in formulas or definable predicates, depending on what the condition was built from.

To prove that we can infact quantify variables over stable $\mathcal{L}$-formulas, we first need a proposition.

Proposition 1.35. *Let $F, G : X \to [0,1]$ be arbitrary functions such that for all $\epsilon > 0$ there is $\delta >$ such that for all $x \in X$ we have*

$$F(x) \le \delta \implies G(x) \le \epsilon.$$

If this is true, then we there exists an increasing, continuous function $\alpha : [0,1] \to [0,1]$ such that $\alpha(0) = 0$ and for all $x \in X$ we have

$$G(x) \le \alpha(F(x)).$$

Proof. For a proof, see Proposition 2.10 of [1]. □

We now prove that we can indeed quantify variables over stable conditions.

Lemma 1.36. *Given a stable $\mathcal{L}$-formula $\psi(x_1,\dots,x_n)$ and $\mathcal{L}$-formula $\phi(x_1,\dots,x_n,y_1,\dots,y_m)$, there exists an $\mathcal{L}$-formula $\theta(y_1,\dots,y_m)$ such that*

$$\theta(\bar{y}) = \inf_{\psi(\bar{x})=0} \phi(\bar{x},\bar{y}).$$

Proof. First we will show that there exist an $\mathcal{L}$-formula for $d(x,D)$ where D the zero set of ψ. Since ψ is a stable formula, by definition for all $\epsilon > 0$ there is $\delta > 0$ such that for all x with $\psi(x) < \delta$, we can find $x' \in M^n$ such that $\|x - x'\| < \epsilon$ and $\psi(x') = 0$. Since $x' \in D$, we have

$$d(x,D) = \inf_{x \in D} \|x - y\| \le \|x - x'\| < \epsilon.$$

Now by Proposition 1.35, there exists a continuous increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that $d(x, D) \leq \alpha(\psi(x))$.

Define the follwing $\mathcal{L}$-formula by

$$F(x) := \inf_y \min\{\alpha(\psi(y)) + \|x - y\|, 2\}.$$

Note that $\|x - y\| \leq 2$ since $x, y \in B_1$. We wil now show that $F(x) = d(x, D)$.

Since $D \subset M^m$, we have

$$\begin{aligned} F(x) &= \inf_y \min\{\alpha(\psi(y)) + \|x - y\|, 2\} \\ &\leq \inf_{y \in D} \min\{\alpha(\psi(y)) + \|x - y\|, 2\} \\ &= \inf_{y \in D} \min\{\alpha(0) + \|x - y\|, 2\} \\ &= \inf_{y \in D} \|x - y\| = d(x, D). \end{aligned}$$

For the reverse inequlity, recall that $d(x, D) \leq \alpha(\psi(y))$, thus by the triangle inequality

$$F(x) \geq \inf_y \min\{d(y, D) + \|x - y\|, 2\} \geq \inf_y \min\{d(x, D), 2\} = d(x, D).$$

Therefor, we have an $\mathcal{L}$-formula $F(x) = d(x, D)$ as desired.

To show that we can quantify over D, we use Proposition 1.35 with the modulus of uniform continuity for $\phi(x, y)$ over x. In other words, let $\beta : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function with $\beta(0) = 0$ such that $|\phi(x, y) - \phi(z, y)| \leq \beta(\|x - z\|)$.

We will show that the $\mathcal{L}$-formula

$$\theta(y) := \inf_x \phi(x, y) + \beta(F(x))$$

is equal to $\inf_{x \in D} \phi(x, y)$.

By definition of β,

$$\phi(x, y) \leq \phi(z, y) + \beta(\|x - z\|)$$

for all $x, z \in M^n$ and $y \in M^m$.

By taking the infimum over $x \in D$ of both sides, since β is increasing, we obtain

$$\inf_{x \in D} \phi(x, y) \leq \phi(z, y) + \beta(\inf_{x \in D} \|x - z\|) = \phi(z, y) + \beta(d(z, D)). \tag{1.1}$$

Now if we take infimum over $z \in D$ of the right hand side of (1.1)

$$\inf_{z \in D} \phi(z, y) + \beta(d(z, D)) = \inf_{z \in D} \phi(z, y).$$

Since $D \subseteq M^n$,

$$\inf_z \phi(z, y) + \beta(d(z, D) \leq \inf_{z \in D} \phi(z, y) + \beta(d(z, D) = \inf_{z \in D} \phi(z, y).$$

Thus, since the inequality (1.1) holds for any z, it certainly holds for $\inf_z$, so we have

$$\inf_{x \in D} \phi(x, y) \leq \inf_z \phi(z, y) + \beta(F(z)) \leq \inf_{z \in D} \mathrm{RHS} = \inf_{x \in D} \phi(x, y).$$

Therefore,

$$\inf_{x \in D} \phi(x, y) = \inf_z \phi(z, y) + \beta(F(z)) = \theta(y)$$

where θ is a $\mathcal{L}$-formula as desired. $\square$

Corollary 1.37. *The* sup *case of Lemma 1.36 also holds, i.e. there exists $\mathcal{L}$-formula θ' such that*

$$\theta'(y) = \sup_{\psi(x)=0} \phi(x,y)$$

Proof. Simply consider $\phi(x) = -\phi(x)$ in Lemma 1.36

$$\begin{aligned} \inf_{x\in D}(-\phi(x,y)) &= \inf_z(-\phi(z,y)) + \beta(F(z)) \\ \implies -\inf_{x\in D}(-\phi(x,y)) &= -\inf_z(-\phi(z,y)) - \beta(F(z)) \\ \implies \sup_{x\in D}\phi(x,y) &= \sup_z \phi(z,y) - \beta(F(z)) = \theta'(y) \end{aligned}$$

□

2. C^*-Algebras

2.1. **Introduction to C^*-algebras.** At the end of this paper, we would like to use the continuous model theory outlined above, in conjunction with C^*-algebra theory, to describe C^*-algebras. Thus, a brief introduction to C^*-algebras is necessary; this will unfortunately be a definition heavy section, along with theorems which are necessary to complete the goals that we wish to attain.

We begin with the usual definition of a C^*-algebra, which we will describe inductively.

Definition 2.1. An *algebra* over $\mathbb{C}$ is a complex vector space A over $\mathbb{C}$ with an associative multiplication $(a,b) \in A\times A \mapsto ab \in A$ which is compatible with the vector space structure.

A *Banach algebra* is a Banach space A over $\mathbb{C}$ and an algebra over $\mathbb{C}$ in which the multiplication satisfies

$$\|ab\| \le \|a\|\|b\| \text{ for } a,b \in A.$$

Note that we know we have a norm on A as A is a Banach space (a vector space which is complete in norm).

A *C^*-algebra* is a Banach algebra A over $\mathbb{C}$ with an involution $a \mapsto a^*$ which is conjugate linear and satisfies the following properties for all $a,b \in A$:

- $(a^*)^* = a$,
- $(ab)^* = b^*a^*$,
- $\|a^*\| = \|a\|$,
- and the C^*-identity, $\|a^*a\| = \|a\|^2$.

Another important definition is what we mean by homomorphism in the language of C^*-algebras.

Definition 2.2. A *$*$-homomorphism* is a homomorphism ϕ between two C^*-algebras A and B satisfying $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

There are important elements of C^*-algebras we often work with. The names given to these elements will become apparent after reading teh section on bounded operators.

Definition 2.3. Let A be a C^*-algebra.

An element $a \in A$ is *self-adjoint* if $a = A^*$.

An element $p \in A$ is a *projection* if $p = p^* = p^2$.

If $1 \in A$, an element $t \in A$ is an *isometry* if $tt^* = 1$.
An element $s \in A$ is a *partial isometry* if $ss^*s = s$.
An element $u \in A$ is *unitary* if $uu^* = u^*u = 1$.

We wish to introduce a heavy theorem known as The Continuous function calculus. This theorem will allow us to describe elements of a C^*-algebra in terms of functions on $\mathbb{C}$. To do this, we must begin with the notion of invertibility.

Definition 2.4. Let A be a C^*-algebra with 1. An element $a \in A$ is *invertible* if there exists $b \in A$ such that $ab = ba = 1$. If this element exists, we write $b = a^{-1}$, the *inverse* of a. We write A^{-1} for the set of invertible elements in A.

These invertible elements behave exactly as we expect them to.

Lemma 2.5. *Let A be a C^*-algebra with 1. Let $a \in A^{-1}$, then a^{-1} is unique and $(ab)^{-1} = b^{-1}a^{-1}$ for all $b \in A^{-1}$.*

Proof. Suppose $c, d \in A$ such that $ac = ca = 1$ and $ad = da = 1$. The we have

$$c = c1 = c(ad) = (ca)d = 1d = d.$$

For the second claim, we simply have

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}) = a1a^{-1} = 1$$

Similarly for the left inverse. □

We now introduce a concept akin to eigenvalues of matrices.

Definition 2.6. Let A be a C^*-algebra with 1 and let $a \in A$. Then the *spectrum* of a denoted $\sigma(a)$ is the subset of $\mathbb{C}$ defined by

$$\sigma(a) := \{\lambda \in \mathbb{C} : (a - \lambda 1) \notin A^{-1}\}.$$

The continuous functional calculus can only be applied to certain elements. Fortunately though, these types of elements include a lot of the important elements that we most often work with.

Definition 2.7. Let A be a C^*-algebra. We say $a \in A$ is *normal* if $aa^* = a^*a$.

Example 2.8. All self adjoint elements are obviously normal; these fortunately include projections. Recall that an element u of a C^*-algebra is unitary if $uu^* = u^*u = 1$, thus all unitary elements are normal.

The last definition needed before stating the continuous functional calculus is what we mean when we talk about an algebra generated by an element.

Definition 2.9. Let A be a C^*-algebra with 1. Let $a \in A$, then $C^*(a)$ is the C^*-algebra generated by a and 1, and is the smallest C^*-subalgebra of A containing both a and 1.

Alternatively, it is the norm closure of the $*$-subalgebra of A made of linear combinations of elements of the form

$$a^{m_1}(a^*)^{n_1}a^{m_2}(a^*)^{n_2}\cdots a^{m_p}(a^*)^{n_p}$$

where $m_i, n_i \geq 0$.

The 'small' requirement that a is a normal element simplifies this expression to

$$C^*(a) = \overline{\operatorname{span}}\left\{\sum \lambda_{m,n} a^m (a^*)^n : m, n \geq 0\right\}.$$

It is important to note that $C^*(a) \subseteq A$.

We can now state the continuous function calculus theorem. Unfortunately to provide a proof, we would have to go into a lot more C^*-algebra theory, which is not the aim of this project.

To motivate this, we pose the following. Suppose we have a C^*-algebra A and $a \in A$. We want to know, when can we write $a = b^2$ for $b \in A$? i.e. can we form $\sqrt{a}$?

Consider the function $f(z) = \sum c_n z^n$, we wish to be able to write $f(a) = \sum c_n a^n$ for $a \in A$, similarly for the function $f(z) = 1/z$, then we want $f(a) = a^{-1}$. Obviously this is not always defined, the continuous functional calculus tells us when it is. We can also take the function $f(z) = \sqrt{z}$, the continuous functional caluclus is the answer to our problems.

Theorem 2.10. (The continuous functional calculus). *Let A be a C^*-algebra with 1. Let $a \in A$ be a normal element. Then there is a unique unital $*$-homomorphism $\Gamma : C(\sigma(a) \to A$ which takes the identity function, $\iota : z \to z$ to a, i.e. $\Gamma(\iota) = a$. Furthermore, $\Gamma : C(\sigma(a)) \to C^*(a)$ is an isomorphism.*

We normally will just write $f(a)$ when we mean $\Gamma(f)$, but we will say $f(a)$ is defined by the continuous functional calculus for a. Lets see this in action.

Example 2.11. Let $f(z) = 1/z$. Let A be a C^*-algebra with normal element a such that f is well defined on $\sigma(a)$. Since multiplication in $C(\sigma(a))$ is simply pointwise multiplication, we have that $f = \iota^{-1}$, ie, it is the inverse of ι in $C(\sigma(a))$. Now, since Γ is unital, it takes inverses to inverses, so we have

$$f(a) = \Gamma(f) = \Gamma(\iota^{-1}) = \Gamma(\iota)^{-1} = a^{-1}$$

2.2. **Bounded operators.**

2.3. **Bounded operators.** We often talk of the bounded linear operators over a Hilbert space. It turns out that this is 'the' example of a C^*-algebra, we'll talk a little bit about why soon. So in this section, we will simply introduce what we mean when we say bounded operators, we will show why they are an example of a C^*-algebra, and we will discuss what we mean by the above statement.

Definition 2.12. Let H be a Hilbert space. The map $T : H \to H$ is a *bounded linear operator* if T is linear and there exists $C \in \mathbb{R}$ such that

$$\|Th\| \leq C\|h\| \quad \text{for all } h \in H.$$

We may ask the question why we are interested in these operators with a bound; it turns out that they are infact the continuous operators on H.

Proposition 2.13. *Suppose $H$is a Hilbert space and $T : H \to H$ is a linear operator. Then the following statements are equivalent:*

(1) *T is continuous on H.*
(2) *T is continuous at 0.*
(3) *There exists a $C \in \mathbb{R}$ such that $\|Th\| \leq C\|h\|$ for all $h \in H$.*

Proof. ($1 \implies 2$) is trivial.
($2 \implies 3$) We note that T continuous at 0 says that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|h - 0\| = \|h\| \leq \delta \implies \|Th - T0\| = \|Th\| < \epsilon.$$

Take $h = 0$, then we have for a fixed $h_0 \in H$, $\|T0\| = 0\|Th_0\| = 0\|h_0\|$. This shows that when $h = 0$, T is bounded.

Now assume $h \neq 0$ and fix $\epsilon > 0$. If we take the vector $\frac{\delta h}{\|h\|}$, the norm of this vector is given by

$$\left|\frac{\delta h}{\|h\|}\right| = \frac{1}{\|h\|}|\delta\||h\| = \delta.$$

So this vector satisfies our condition for continuity, so we know that

$$\left\|T\left(\frac{\delta h}{\|h\|}\right)\right\| < \epsilon \implies \frac{\delta}{\|h|}\|Th\| \leq \epsilon \implies \|Th\| \leq \frac{\epsilon\|h\|}{\delta}$$

So now take $C = \frac{\epsilon}{\delta}$, and we have that $\|Th\| \leq C\|h\|$ as required.

($3 \implies 1$) Suppose C satisfies $\|Th\| < C\|h\|$ and $\{h_n\}$ is a sequence in H satisfying $h_n \to h \in H$.

$$\|Th_n - Th\| = \|T(h_n - h)\| \leq C\|h_n - h\| \to 0$$

$$\implies Th_n \to Th \quad \text{for all } h_n \to h \in H.$$

Which tells us that T is continuous on H. □

What is very interesting of this proposition above is that continuous everywhere is equivalent to just continuous at 0. This proposition outlines why we were interested in bounded linear operators. We will now begin to discuss them as C^*-algebras. We begin this by defining a norm on the operator; the natural choice would be the lowest bound for the operator.

Proposition 2.14. *If T is a boudned linear operator on a Hilbert space H, we define the operator norm as the greatest lower bound of bounds on T; that is,*

(1) $\|T\|_{op} = \inf\{C : \|Th\| \leq C\|h\| \quad \text{for all } h \in H\}$
Equivalently, we can define $\|T\|_{op}$ by
(2) $\|T\|_{op} = \sup\{\|Th\| : \|h\| \leq 1\}$.

Moreover, $\|T\|_{op}$ is a bound for T.

Proof. Take (2), fix $h \in H$ such that $\|h\| \leq 1$. Since T is bounded, we know the set defined in (2) is non-empty, and that there exists C such that:

$$\|Th\| \leq C\|h\| \leq C.$$

So the set is bounded by C, so the least upper bound is well defined.

Since C is an upper bound for (2), we have that $\|T\| \leq C$, as $\|T\|$ is the least upper bound. We can also see that this C lies in the set $\{C : \|Th\| \leq C\|h\| \quad \text{for all } h \in H\}$. Since $\|T\| \leq C$, $\|T\|$ is a lower bound for this set.

Now, for any $h \in H$, we have that

$$\begin{aligned} &\frac{1}{\|h\|}\|Th\| = \|T\frac{h}{\|h\|}\| \leq \|T\| \\ \implies &\|Th\| \leq \|T\|\|h\| \\ \implies &\|T\| \in \{C : \|Th\| \leq C\|h\| \quad \text{for all } h \in H\}. \end{aligned}$$

Now, since $\|T\|$ is a lower bound of the set, and it is in the set, it must be the greatest lower bound. So we have

$$\|T\| = \inf\{C : \|Th\| \leq C\|h\| \quad \text{for all } h \in H\}.$$

□

We now need to show that the operator norm defined above, is infact a norm; i.e. it satisfies the norm axioms. But first we need to ensure that the bounded operators form a vector space.

Proposition 2.15. *If H is a Hilbert space, then the set $B(H)$ of bounded linear operators is a vector space under operations*

$$(S+T)(h) = Sh + Th \quad \textit{for all } h \in H$$

$$(cT)(h) = c(Th) \quad \textit{for all } c \in \mathbb{C},\ h \in H$$

Proof. Suppose $S, T \in B(H)$. Then $(S+T)$ is linear since

$$\begin{aligned} (S+T)(ah+k) &= S(ah+k) + T(ah+k) \\ &= aSh + Sk + aTh + Tk \\ &= a(Sh+Th) + (Sk+Tk) \\ &= a(S+T)h + (S+T)k. \end{aligned}$$

It is bounded since

$$\begin{aligned} \|(S+T)h\| &= \|Sh+Th\| \\ &\leq \|Sh\| + \|Th\| \\ &\leq \|S\|_{op}\|h\| + \|T\|_{op}\|h\| \\ &= (\|S\|_{op} + \|T\|_{op})\|h\|. \end{aligned}$$

So $S+T$ is linear and bounded, hence the set is closed under vector addition. Now

$$\begin{aligned} (cT)(ah+k) &= c(T(ah+k)) \\ &= c(aTh+Tk) \\ &= acTh + cTk \\ &= a(cT)h + (cT)k \end{aligned}$$

And so (cT) is linear. Now to see that (cT) is bounded,

$$\begin{aligned} \|(cT)h\| &= \|cTh\| \\ &= |c|\|Th\| \\ &\leq |c|\|T\|_{op}\|h\| \end{aligned}$$

So cT is linear and bounded, hence, the set is closed under scalar multiplication.

Now observe that $(0)h = 0$. We have

$$(0)(ah + k) = 0 = a(0)h + (0)k$$
$$\|0h\| = \|0\| \leq |c|\|h\| \quad \text{for all } c \in \mathbb{R}$$

So the zero operator is contained in the bounded operators, so $B(H)$ is a vector space. □

Now that we know $B(H)$ forms a vector space, we must show that the operator norm is infact a norm.

Lemma 2.16. *The operator norm is a norm in the sense that*

(1) $\|\lambda T\|_{op} = |\lambda|\|T\|_{op}$
(2) $\|S + T\|_{op} \leq \|S\|_{op} + \|T\|_{op}$
(3) $\|T\|_{op} = 0 \implies T = 0$

Proof. The proofs are relatively simple, by manipulation of the definition of the operator norm. For (1),

$$\|\lambda T\|_{op} = \sup_{\|h\|\leq 1} |\lambda|\|Th\| = |\lambda| \sup_{\|h\|\leq 1} \|Th\| = |\lambda|\|T\|_{op}.$$

For (2),

$$\|S+T\|_{op} = \sup_{\|h\|\leq 1} \|(S+T)h\| \leq \sup_{\|h\|\leq 1} \left(\|Sh\| + \|Th\|\right) = \sup_{\|v\|\leq 1} \|Sh\| + \sup_{\|h\|\leq 1} \|Th\| = \|S\|_{op} + \|T\|_{op}.$$

And finally for (3),

$$\|T\| = 0 \implies \sup_{\|h\|\leq 1} \|Th\| = 0 \implies Th = 0 \quad \text{for all } h \text{ with } \|h\| \leq 1$$

Now by linearity, we can write any vector as a unit vector times its length, so

$$Th = T\|h\|\frac{h}{\|h\|} = \|v\|T\left(\frac{h}{\|h\|}\right) = 0 \quad \text{for all } h \in H \implies T = 0.$$

We now show that $B(H)$ is infact a Banach space under the operator norm. □

Proposition 2.17. *Let H be a Hilbert space, then $B(H)$ is complete in the operator norm.*

Proof. To show completeness, we wish to show that every Cauchy sequence in $B(H)$ converges to an element in $B(H)$. That is, for $(T_n) \in B(H)$ with $\|T_n - T_m\|_{op} \to 0$ as $m, n \to \infty$, there exists $T \in B(H)$ such that $\|T_n - T\|_{op} \to 0$ as $n \to \infty$

Fix $h \in H$

$$\|T_m h - T_n h\| \leq \|T_m - T_n\|_{op}\|h\| \to 0.$$

Which says $\{T_n h\}$ is Cauchy in H. Since H is complete, there exists $k \in H$ such that $T_n h \to k$. Define $T : H \to H$ by $Th = k$, so $T_n h \to Th$. We want to show that this T is in $B(H)$ and then that $T_n \to T$.

Now, for $h, k \in H$, $c \in \mathbb{C}$, we have

$$T_n(ch + k) \to T(ch + k)$$

but, since T_n is linear, we have

$$T_n(ch+k) = cT_nh + T_nk \to cTh + Tk.$$

But since T_n converges to a unique element, we have that

$$T(ch+k) = cTh + Tk,$$

proving T is linear.

Since T_n is Cauchy, for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$m, n \geq N \implies \|T_n - T_m\| < \epsilon.$$

Fix $n \geq N$, we have that

$$\|T_n\| - \|T_N\| < \|T_n - T_N\| < \epsilon \implies \|T_n\| < \epsilon + \|T_N\|.$$

Which shows that T_n is bounded, which means there is some C satisfying

$$\|T_n\| \leq C \implies \|T_nh\| \leq C\|h\| \quad \text{for all } h \in H$$

$$\|Th\| = \lim_{n\to\infty} \|T_nh\| \leq C\|h\| \quad \text{for all } h \in H.$$

So T is bounded. Telling us that $T \in B(H)$.

Now we want to show that $T_n \to T$. Fix $\epsilon > 0$, since T_n is cauchy, there is an $N \in \mathbb{N}$ such that

$$m, n \geq N \implies \|T_m - T_n\|_{op} < \frac{\epsilon}{2}.$$

Fix $n \geq N$, for all $h \in H$ and for $m \geq N$, we have

$$\begin{aligned} \|T_mh - T_nh\|_{op} &< \frac{\epsilon}{2}\|h\| \\ \|Th - T_nh\|_{op} &\leq \frac{\epsilon}{2}\|h\| \\ \|(T - T_n)h\|_{op} &\leq \frac{\epsilon}{2}\|h\| \\ \|T - T_n\|_{op} &\leq \frac{\epsilon}{2} < \epsilon \\ \implies T \to T_n &\text{ in operator norm.} \end{aligned}$$

□

We now define a multiplication on $B(H)$, we show that under this multiplication, $B(H)$ forms a Banach algebra.

Proposition 2.18. *Let H be a Hilbert space and fix $S, T \in B(H)$. Then the composition $ST = S \circ T$ defined by $(ST)v = S(Tv)$ is a boudned linear operator on H with $\|ST\|_{op} \leq \|S\|_{op}\|T\|_{op}$.*

Proof. For each $h, k \in H$ and $c \in \mathbb{C}$, we have

$$\begin{aligned} (ST)(ch+k) &= S(T(ch+k)) \\ &= S(cTh + Tk) \\ &= ScTh + STk \\ &= c(ST)h + (ST)k \end{aligned}$$

Hence ST is linear. Now we must show ST is bounded. We have

$$\|ST\|_{op} = \sup_{\|h\|\le 1\|} \|STh\| \le \sup_{\|h\|\le 1\|} \|S\|_{op}\|Th\| = \|S\|_{op}\|T\|_{op},$$

which tells us ST is bounded. So $ST \in B(H)$ and $\|ST\|_{op} \le \|S\|_{op}\|T\|_{op}$. □

We have now shown that $B(H)$ is infact a Banach algebra. To show that $B(H)$ is infact a C^*-algebra, we must show that adjoint elements exist, and that they satisfy the C^*-algebra axioms. Since we are working on bounded operators over Hilbert spaces, we have an inner product defined. This is the one property we have not yet used, infact, it turns out everything up until now works for bounded operators over any Banach space, but this is where the Hilbert space criteria is essential.

Theorem 2.19. *Let H be a Hilbert space. For every bounded linear operator T on H, there is a unique bounded linear operator T^* on H such that*

$$(Th|k) = (h|T^*k) \quad \text{for all } h,k \in H$$

Furthermore, the adjoint operation $T \mapsto T^*$ satisfies

(a) $(cS + dT)^* = \bar{c}S^* + \bar{d}T^*$,
(b) $(ST)^* = T^*S^*$,
(c) $(T^*)^* = T$,
(d) $\|T^*\|_{op} = \|T\|_{op}$,
(e) $\|T^*T\|_{op} = \|T\|_{op}^2$.

Proof. To show the existence of T^*, fix $k \in H$. Let $\phi_{T,k} : H \to \mathbb{C}$ by $\phi_{T,k}(h) = (Th|k)$ We want to show $\phi_{T,k}$ is a bounded linear functional.

$$\begin{aligned}\phi_{T,k}(ch_1 + h_2) &= (T(ch_1 + h_2)|k)\\ &= (cTh_1 + Th_2|k)\\ &= c(Th_1|k) + (Th_2|k)\\ &= c\phi_{T,k}(h_1) + \phi_{T,k}(h_2).\end{aligned}$$

Which shows $\phi_{T,k}$ is linear.

To show that $\phi_{T,k}$ is bounded,we note that

$$|\phi_{T,k}(h)| = |(Th|k)| \le \|Th\|\|k\| \le \|T\|_{op}\|h\|\|k\|.$$

This says that $\phi_{T,k}$ is bounded, and hence is a bounded linear functional.

Now, by Riesz Representation Theorem, there exists $Sk \in H$ such that

$$\phi_{T,k}(h) = (h|Sk) \quad \text{for all } h \in H,$$

and $\|\phi_{T,k}\|_{op} = \|Sk\|$.

Since this is for any arbitrary k, there is a function $S : H \to H$ satisfying

$$(h|Sk) = \phi_{T,k}(h) = (Th|k) \quad \text{for all } h,k \in H.$$

We now let $S = T^*$.

To show $T^* \in B(H)$, fix $k_1, k_2 \in H$, $c \in \mathbb{C}$ and let $h \in H$.

$$\begin{aligned}
(T^*(ck_1 + k_2)|h) &= \overline{(h|T^*(ck_1 + k_2)} \\
&= \overline{(Th|ck_1 + k_2)} \\
&= \overline{\bar{c}(Th|k_1) + (Th|k_2)} \\
&= c\overline{(Th|k_1)} + \overline{(Th|k_2)} \\
&= c\overline{(h|T^*k_1)} + \overline{(h|T^*k_2)} \\
&= c(T^*k_1|h) + (T^*k_2|h) \\
&= (cT^*k_1 + T^*k_2|h)
\end{aligned}$$

This says that $T^*(ck_1 + k_2) = cT^*k_1 + T^*k_2$, which says T^* is linear.

To show T^* is bounded, we recall that $\|\phi_{T,k}\|_{op} = \|Sk\| = \|T^*k\|$. So we have

$$\|T^*k\| = \|\phi_{T,k}\|_{op} \leq \|T\|_{op}\|k\| \implies T^* \text{ is bounded and } \|T^*\|_{op} \leq \|T\|_{op}.$$

Now, to show uniqueness, fix $h, k \in H$ and suppose $(h|T^*k) = (h|Sk)$.

$$\begin{aligned}
(h|T^*k) = (h|Sk) &\implies (h|T^*k) - (h|Sk) = 0 \\
&\implies (h|T^*k) - (h|Sk) = 0 \\
&\implies (h|T^*k - Sk) = 0 \\
&\implies (h|(T^* - S)k) = 0 \\
&\implies \|(T^* - S)k\| = 0 \\
&\implies (T^* - S)k = 0 \\
&\implies T^* - S = 0 \\
&\implies T^* = S
\end{aligned}$$

For all of the following, fix $h, k \in H$.

Now for the proof of (a), we have

$$\begin{aligned}
((cS + dT)h|k) &= (cSh|k) + (dTh|k) \\
&= c(Sh|k) + d(Th|k) \\
&= c(h|S^*k) + d(h|T^*k) \\
&= (h|\bar{c}S^*k) + (h|\bar{d}T^*k) \\
&= (h|\bar{c}S^*k + \bar{d}T^*k) \\
&= (h|(\bar{c}S^* + \bar{d}T^*)k).
\end{aligned}$$

Which says that $(cS + dT)^* = \bar{c}S^* + \bar{d}T^*$.

For (b), we calculate

$$(ST(h)|k) = (S(Th)|k) = (Th|S^*k) = (h|T^*S^*k).$$

which gives us $(ST)^* = T^*S^*$.

For (c), we simply note

$$(T^*h|k) = \overline{(k|T^*h)} = \overline{(Tk|h)} = (h|Tk) \implies (T^*)^* = T.$$

For (d), remembering that we already showed $\|T^*\| = \|T\|$, we then use part (c) to show

$$\|T\| = \|(T^*)^*\| \leq \|T^*\| \leq \|T\| \implies \|T\| = \|T^*\|.$$

And for (e), we begin by

$$\|T^*T\| \leq \|T\|\|T^*\| = \|T\|\|T\| = \|T\|^2.$$

Now for the reverse inequality

$$\|Th\|^2 = (Th|Th) = (h|T^*Th) \leq \|h\|\|T^*Th\| \leq \|T^*T\|\|h\|^2.$$

Which gives

$$\|T\|^2 = \sup_{\|h\|\leq 1} \|Th\|^2 \leq \sup_{\|h\|\leq 1} \|T^*T\|\|h\| \leq \|T^*T\|.$$

And so $\|T\|^2 \leq \|T^*T\| \leq \|T\|^2 \implies \|T\|^2 = \|T^*T\|$. □

We can see from this theorem that $B(H)$ satisfies all the final axioms of a C^*-algebra; thus $B(H)$ is infact a C^*-algebra. But there is more to the story.

Definition 2.20. Let A be a C^*-algebra. A *representation* of A on a Hilbert space H is a homomorphism π of A into the algebra $B(H)$. When A has an identity 1, we say that π is *nondegenerate* if $\pi(1) = 1$, in general π is nondegenerate if

$$\operatorname{span}\{\pi(a)h : a \in A, h \in H\}$$

is dense in H. A representation π is *faithful* if it is injective.

Theorem 2.21 (Gelfand-Naimark)**.** *Every C^*-algebra A has a faithful nondegenerate representation.*

So basically, this is saying that every C^*-algebra is isomorphic to $B(H)$ for some Hilbert space H. This is a big result, and is due to the Gelfand-Naimark-Segal theorem, which is a constructive proof. The GNS-construction provides the method to obtain what is known as the GNS-representation associated to f, where f is a positive functional on a C^*-algebra. Furthermore, it can be shown that for each element a of a C^*-algebra A, there is a functional f satisfying $f(a^*a) = \|a\|^2$. By taking the direct product of each functional on all elements over the C^*-algebra, we obtain the faithful nondegenerate representation. The proof of all this is outside of the focus of this paper.

2.4. Projections and Partial Isometries.

I have some stuff to put in here, plus some stuff further down will go in here...still editing, alex(aus)

2.5. **Matrix Algebras.** In this section, we seek to develop important aspects of matrix algebras that will aid our analysis of UHF and AF algebras in later sections. For the majority of this section, we will be in $M_n\mathbb{C}$. We first make note the following well known theorem, whose proof we omit:

Theorem 2.22. *A matrix A is unitarily diagonalizable if and only if it is normal (i.e., if $A^*A = AA^*$, then $A = UDU^*$ for some unitary U and diagonal matrix D).*

Furthermore, we state the following definition which is plays an important role in many algebras along with $M_n(\mathbb{C})$.

Definition 2.23. We say that two projections p and q in an algebra A are $Murray\text{-}von$ $Neumann\ Equivalent$ and write $p \sim q$ if there exists a $v \in A$ such that

$$vv^* = p \quad \text{and} \quad v^*v = q$$

Next, we define a standard map from linear algebra, the function *trace*. Here we would like to make a strong warning to the reader not to confuse the *trace* function with a function that is a trace, especially since the *trace* function is not a trace! This unfortunate convention is very standard and so to try and cause little confusion to the reader, who probably has some familiarity with the *trace* function, we follow said conventions.

Definition 2.24. Let $A = (a_{ij})_{1\leq i,j\leq n}$. Then we define

$$trace(A) = \sum_{i=1}^{n} a_{ii}$$

Remark 2.25. It follows from the definition that $trace : M_n(\mathbb{C}) \to \mathbb{C}$ is a linear map, and furthermore it is not hard to check that $trace(AB) = trace(BA)$. We also note that if A is a normal matrix, then if $A = UDU^*$ is the unitary diagonalization of A, $trace(A) = trace(UDU^*) = trace(U(DU^*)) = trace(DU^*U) = trace(D)$.

Lemma 2.26. *Let $P \in M_n(\mathbb{C})$ be an orthogonal projection matrix (i.e. $P^2 = P^* = P$). Then*

$$\mathbb{C}^n = range(P) \bigoplus ker(P)$$

Proof. We only need to show that

$$(\forall v \in \mathbb{C}^n)(\exists! u, w) \text{ such that } (v = u + w) \wedge (u \in range(P), w \in ker(P))$$

Let $v \in \mathbb{C}^n$ and let $w = v - Pv$. Then

$$Pw = P(v - Pv) = Pv - P^2v = Pv - Pv = 0 \Rightarrow w \in ker(P)$$

Therefore we have that $v = Pv + w$, where $Pv \in range(P)$ and $w \in ker(p)$ for an arbitary $v \in \mathbb{C}^n$.

We now show this representation is unique. First note that $range(P)$ and $ker(P)$ are subspaces of $\mathbb{C}^n$, and furthermore that $range(P) \cap ker(P) = \{0\}$. Suppose we have that

$$\exists u', w' \in \mathbb{C}^n \text{ such that } v = u + w = u' + w'.$$

Then

$$u - u' = w' - w \Rightarrow u - u' = w' - w = 0 \Rightarrow u = u', w = w'$$

□

Lemma 2.27. *Let $P \in M_n(\mathbb{C})$ be an orthogonal projection matrix. Then the eigenvalues of P are restricted to the values 0 and 1.*

Proof. Suppose we have, for some non-zero $x \in \mathbb{C}^n$, that $Px = \lambda x$ for some $\lambda \in \mathbb{C}$. Then let by the previous lemma, $x = u + v$, where $u \in ker(p)$, $v \in range(p)$. We then have that $Px = P(u+v) = u = \lambda x$, so x is a scalar multiple of u, and since the range of P is a subspace, $\lambda x \in range(P)$. Therefore

$$\lambda x = P(\lambda x) = \lambda P x = \lambda^2 x \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = 1$$

□

Lemma 2.28. *Let P be a projection. Then $rank(P) = trace(P)$. This also implies that for any projections P and Q, $trace(Q) = trace(P)$ if and only if $rank(P) = rank(Q)$.*

Proof. P is normal so we can write $P = UDU^*$ where U is a unitary and D is a diagonal matrix with the entries being restricted to the eigenvalues of P.

Note that

$$P(range(P)) = range(P) \text{ and } P(ker(P)) = \{0\}$$

so by lemma 2.26, the multiplicity of 1's in D is $m := dim(range(P))$. Since the rest of the entries in D have to be 0's, it follows that

$$trace(D) = m = dim(range(P)) = rank(P)$$

□

We next define what a trace function is.

Definition 2.29. Let $\tau : A \to \mathbb{C}$, where A is some unital C*-algebra. Then we call τ a trace function if it has the following properties:

(1) τ is linear
(2) For all $a \in A$, $\tau(a^* a) \subseteq [0, \infty)$, i.e. τ is positive.
(3) For all $a, b \in A$, $\tau(ab) = \tau(ba)$
(4) $\tau(I) = 1$

Remark 2.30. It follows directly from the definitons that for two projections P, Q in a C*-algebra, and for any trace τ, $\tau(P) = \tau(Q)$ if $P \sim Q$.

We now define an actual trace on a matrix algebra.

Proposition 2.31. *For $A = (a_{ij})_{1 \le i,j \le n}$, let*

$$tr(A) = \frac{1}{n} trace(A) = \frac{1}{n} \sum_{i=1}^{n} a_{ii}$$

Then tr is a trace function.

Proof. The linearity of tr follows from the linearity of $trace$, and $tr(I) = 1$ is also clear. Also, we get that

$$tr(AB) = \frac{1}{n} trace(AB) = \frac{1}{n} trace(BA) = tr(BA)$$

To show positivity, we multiply out and see that:

$$tr(A^* A) = \frac{1}{n} \sum_{i=1}^{n} \bar{a}_i a_i \ge 0$$

where a_i denotes the ith column of A, and $\bar{a}_i a_i$ denotes the standard inner product of a_i with itself in $\mathbb{C}^n$. This shows tr is indeed a trace function.

□

We note that one can easily see that again we have that for projections P and Q, $rank(P) = rank(Q)$ if and only if $tr(P) = tr(Q)$.

Proposition 2.32. *Let P and Q be projection matrices. Then $P \sim Q$ if and only if $tr(P) = tr(Q)$, which is itself equivalent to $rank(P) = rank(Q)$ Furthermore, $tr(P) = k/n$ for some $0 \le k \le n$, which holds for any projection matrix in general.*

Proof. Suppose $P \sim Q$. Then there exists a partial isometry V such that $VV^* = P$ and $V^*V = Q$. Therefore,

$$tr(P) = tr(VV^*) = tr(V^*V) = tr(Q)$$

Now suppose $tr(P) = tr(Q)$, and that $P, Q \in M_n(\mathbb{C})$. Then it follows from the definition that $trace(Q) = trace(P)$ and so by lemma 2.28, $rank(P) = rank(Q) = r$.

Let $P = UDU^*$ and $Q = WCW^*$ be the unitary diagonalization of P and Q. Then it follows that D and C are diagonal matrices with r 1's down the diagonal. Therefore they can be written as

$$D = \sum_{i=1}^{r} e_{P(i)P(i)} \quad \text{and} \quad C = \sum_{i=1}^{r} e_{Q(i)Q(i)}$$

where we let $\{e_{ij}\}_{i,j=1}^{n} = \{\{\delta_{ik}\delta_{jl}\}_{k,l=1}^{n}\}_{i,j=1}^{n}$ be simply the set of matrices with a 1 in the $(i,j)th$ entry, and 0's otherwise and that $1 \le P(i), Q(i) \le n$, and $P(i) = P(j)$ or $Q(i) = Q(j)$ implies that $i = j$. We note that $\{e_{ij}\}_{i,j=1}^{n}$ do indeed form a set of matrix units.

Now define the matrix

$$V = \sum_{i=1}^{r} e_{P(i)Q(i)}$$

Then since $e_{ij}^* = e_{ji}$ and $(e_{ij} + e_{kl})^* = e_{ij}^* + e_{kl}^*$ for all i and j, it follows that

$$VV^* = (\sum_{i=1}^{r} e_{P(i)Q(i)})(\sum_{i=1}^{r} e_{P(i)Q(i)})^* = (\sum_{i=1}^{r} e_{P(i)Q(i)})(\sum_{i=1}^{r} e_{Q(i)P(i)}) = \sum_{i=1}^{r} e_{P(i)P(i)} = D$$

and

$$V^*V = (\sum_{i=1}^{r} e_{P(i)Q(i)})^*(\sum_{i=1}^{r} e_{P(i)Q(i)}) = (\sum_{i=1}^{r} e_{Q(i)P(i)})(\sum_{i=1}^{r} e_{P(i)Q(i)}) = \sum_{i=1}^{r} e_{Q(i)Q(i)} = C$$

Now let $Y = UVW^*$. Then

$$YY^* = (UVW^*)((W^*)^*V^*U^*) = U(VV^*)U^* = UDU^* = P$$

and

$$Y^*Y = ((W^*)^*V^*U^*)(UV(W^*)) = W(V^*V)W^* = WCW^* = Q$$

Therefore $P \sim Q$. Also, it's clear that $tr(P) = tr(D) = r/n$, $0 \le r \le n$.

□

We now prove a fundamental fact about matrix algebras which will be important in the discussion of UHF algebras.

Lemma 2.33. *The function $tr : M_n(\mathbb{C}) \to \mathbb{C}$ is the unique trace on the algebra $M_n(\mathbb{C})$.*

Proof. By our previous lemma we have that tr is a trace function, so it remains to show that tr is the unique trace function on $M_n(\mathbb{C})$.

Suppose that we have some other trace function $\tau : M_n(\mathbb{C}) \to \mathbb{C}$. By previous lemmas, we know that all rank one projections are Murray-von Neumann equivalent and therefore have equal traces. We then find that, since each e_{ii} is a rank one projection,

$$1 = \tau(I) = \tau(\sum_{i=1}^{n} e_{ii})) = \sum_{i=1}^{n} \tau(e_{ii}) = n\tau(e_{11}) \Rightarrow \tau(e_{11}) = 1/n$$

and so all rank one projections have trace $1/n$, and more importantly, $\tau(e_{ii}) = 1/n$ for all i. Now we show $\tau(e_{ij}) = 0$ for $i \neq j$.

Note that we have that $e_{i1}e_{1j} = e_{ij}$, so for $i \neq j$,

$$\tau(e_{ij}) = \tau(e_{i1}e_{1j}) = \tau(e_{1j}e_{i1}) = \tau(0) = 0$$

since τ is linear. Therefore, $\tau(e_{ij}) = 0$ for $i \neq j$. So we find that for $A = \sum_{i,j=1}^{n} \lambda_{ij}e_{ij} \in (M_n\mathbb{C})$,

$$\tau(A) = \tau(\sum_{i,j=1}^{n} \lambda_{ij}e_{ij}) = \frac{1}{n}\sum_{i=1}^{n} \lambda_{ii} = tr(A)$$

so that $\tau(A) = tr(A)$ for all $A \in M_n(\mathbb{C})$, completing the proof. □

Proposition 2.34. *For $n, k \in \mathbb{N}$, there exists a unital *-homomorphism $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ if and only if $n|k$.*

Proof. First suppose that we have a unital *-homomorphism $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$. Then consider the function $\tau(A) = tr(\Phi(A))$. We want to show it is a trace on $M_n(\mathbb{C})$. For $A, B \in M_n(\mathbb{C}), \alpha, \beta \in \mathbb{C}$,

$$\tau(\alpha A + \beta B) = tr(\alpha\Phi(A) + \beta\Phi(B)) = \alpha tr(\Phi(A)) + \beta tr(\Phi(B)) = \alpha\tau(A) + \beta\tau(B)$$

so τ is linear. Furthermore we get that

$$\tau(I) = tr(\Phi(I_n)) = tr(I_k) = 1$$

and

$$\tau(AB) = tr(\Phi(AB)) = tr(\Phi(A)\Phi(B)) = tr(\Phi(B)\Phi(A)) = tr(\Phi(BA)) = \tau(BA)$$

Finally,

$$\tau(a^*a) = tr(\Phi(a^*a)) = tr(\Phi(a^*)\Phi(a)) = tr((\Phi(a))^*(\Phi(a)) \geq 0$$

so that τ is a trace on $M_n(\mathbb{C})$. But by the previous lemma, tr is the unique trace on $M_n(\mathbb{C})$, so it must be true that $tr(a) = \tau(a)$. Since *-homomorphisms send projections to projections, if we consider a rank one projection $q \in M_n(\mathbb{C})$, we get that

$$1/n = tr(q) = tr(\Phi(q)) = m/k \Rightarrow nm = k \Rightarrow n|k$$

Now suppose $n|k$, so that $nm = k$. Let $\{e_{ij}\}_{1\leq i,j\leq n}, \{f_{ij}\}_{1\leq i,j\leq k}$ be matrix units for $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$ respectively. Then let

$$\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C}) \text{ take } e_{ij} \longmapsto \sum_{l=1}^{m} e_{(j+nl)(i+nl)}$$

Our mapping Φ could be thought of as mapping matrices in $A \in M_n(\mathbb{C})$ to diagonal matrices in $M_m(M_n(\mathbb{C}))$ with the diagonal entries being the matrix A, i.e.

$$\Phi : A \longmapsto \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}$$

The linearity of Φ follows immediately, as well as the fact that $\Phi(I_n) = I_k$. The preservation of multiplication and the involution readily follow as well, showing Φ is as desired.

□

We now would like to show an important property of all matrix algebra automorphisms.

Theorem 2.35. *Every *-automorphism of $M_n(\mathbb{C})$ is inner, i.e. if $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a *-automorphism then there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $\Phi(A) = U^*AU$ for all $A \in M_n(\mathbb{C})$.*

Proof. We give two proofs: one abstract and standard, one concrete and not previously known to us.

Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a *-automorphism, so that we have M_n as a left module over itself in two ways: under left multiplication and under left multiplication after Φ. Since both M_n-modules are n^2-dimensional complex vector spaces, there exists a vector space isomorphism between them. The linear isomorphisms of $\mathbb{C}^{n^2}$ are just conjugation by units in M_n; furthermore, a unit thus producing Φ must preserve the standard Hermitian inner product to give rise to a $*$-automorphism, which requires that it be unitary.

Now we will work with bare hands to clarify that this representation is not unique.

Take Φ as above. Then Φ sends each matrix unit e_{ij} to another partial isometry with 1-dimensional coimage and range, and in particular sends e_{ii} to a 1-dimensional projection. With this fact, and noting that the images of the e_{ii} under Φ remain orthogonal, we may naturally define $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ as a Hilbert space automorphism taking each basis vector e_i to $\varphi(e_i)$ any unit-length element of the range of $\Phi(e_{ii})$.

This gives φ as any of a family of unitary matrices parameterized by n unit-magnitude complex numbers. We can reduce this underdetermination by requiring $\Phi(e_{ij})(\varphi(e_j)) = \varphi(e_i)$. Since the image of the matrix units under Φ observes $\Phi(e_{ij})\Phi(e_{kl}) = \delta_{jk}\Phi(e_{il})$, this restriction is satisfied everywhere as soon as it holds that $\Phi(e_{1j})(\varphi(e_j)) = \varphi(e_1)$ for each j.

This has reduced φ to a unitary matrix with a single free parameter on the unit circle: $\varphi : \mathbb{C}^n \to \mathbb{C}^n, v \mapsto \lambda U v, |\lambda| = 1$, since we have ratios between $\varphi(e_i), \varphi(e_j)$ but have not fixed any one basis vector. By the construction of $U_\lambda = \lambda U$, $U_\lambda e_{ij} U_\lambda^*(Ue_k) = \delta_{jk} U e_i$. That is, the system of matrix units of M_n under conjugation by each U_λ acts on the basis of $\mathbb{C}^n$ mapped under U in the same way as the matrix units under Φ do, which is enough to show Φ is represented by conjugation under U. □

Remark 2.36. The second proof above shows a bit more. Along the same line as the underdetermined unitary it produced to represent Φ, we can see directly that any unitary conjugation on M_n acts only up to multiplication by an element of the unit circle: for $A \in M_n$, $(\lambda U)A(\lambda U)^* = \lambda UA\bar{\lambda}U^* = UAU^*$. Thus the $*$-automorphism group of $M_n(\mathbb{C})$ is exactly the projective unitary group $PU_n = U_n/U_1$.

2.6. **UHF and AF algebras.**

Definition 2.37. A direct system is a set X with a preorder and pairwise upper bounds, that is, a relation $\leq$ that's transitive and reflexive and such that for every $x, y \in X$, $x \leq z$ and $y \leq z$ for some z. One convenient interpretation takes X as a category and $\leq$ as its morphisms; if there's at most one morphism between any two objects, then we need only further require the upper bounds.

Definition 2.38. A categorical direct limit over the direct system X is an object L equipped with morphisms φ_x from each $x \in X$ such that (1) whenever $\psi_{xy} : x \to y$ is a morphism in X, $\varphi_y \circ \psi_{xy} = \varphi_x$ and (2) L is initial with respect to this property: given any other L' satisfying (1), we have a unique map from L to L' commuting with the maps from X to each limit object.

Definition 2.39. Given a direct system S of C^* algebras and $*$-homomorphisms $\psi_{AB} : A \to B$, the C^*-direct limit is constructed as a quotient of the disjoint union of algebras is S modulo a certain equivalence relation $\sim$. We set $a \sim b$ for $a, b \in \sqcup_S A$ if a or b is an image of the other under some homomorphism in S. Since S contains the identity homomorphisms, it's immediate that $\sim$ gives an equivalence relation. The upper bound property of S allows us to define convergent sequences in the direct limit L: given $a \in A \in S, b \in B \in S$, $\|a - b\|_L = \|a - b\|_C$, where both A and B map into C. Then the completion of L as a C^*-algebra is with respect to this norm.

Proposition 2.40. *C^*-direct limits are categorical direct limits, and from here on we will refer to both as simply "direct limits."*

Proof. Let L be the C^*-direct limit of a direct system S. Since $\sim$ specifically identifies $A \in S$ with its images, it's immediate that the maps $\varphi_A : A \to L$ commute with the maps of S. So we'll check the universal property. Let L' have maps $\theta_A : A \to L'$ that commute with maps in S.

We'll define a map $u : L \to L'$ demonstrating L's universality. For points of L in the quotient of the original union, i.e., $a \in L \cap A, A \in S$, we simply set $u(a) = \theta(a)$. For limits of Cauchy sequences, $s = \lim_S s_A$, set $u(s) = \lim_S \theta(s_A)$. The limit exists since L' is C^*. It's immediate that for $a \in A \in S$, $u\varphi_A(a) = \theta_A(a)$ and that $\lim u(a) = u(\lim(a))$. Finally, $u\phi_B\psi_{AB}(a) = u\phi_A(a) = \theta_A(a) = \theta_B\psi AB(a)$, so the diagram induced by u commutes. □

Definition 2.41. We say that a C*-algebra A is AF (approximately finite dimensional) if it is the direct limit of a countable system of finite dimensional C*-algebras. In particular, we call A UHF, for uniformly hyperfinite, if it the direct limit of a countable system of full complex matrix algebras under their uniform norm.

The following two facts will make the classes AF and UHF greatly more tractable by making precise the objects which may be included in our direct systems and by obtaining a very simple direct system representing each isomorphism class.

Theorem 2.42. *Every finite dimensional C*-algebra B is isomorphic to a finite direct sum over matrix algebras, i.e.*

$B = \bigoplus_{k=1}^{m} M_{n(k)}(\mathbb{C})$

Theorem 2.43. *In any category, let L_1 be the direct limit of a direct system S_1 and L_2 of the system S_2. Suppose further that S_1 and S_2 are both countable. Take enumerations $n_1 : \mathbb{N} \to S_1, n_2 : \mathbb{N} \to S_2$, and suppose given for every i and some $j, k > i$ $\chi_{ij} : n_1(i) \to n_2(j)$ and $\omega_{ik} : n_2(i) \to n_1(k)$. Then $L_1 \cong L_2$.*

Proof. Blindingly obvious to anyone who's finished elementary school. □

Corollary 2.44. *Let A be an AF algebra. Then A is isomorphic to the direct limit of any totally ordered direct system of algebras cofinal with the underlying direct system of A.*

Proof. Take some such cofinal system as S_1, S_2 the underlying direct system of A, n_2 any enumeration of S_2, and define n_1 to agree with n_2 on $S_1 \subset S_2$. To make n_1 total, we can add redundant copies of the elemnts of S_2 so that for $a, b \in S_1$ if $a = n_2(k), b = n_2(k+m)$ and no other element of S_2 occurs in between, $n_2(j) = a$ for each $j \in \{k, k+1, ..., k_m\}$. Then the previous theorem applies. □

This gives us a very nice characterization of isomorphism classes of UHF algebras.

Definition 2.45. For $\{k_n\}$, an increasing sequence of natural numbers, we define a *generalized integer* by the formal product

$\kappa = \prod_{p\ prime} p^{k(p)}$

where $k(p) = \sup\{i : p^i | k_n \text{ for some } n \in \mathbb{N}\} \in \mathbb{N} \cup \{\infty\}$

Given a UHF algebra $A = \varinjlim M_{n_i}(\mathbb{C})$, the *generalized integer* of A is

$\kappa_A = \prod_{p\ prime} p^{k_A(p)}$

where $k_A(p) = k(p)$ with $\{k_i\} = \{n_i\}$.

An example is now in order.

Example 2.46. Let $A = \varinjlim M_{2^n}(\mathbb{C})$. Then we see that $\kappa_A = 2^\infty$, and that $k_A(p) = 0$ for primes $p \neq 2$. It also follows that only numbers of the form 2^n, for some $n \in \mathbb{N}$, have that $2^n | \kappa_A$. A is called the CAR algebra and is often denoted $A = M_{2^\infty}$. Similarly we have that $M_{3^\infty} = \varinjlim M_{3^n}(\mathbb{C})$

Proposition 2.47. *There exists a UHF algebra with each generalized integer k.*

Proof. The "greatest" generalized integer $\prod_p p^\infty$ characterizes the universal UHF algebra, which we may construct as the limit of the system

$$M_2 \to M_6 \to M_12 \to M_36 \to M_180 \to ...$$

The pattern here is the same as that used to enumerate $\mathbb{N}^2$: add a 2, then a 3, a 2, a 3, a 5, a 2, 3, 5, 7, and so on. Certainly this will give the universal UHF algebra in the limit.

It's apparent how we can use this construction for a "smaller" generalized integer: simply leave out stages of the above limit at a matrix algebra divisible by too high a power of any prime, and divide the later dimensions by the prime that would have been factored in. □

Corollary 2.48. *The isomorphism classes of UHF algebras are in one-to-one correspondence with the generalized integers.*

Proof. We have only to point out that each limit of a totally ordered sequence of matrix algebras gives rise uniquely to a generalized integer. □

Lemma 2.49. *All UHF algebras have a unique trace.*

Proof. Let $A = \varinjlim M_{n_i}(\mathbb{C})$ be a UHF algebra. By lemma **??**, each $M_{n_i}(\mathbb{C})$ has a unique trace τ_i. Furthermore, by definition of the UHF algebra , we see that $\tau_i = \tau_{i-1}$ when restircted to $M_{n_{(i-1)}}(\mathbb{C})$. Therefore we can get a well-defined trace τ' on the dense subset $\bigcup_{i\in\mathbb{N}} M_{n_i}(\mathbb{C})$ of A, where $\tau' = \tau_i$ when restricted to $M_{n_i}(\mathbb{C})$. Since the trace function is norm -continuous, it follows that τ' extends uniquely to a trace τ on all of A.

Now we prove uniqueness of τ. Suppose θ is a trace function on A. Then it follows that for all $i \in \mathbb{N}$, $\theta = \tau_i$ when restriced to $M_{n_i}(\mathbb{C})$ due to the uniquenes s of the trace on each matrix algebra. Therefore, θ agrees with τ on a a dense subset of A, and again we use that the trace function is norm continuous to see that $\theta = \tau$ on all of A.

□

3. Applications of Model Theory to C^*-algebras

3.1. **Types and stability.** These few pages will deal with the stability of a few important formulas. The motivation and goal is to show that our matrix units formula is indeed stable, and similarly we have the same is true about our direct sum matrix units formula.

We note that we always quantify over balls of radius 1 so all elements in this section will be assumed to have at most norm 1. We also note that given two n-tuples $\bar{x}, \bar{y}$ in some normed algebra, we use the convention that $\|\bar{x} - \bar{y}\| = \max_{1\le i\le n} \|x_i - y_i\|$.

It should also be noted that in a few of the proofs, the calculation of δ is omitted, the meaning of this being clear from the context. The proofs were written so that the calculation might be accessible to an interested reader. The importance of the δ calculation is not very high in the case of this paper.

The techniques in the proofs that follow will rely heavily on continuous functional calculus and facts about partial isometries and projection.

We begin by showing that orthogonal projections are stable. We define the formula $\rho(p) = \|p^2 - p\| + \|p^* - p\|$ and note that it equals 0 if and only if p is a projection.

Proposition 3.1. *Let A be a C^*-algebra. Then orthogonal projections are stable, i.e. $\forall\epsilon > 0\ \exists\delta(\epsilon) = \delta > 0$ such that if $x \in A$, $\rho(x) = \|x - x^*\| + \|x - x^2\| \le \delta$, then $\exists q \in A$ such that $\rho(q) = 0$ and $\|q - x\| \le \epsilon$.*

Proof. Fix $\epsilon > 0$ and let $\delta = \min\{(1/4)^2, (\epsilon/4)^2\}$. Suppose we have that $\rho(x) \le \delta$.

First note that $a = \frac{x+x^*}{2} \in A$ is self-adjoint and $\|a - x\| = (1/2)\|x^* - x\| \le \delta/2 < \epsilon/2$. We want to show that $\|a^2 - a\|$ is small so that it will also be "close" to satisfying the conditions of a projection. Then since it is self-adjoint we will be able to apply continuous functional calculus to its spectrum to find a projection.

First, we get the following two inequalities:

$$\|xa - x\| = (1/2)\|x^2 + xx^* - 2x\| \le (1/2)(\|x^2 - x\| + \|xx^* - x^2 + x^2 - x\|) \le \frac{3}{2}\delta$$

Similarly, it is not hard to show $\|ax - x\| \le \frac{3}{2}\delta$. Now we get an estimate on $\|a^2 - a\|$:

$$\begin{aligned}
\|a^2 - a\| = \|a^2 - x + x - a\| &\le \|a^2 - x\| + \delta/2 \\
&\le \|a^2 - ax\| + \|ax - x\| + \delta/2 \\
&\le \|a^2 - ax - xa + xa + x^2 - x^2\| + 2\delta \\
&\le \|(a - x)^2\| + 3\delta \\
&\le \delta^2 + 3\delta < 4\delta.
\end{aligned}$$

Now, by the continuous function calculus, we have that $f(t) = t$ is the continuous functional representation of a on $\sigma(a)$, and so we get that

$$\|a^2 - a\| = \|t^2 - t\|_\infty = \sup_{t\in\sigma(a)} |t^2 - t| \le 4\delta$$

so that $|t^2 - t| < 4\delta$ for all $t \in \sigma(a)$. From this inequality it follows that $\sigma(a) = range(f) \cap [\,\frac{1}{2} - \sqrt{\frac{1}{4} - 4\delta}, \frac{1}{2} + \sqrt{\frac{1}{4} - 4\delta}\,] = \varnothing$, and we note that $\frac{1}{2} - \sqrt{\frac{1}{4} - 4\delta} \le 2\sqrt{\delta}$ and $\frac{1}{2} + \sqrt{\frac{1}{4} - 4\delta} \ge 1 - 2\sqrt{\delta}$.

Next, we bound $f(t)$ above and below. Since we are working in the unit ball of radius 1, the spectral radius of a must be less than or equal to 1, so that $\|t\| \le 1$ $\forall t \in \sigma(a)$. For a lower bound, t^2 is non-negative for all $t \in \sigma(a)$, and so

$$t > t^2 - 4\delta \ge -2\sqrt{\delta}$$

It now follows that $\sigma(a) \subset (-2\sqrt{\delta}, 2\sqrt{\delta}) \cup (1 - 2\sqrt{\delta}, 1)$. Now let g be the continuous function which maps the first interval to 0 and the second to 1. Let $q = g(a) \in C^*(a) \subset A$. Since $\sigma(q) = \{0, 1\}$ we have that q is a projection. Furthermore since $\|a - q\| = \|f - g\|_\infty$, and $\|g(t) - f(t)\| < 2\sqrt{\delta} = \epsilon/2$ for all $t \in \sigma(a)$,

$$\|x - q\| \le \|x - a\| + \|a - q\| \le \epsilon/2 + \epsilon/2 = \epsilon$$

completing the proof.

□

We next recall the following formula, which is meant to define a set of nxn matrix units in our algebra.

$$\psi_n(\bar{x}) = \|\sum_{i}^{n} x_{ii} - I\| + \sum_{i,j,k,l}^{n} \|x_{ij}x_{kl} - \delta_{jk}x_{il}\| + \sum_{i,j}^{n} \|x_{ij}^* - x_{ji}\|$$

We would like to show $psi_n(\bar{x}) = 0$ is a stable condition, and we will use a sequence of lemmas to do so. First we will show that if we have a set in our unital C*-algebra of elements which are nearly orthogonal projections, almost sum up to the identity, and nearly multiply pairwise to 0, we can find an actual set of such elements that actually have all of those properties. These are meant to satisty important properties of the diagonal elements of a potential copy of a matrix algebra in our C*-algebra.

Lemma 3.2. *Let A be a unital C^*-algebra. Then let*

$$\tau_n(\bar{x}) = \|\sum_{i=1}^{n} x_i - I\| + \sum_{i,j=1}^{n} \|x_ix_j - \delta_{ij}x_i\| + \sum_{i=1}^{n} \|x_i^* - x_i\|$$

Then $\tau_n(\bar{x}) = 0$ is stable.

Proof. Fix $\epsilon > 0, n \in \mathbb{N}$. Suppose we have that $\tau_n(\bar{a}) \leq \delta$ for some $\bar{a} \in A$, and some δ which will be very small. First note that we have that

$$\|a_1 - a_1^*\| + \|a_1 - a_1^2\| \leq \delta$$

so by our previous lemma we have that there exists an orthogonal projection e_1 such that $\|e_1 - a_1\| \leq \delta_1$. consider the subalgebra $A_1 = (I - e_1)A(I - e_1)$ of A (a subalgebra since it is a closed subspace and is closed under all *-polynomials), and note that all vectors in it are orthogonal to e_1. Furthemore, it has its own subalgebra "identity" $I_1 = I - e_1$.

Now suppose we have found a set of $\{e_i\}_{i=1}^m$, $1 < m < n-1$, such that each e_i is an orthogonal projection, $e_ie_j = e_je_i = 0$ for $i \neq j$, and $\|e_i - a_i\| \leq \delta_i$ for all $i \leq m$. Assume also that we have a set of $\{I_i\}_{i=1}^m$, where $I_1 = I - e_{11}$, and $I_i = I - \sum\limits_{j=1}^{i} e_j$.

We then consider the subalgebra (a subalgebra since it is a closed subspace and is closed under all *-polynomials) $A_m = I_mAI_m$, and note that all of its elements multiply with each e_i for $i \leq m$ to 0. Consider $y_{m+1} = I_m a_{m+1} I_m$. We want to show that y_{m+1} is very close to a_{m+1} and that it is also very close to being a projection. First we look at:

$$\begin{aligned}
\|y_{m+1} - a_{m+1}\| &= \|(I - \sum_{j=1}^{m} e_j)a_{m+1}(I - \sum_{j=1}^{m} e_j) - a_{m+1}\| \\
&= \|(\sum_{j=1}^{m} e_j)a_{m+1}(\sum_{j=1}^{m} e_j) - (\sum_{j=1}^{m} e_j)a_{m+1} - a_{m+1}(\sum_{j=1}^{m} e_j)\| \\
&\leq \|(\sum_{j=1}^{m} e_j)\|(\sum_{i=1}^{m} \|e_ia_{m+1}\|) + (\sum_{i=1}^{m} \|e_ia_{m+1}\|) + (\sum_{i=1}^{m} \|a_{m+1}e_i\|)
\end{aligned}$$

Note that, for $1 \leq i \leq n$,

$$\|e_ia_{m+1}\| = \|e_ia_{m+1} - a_ia_{m+1} + a_ia_{m+1}\| \leq 2\delta_m$$

and likewise we find that $\|a_{m+1}e_i\| \leq 2\delta_m$. Therefore

$$\|(\sum_{j=1}^{m} e_j)\|(\sum_{i=1}^{m} \|e_ia_{m+1}\|) + (\sum_{i=1}^{m} \|e_ia_{m+1}\|) + (\sum_{i=1}^{m} \|a_{m+1}e_i\|) \leq 6m^2\delta_m$$

Now we just need to show that y_{m+1} is very close to being a projection. First for the adjoint,

$$\|y_{m+1} - y_{m+1}^*\| = \|(I - \sum_{j=1}^{m} e_j)(a_{m+1} - a_{m+1}^*)(I - \sum_{j=1}^{m} e_j)\| \leq \|a_{m+1} - a_{m+1}^*\| \leq \delta \leq \delta_m$$

And now to show it is nearly idempotent,

$$\begin{aligned}\|y_{m+1}-y_{m+1}^2\| &= \|(I-\sum_{j=1}^{m}e_j)[a_{m+1}-a_{m+1}(I-\sum_{j=1}^{m}e_j)a_{m+1}](I-\sum_{j=1}^{m}e_j)\| \\ &\leq \|a_{m+1}-a_{m+1}(I-\sum_{j=1}^{m}e_j)a_{m+1}\| \\ &\leq \|a_{m+1}-a_{m+1}^2+a_{m+1}\sum_{j=1}^{m}e_ja_{m+1}\| \\ &\leq (m+1)\delta+\sum_{i=1}^{m}\delta_i \leq \delta_m(2m+1)\end{aligned}$$

so that $\|y_{m+1}-y_{m+1}^2\|+\|y_{m+1}-y_{m+1}^*\| \leq \delta_m(2m+1)+\delta_m = 2\delta_m(m+1)$.

Now, we can again find an orthogonal projection e_{m+1} with $\|e_{m+1}-a_{m+1}\| \leq \delta_{m+1}$, in our subalgebra, by lemma 3.1. Since it is in the subalgebra I_mAI_m it will multiply with e_i to 0 on the left and right for $i \leq m$.

Using the induction we just proved above, we find a set $\{e_i\}_{i=1}^{n-1}$ of orthogonal projections such that $\|e_i-a_i\| \leq \delta_i$ for all i and each multiplies with one another to 0 on the left and right. Let $e_n = I-\sum_{i=1}^{n-1}e_i$. This will give an orthogonal projection which multiplies on the left and right to 0 with each e_i for $i<n$, and gives that $\sum_{i=1}^{n}e_i = I$. Finally, we will show it is very close to a_n.

$$\|a_n-e_n\| = \|\sum_{i=1}^{n}a_i-I+\sum_{i=1}^{n-1}e_i-\sum_{i=1}^{n-1}a_i+I-\sum_{i=1}^{n}e_i\| \leq \delta+\sum_{i=1}^{n-1}\|a_i-e_i\| \leq n\delta_{n-1}$$

One can see that each δ_i ultimately only was dependent on the original choice of δ, which itself was only dependent on ϵ and n, and also that $\delta_i \leq \delta_j$ for $i \leq j$. Although we do not explicitly calculate δ we just make it small enough so that, assuming ϵ is also very small, $n\delta_{n-1} \leq \epsilon$, and then $\|a_i-e_i\| \leq \epsilon$ should follow for all i. Now have the desired set of e_i's.

□

We next define the formula $\nu(v) = \|(vv^*)-(vv^*)^2\|$, which we note has that $\nu(v)=0$ if and only if v is a partial isometry.

Lemma 3.3. *Let A be a C^*-algebra. Then for all $\epsilon>0$ there exists a $\delta>0$ such that if we have, for some $a \in A$, $\nu(a) = \|(aa^*)^2-aa^*\| \leq \delta$, then there exists a $w \in C^*(a)$ such that $\nu(w)=0$ and is a partial isometry, and has that $\|w-a\| \leq \epsilon$. We therefore also have that the condition of being a partial isometry is stable.*

Proof. Fix $\epsilon>0$ and let $\delta = \min\{(\epsilon/4)^8/4, (1/4)^8/4\}$. Suppose that we have, for some $a \in A$, $\|(aa^*)^2-aa^*\| \leq \delta$, for some very small δ. We want to show that we also get that $\||a|^2-|a|\| \leq (2\sqrt{\delta})^{1/2}$, where $|a| = (aa^*)^{1/2}$.

Since aa^* is clearly self adjoint it satisfies the hypotheses of proposition 3.1, and so if we apply continuous functional calculus to its spectrum, we find in the same way that

$$\sigma(aa^*) \subseteq [0,2\sqrt{\delta}] \cup [1-2\sqrt{\delta},1]$$

with the bound below being now 0 since aa^* is positive and so can have only non-negative spectral values.

Therefore, since $\sigma(aa^*) \subset [0,\infty)$, $g : \sigma(aa^*) \to \mathbb{C}$ which takes $t \longmapsto \sqrt{t}$ is a continuous, real-valued function on the spectrum of aa^*, and $g(aa^*) = |a|$. So, we get that $\||a|^2 - |a|\| = \|t - \sqrt{t}|_\infty = \sup_{t\in\sigma(aa^*)} |t - \sqrt{t}|$. For $t \in [1 - 2\sqrt{\delta}, 1]$,

$$|t - \sqrt{t}| = \sqrt{t} - t \leq 2\sqrt{\delta}$$

For $t \in [0, 2\sqrt{\delta}]$,

$$|t - \sqrt{t}| = \sqrt{t} - t \leq \sqrt{2\sqrt{\delta}} - 0 = (2\sqrt{\delta})^{1/2}$$

So we get that $\||a|^2 - |a|\| \leq (2\sqrt{\delta})^{1/2}$. Since $|a|$ is also self-adjoint, we can now fully apply proposition 3.1 and get a projection p such that $\||a| - p\| \leq \epsilon$, since $(2\sqrt{\delta})^{1/2} = \min\{(\epsilon/4)^2, (1/4)^2\}$.

Now we consider the polar decomposition of $a = |a|v$. We will find a partial isometry $w \in C^*(a)$ very close to a.

Our strategy will be to use continuous functional calculus. Because we need a normal element to apply the theory, we will use $|a| = (aa^*)^{1/2}$, and note that $C^*((aa^*)^{1/2}) \subset C^*(a)$. By the above argument, we have that

$$\sigma(|a|) \subset [0, \sqrt{\delta}) \cup (1 - \delta, 1]$$

Let $f : \sigma(|a|) \to \mathbb{C}$ be such that $f(t) = 0$ for $t \in [0, \sqrt{\delta})$ and $f(t) = 1/t$ for $t \in (1 - \delta, 1]$. Then f is clearly continuous, and we see that since the functional representation of $|a|$ is simply t so that $f(|a|)|a|$ is a projection. Furthermore, it is not hard to see that $f(|a|)|a| = p$.

Now, let $w = pv$. Then

$$w = pv = f(|a|)|a|v = f(|a|)a \in C^*(a)$$

And w is a partial isometry because

$$ww^* = f(|a|)aa^*f(|a| = f(|a|)|a|^2 f(|a|) = p^2 = p$$

which is a projection, and

$$(w^*w)^2 = a^*f(|a|)^2|a|^2 f(|a|)^2 a = a^*(pf(|a|)^2)a = a^*f(|a|)^2 a = w^*w$$

We see that $pf(|a|)^2 = f(|a|)^2$ follows from the fact that whenever $t < 1/2$, $p = 0 = f(|a|)^2$, and when $t > 1/2$, $p = 1$, and so $pf(|a|)^2 = f(|a|)^2$. So $w \in C^*(a)$ and ww^* and w^*w are both projections and therefore w is a partial isometry. It only remains to check that it is very close to a.

$$\|a - w\| = \||a|v - pv\| \leq \||a| - p\| \leq \epsilon$$

completing the lemma. □

Lemma 3.4. *Suppose P and Q are projections in a unital C^*-algebra. Then $P - Q$ is a projection if either $PQ = Q$ or $QP = Q$, i.e. if either range$(P) \subseteq$ range(Q) or range$(Q) \subseteq$ range(P).*

Proof. Suppose $PQ = Q$. Then $P = P - Q + PQ$, and so

$$P - Q + PQ = (P - Q + PQ)^* = P^* - Q^* + Q^*P^* \Rightarrow PQ = QP$$

Therefore

$$(P-Q)^2 = (P-PQ)^2 = P^2 - P^2Q - PQP + (PQ)^2 = P - PQ - PQ + PQ = P - PQ = P - Q$$

Since $P - Q$ is self-adjoint, we get that $P - Q$ is a projection. The case of $QP = P$ is similar.

□

Corollary 3.5. *Let A be a unital C*-algebra. Then $\forall \epsilon > 0\ \exists \delta > 0$ such that if $\|a^*a - I\| + \|(aa^*)^2 - (aa^*)\| \le \delta$ then there exists a $u \in C^*(a)$ such that $\|u^*u - I\| + \|(uu^*)^2 - (uu^*)\| = 0$ and $\|u - a\| \le \epsilon$. This implies that the condition of being an isometry is stable.*

Proof. Fix $\epsilon > 0$, let $\delta = \min\{(\epsilon/4)^8/16, (1/4)^8/16\}$. By lemma **??** we can find a partial isometry $u = f(|a|)a \in C^*(a)$ (where f is the function defined in the previous lemma) such that $\|u - a\| \le \epsilon/4 < \epsilon$ and $\|u - a\| \le 1/4$.

We furthermore want to show that $u^*u = I$ We immediately see that $(u^*u)I = u^*u$, and since u is a partial isometry, by the previous lemma we get that $I - u^*u$ is a projection. With the help of the identity

$$u^*a = (f(|a|)|a|v)^*a = v^*|a|f(|a|)a = (|a|v)^*u = a^*u$$

we get that

$$\|u^*u - I\| \le \|u^*u - u^*a + a^*u - a^*a\| + 1/4 \le 3/4 < 1$$

Since the operator norm of any non-zero projection is always 1, it follows that $I - u^*u = 0 \Rightarrow u^*u = I$, completing the proof.

□

Theorem 3.6. *Let A be a C*-algebra. Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that is $\bar{a} \in A$ is such that $\psi_n(\bar{a}) \le \delta$ then there exists an $\bar{e} \in A$ with $\|\bar{a} - \bar{e}\| \le \epsilon$ such that $\psi_n(\bar{e}) = 0$.*

Proof. Fix $\epsilon > 0, n \in \mathbb{N}$. In what follows we will use δ's and δ_x's purely for helping to see the steps and inferences within the inequalities. They are each only dependent on the originial δ which is itself only dependent on ϵ and n. We do not calculate δ, for it is tedious and it is not our main motivation. We simply attempt to make it clear that so long as we make δ very very small, everything will work out nicely.

We first notice that the set $\{a_{ii}\}_{i=1}^n$ satisfy the hypotheses of lemma 1.3, and so we can find a set $\{e_{ii}\}_{i=1}^n$ of orthogonal projections who sum up to the identity, are very close to their respective a_{ii}, and have that $e_{ii}e_{jj} = 0$ for $i \neq j$.

Next we need to find a set of e_{ij}'s, $i \neq j$, close to the set of $a_{ij}'s$, such that

$$\sum_{i,j,k,l}^{n} \|e_{ij}e_{kl} - \delta_{jk}e_{il}\| + \sum_{i,j}^{n} \|e_{ij}^* - e_{ji}\| = 0$$

To do so we will find a set of e_{1i}'s, each of which is a partial isometry closely related to a_{1i} and have that $e_{1i}e_{1j} = 0$ for $i \neq j$.

We consider the subspace $e_{11}Ae_{ii}$, and look at the element $y_{1i} = e_{11}a_{1i}e_{ii}$. We will show $\|y_{1i} - a_{1i}\|$ can be made very small:

$$\begin{aligned}\|y_{1i}-a_{1i}\| = \|e_{11}a_{1i}e_{ii}-a_{1i}\| &= \|e_{11}a_{1i}e_{ii}-e_{11}a_{1i}a_{ii}\| + \|e_{11}a_{1i}a_{ii}-a_{1i}\| \\ &\leq \delta_i + \|e_{11}a_{1i}a_{ii}-a_{11}a_{1i}a_{ii}\| + \|a_{11}a_{1i}a_{ii}-a_{11}a_{1i}\| + \|a_{11}a_{1i}-a_{1i}\| \\ &\leq \delta_1+\delta_i+2\delta = \delta_{yi}\end{aligned}$$

Our goal is to now show that y_{1i} is nearly a partial isometry. First we show that a_{1i} is, and then show that the same will be true of y_{1i} due to how close they are to one another. Clearly $a_{1i}a_{1i}^*$ is self adjoint, so we need only to show that it is nearly idempotent.

First off, two inequalities relating a_{1i} and a_{11}:

$$\|a_{1i}a_{1i}^* - a_{11}\| = \|a_{1i}a_{1i}^* - a_{1i}a_{i1} + a_{1i}a_{i1} - a_{11}\| \leq 2\delta$$

$$\|(a_{1i}a_{1i}^*)^2 - a_{11}^2\| =\leq \|(a_{1i}a_{1i}^* - a_{11})^2\| + \|a_{11}a_{1i}a_{1i}^* - a_{11}^2\| + \|a_{1i}a_{1i}^*a_{11} - a_{11}^2\| \leq \delta^2 + 4\delta \leq 5\delta$$

and this gives us that

$$\|(a_{1i}a_{1i}^*)^2 - a_{1i}a_{1i}^*\| = \|(a_{1i}a_{1i}^*)^2 - a_{11}^2 + a_{11}^2 - a_{11} + a_{11} - a_{1i}a_{1i}^*\| \leq 5\delta + \delta + 2\delta = 8\delta$$

So a_{1i} is approximately a partial isometry. We will now show the same with y_{12}. We again do this by beginning with two inequalities:

$$\|y_{1i}y_{1i}^* - a_{1i}a_{1i}^*\| = \|y_{1i}y_{1i}^* - a_{1i}y_{1i}^* + a_{1i}y_{1i}^* - a_{1i}a_{1i}^*\| \leq 2\delta_{yi}$$

$$\|(y_{1i}y_{1i}^*)^2 - (a_{1i}a_{1i}^*)^2\| = \|(y_{1i}y_{1i}^*)^2 - a_{1i}a_{1i}^*y_{1i}y_{1i}^* + a_{1i}a_{1i}^*y_{1i}y_{1i}^* - (a_{1i}a_{1i}^*)^2\| \leq 4\delta_{yi}$$

Since $y_{1i}y_{1i}^*$ is self-adjoint, we will check it is almost idempotent:

$$\|(y_{1i}y_{1i}^*)^2 - y_{1i}y_{1i}^*\| \leq \|(y_{1i}y_{1i}^*)^2 - (a_{1i}a_{1i}^*)^2 + (a_{1i}a_{1i}^*)^2 - a_{1i}a_{1i}^* + a_{1i}a_{1i}^* - y_{1i}y_{1i}^*\| \leq 8\delta + 6\delta_{yi}$$

So now y_{1i} satisfies the hypotheses of the lemma **??**, and so we can find a partial isometry $w_i \in C^*(y_{1i})$ such that $\|w_i - y_{1i}\| \leq \gamma_i$ is very small.

Let $e_{1i} = w_i$. Then e_{1i} is very close to a_{1i}:

$$\|e_{1i} - a_{1i}\| = \|e_{1i} - y_{1i} + y_{1i} - a_{1i}\| \leq \delta_{yi} + \gamma_i$$

Next we will use lemma **??** about when the difference of projections is itself a projection. In what follows, by $f_i(|y_{1i}|)$ we are refering to the function used in lemma **??**.

We first show that $e_{11}e_{1i}e_{1i}^* = e_{1i}e_{1i}^* \iff range(e_{1i}e_{1i}^*) \subseteq range(e_{11})$:

$$\begin{aligned} range(e_{1i}e_{1i}^*) = e_{1i}e_{1i}^*A &= |y_{1i}|f_i(|y_{1i}|)A \\ &\subseteq (y_{1i}y_{1i}^*)^{1/2}A \\ &= (e_{11}a_{1i}a_{1i}^*e_{11})^{1/2}A \\ &= (e_{11}^2e_{11}a_{1i}a_{1i}^*e_{11})^{1/2}A \\ &= e_{11}(e_{11}a_{1i}a_{1i}^*e_{11})^{1/2}A \subseteq e_{11}A = range(e_{11}) \end{aligned}$$

Next we get that: $e_{ii}e_{1i}^*e_{1i} = e_{1i}e_{1i}^* \iff range(e_{1i}^*e_{1i}) \subseteq range(e_{ii})$:

$$range(e_{1i}^*e_{1i}) = e_{1i}^*e_{1i}A = y_{1i}^*f_i(|y_{1i}|)^2y_{1i} \subseteq y_{1i}^*A =\subseteq e_{ii}A = range(e_{ii})$$

So now we have that $e_{11} - e_{1i}e_{1i}^*$ and $e_{ii} - e_{1i}^*e_{1i}$ are projections. First note that $\|a_{1i}a_{1i}^* - a_{11}\|$ can be made very small, and since $\|a_{11} - e_{11}\|$ and $\|a_{1i}a_{1i}^* - e_{1i}e_{1i}^*\|$ can also be made very small, we can get that $\|e_{1i}e_{1i}^* - e_{11}\| < 1$, so that $e_{1i}e_{1i}^* = e_{11}$. Likewise one can show that $e_{1i}^*e_{1i} = e_{ii}$.

Finally, we will show $\|e_{i1} - a_{i1}\|$ can be made very small:

$$\|e_{i1} - a_{i1}\| = \|e_{1i}^* - a_{1i}^* + a_{1i}^* - a_{i1}\| \leq \delta_{ei} + \delta$$

We now note that all partial isometries w are characterized by the equality $w = ww^*w$, so that

$$e_{1i}e_{ii} = e_{1i}e_{1i}^*e_{1i} = e_{1i}$$

and

$$e_{11}e_{1i} = e_{1i}e_{1i}^*e_{1i} = e_{1i}$$

Therefore we also get that, for $k \neq i$, $j \neq 1$,

$$e_{1i}e_{kk} = e_{1i}(e_{ii}e_{kk}) = e_{jj}e_{12} = (e_{jj}e_{11})e_{1i} = 0$$

When we look at e_{1i}^*, we can now use that it is also a partial isometry and so

$$e_{1i}^*e_{11} = e_{1i}^*e_{1i}e_{1i}^* = e_{1i}^*$$

and

$$e_{ii}e_{1i}^* = e_{1i}^*e_{1i}e_{1i}^* = e_{1i}^*$$

Therefore again we see that for $k \neq 1$, $j \neq i$,

$$e_{1i}^*e_{kk} = e_{1i}^*(e_{11}e_{kk}) = e_{jj}e_{1i}^* = (e_{jj}e_{11})e_{1i}^* = 0$$

For e_{1i} and e_{1i}^*, these properties can be summarized by $e_{1i} \in e_{11}Ae_{ii}$ and $e_{1i}^* \in e_{ii}Ae_{11}$. We also note that this implies that $e_{1j}e_{1k}^* = \delta_{jk}e_{11}$, where here δ_{jk} denotes the Kronecker delta.

Now we can finally let $e_{ij} = e_{1i}^*e_{1j}$ for $i \geq 2$, $j \geq 1$ $j \neq i$. We then get the important property that

$$e_{ij}e_{kl} = e_{1i}^*e_{1j}e_{1k}^*e_{1l} = \delta_{jk}e_{1i}^*e_{11}e_{1l} = \delta_{jk}e_{1i}^*e_{1l} = \delta_{jk}e_{il}$$

with δ_{jk} again denoting the Kronecker delta. Since each e_{ij} is defined to satisfy the property $e_{ij}^* = e_{ji}$, we find that our entire set of e_{ij}'s is as desired. Now we need to show that e_{ij} is very close to a_{ij} for $i, j \geq 2$ and $i \neq j$.

$$\begin{aligned}
\|e_{ij} - a_{ij}\| &= \|e_{1i}^*e_{1j} - e_{1i}^*a_{1j} + e_{1i}^*a_{1j} - a_{ij}\| \\
&\leq \delta_{ej} + \|e_{1i}^*a_{1j} - a_{1i}^*a_{1j} + a_{1i}^*a_{1j} - a_{ij}\| \\
&\leq \delta_{ej} + \delta_{ei} + \|a_{1i}^*a_{1j} - a_{i1}a_{1j} + a_{i1}a_{1j} - a_{ij}\| \\
&\leq \delta_{ej} + \delta_{ei} + 2\delta
\end{aligned}$$

We again make note that all δ_x's depend only on our original δ, so we make δ very small, completing the proof.

$\square$

Next, we show the stability of the following formula. Suppose we are in a C*-algebra A, and let $\bar{n} = (n(1), n(2), \ldots, n(m)) \in \mathbb{N}^m$ and $\bar{x} = (\bar{x}^{(1)}, \bar{x}^{(2)}, \ldots, \bar{x}^{(m)}) \in A^\eta$ for some $m \in \mathbb{N}$, where $\eta = \sum_{i=1}^{m} n(i)^2$. The we define

$\Psi_{\bar{n}}(\bar{x}) = \sum_{i=1}^{m} \varphi_{n(i)}(\bar{x}^{(i)}) + \| \sum_{i=1}^{m} \sum_{j=1}^{n(i)} x_{jj}^{(i)} - I\|$

where

$\varphi_{n(i)}(\bar{x}^{(i)}) = \sum_{h,j,k,l=1}^{n(i)} \|x_{hj}^{(i)} x_{kl}^{(i)} - \delta_{jk} x_{hl}^{(i)}\| + \sum_{h,j=1}^{n(i)} \|x_{hj}^{(i)*} - x_{jh}^{(i)}\|$

Proposition 3.7. *The condition $\Psi_{\bar{n}}(\bar{x}) = 0$ is stable, i.e. $\forall \epsilon > 0$ $\exists \delta(\epsilon) = \delta > 0$ such that if $\bar{a} \in A^\eta$, $\Psi_{\bar{n}}(\bar{a}) \leq \delta$, then $\exists \bar{e} \in A^\eta$ such that $\Psi_{\bar{n}}(\bar{e}) = 0$ and $\|\bar{e} - \bar{a}\| \leq \epsilon$.*

Proof. The proof is very similar to that of the previous theorem. We first begin by considering the set of $a_{jj}^{(i)}$'s and noticing that they satisfy the hypotheses of lemma 1.3. So we get a set of orthogonal projections $e_{i_{jj}}$'s being pairwise orthogonal and summing up to the identity.

To find the sets of $e_{jk}^{(i)}$'s, $j \neq k$, $1 \leq i \leq m$, we apply exact same method we used to find off-diagonal elements in the previous proof. For a fixed i, consider the $n(i)xn(i)$ matrix with non-diagonal entries $a_{jk}^{(i)}$ and diagonal entries $e_{jj}^{(i)}$. Then we again consider $y_{1k}^{(i)} = e_{11}^{(i)} a_{1k}^{(i)} e_{kk}^{(i)}$, and through the exact same inequalites find that each $y_{1k}^{(i)}$ is very close to each $a_{1k}^{(i)}$ and is nearly a partial isometry. Applying lemma 1.4, we get a set of partial isometries $e_{1k}^{(i)}$ very close to $a_{1k}^{(i)}$ and again show, through the exact same steps, the the $e_{1k}^{(i)}$'s satisfy all the desired properties. If this is done for every i, we are finished. ∎

Corollary 3.8. *Let A be a unital C*-algebra, and let*

$$\hat{\Psi}_{\bar{n}}(\bar{x}) := \Psi_{\bar{n}}(\bar{x}) + \sum_{i=1}^{n} (1 \dot{-} \|x_{11}^{(i)}\|)$$

Then the condition $\hat{\Psi}_{\bar{n}}(\bar{x}) = 0$ is stable.

Proof. Without loss of generality, assume $0 < \epsilon < 1/2$, and suppose we have that $\hat{\Psi}_{\bar{n}}(\bar{a}) \leq \delta$ for a very small $\delta \leq 1/2$. Then by our previous proposition, we can choose a $\bar{e} \in A$ such that $\|\bar{a} - \bar{e}\| \leq \epsilon$ and has that $\Psi_{\bar{n}}(\bar{e}) = 0$. We then find that for each $1 \leq i \leq n$, since $\|e_{11}^{(i)}\| \leq 1$,

$1 \dot{-} \|e_{11}^{(i)}\| = \max\{1 - \|e_{11}^{(i)}\|, 0\} \leq 1 - \|e_{11}^{(i)}\| = |1 - \|a_{11}^{(i)}\|| + |\|a_{11}^{(i)}\| - \|e_{11}^{(i)}\|| < 1/2 + \|a_{11}^{(i)} - e_{11}^{(i)}\| < 1/2 + 1/2 = 1$

This implies that $1 - \|e_{11}^{(i)}\| < 1$ $\forall i$, and since the operator norm of any non-zero matrix unit is 1, it follows that $1 - \|e_{11}^{(i)}\| = 0$, and so $\hat{\Psi}_{\bar{n}}(\bar{e}) = 0$. ∎

We now show, for their own sake, that being a unitary is a stable condition.

Proposition 3.9. *Let A be a unital C*-algebra. Then $\forall \epsilon > 0$ $\exists \delta > 0$ such that if $\|aa^* - I\| + \|a^*a - I\| \leq \delta$ then there exists a $u \in C^*(a)$ such that $\|uu^* - I\| + \|u^*u - I\| = 0$ and $\|u - a\| \leq \epsilon$. This implies that the condition of being a unitary operator is stable.*

Proof. Fix $\epsilon > 0$ and let $\delta = \min\{1/4, \epsilon\}$. Suppose we have an $a \in A$ such that $\|aa^* - I\| + \|a^*a - I\| \leq \delta$.

Begin by noting that aa^* is a normal operator, so we can apply continuous functional calculus. If we let $t \in C(\sigma(aa^*))$ be the identity function representing aa^*, then we have that

$$\|t - 1\|_\infty = \sup_{t \in \sigma(aa^*)} |t - 1| = \|aa^* - I\| \leq \delta$$

so $\sigma(aa^*) \subset [1 - \delta, 1 + \delta]$.

Now by polar decomposition, we have that $a = |a|u$, where $|a| = (aa^*)^{1/2}$ and u is a partial isometry. We will show u is a unitary operator close to a, and so our desired element.

Since $|a|$ is a normal operator, we can apply continuous functional calculus to its spectrum. We first note that in the continuous function space on $\sigma(aa^*)$,

$$\begin{aligned} \||a| - I\| = \|\sqrt{t} - 1\|_\infty &= \sup_{t \in \sigma(aa^*)} |\sqrt{t} - 1| \\ &\leq \sup_{t \in \sigma(aa^*)} |\sqrt{t} - 1||\sqrt{t} + 1| \\ &= \sup_{t \in \sigma(aa^*)} |t - 1| = \|aa^* - I\| \leq \delta \end{aligned}$$

so that $\sigma(|a|) \subset [1 - \delta, 1 + \delta]$. Let $f : \sigma(|a|) \to \mathbb{C}$ take $t \longmapsto 1/t$ so that $f(|a|)|a| = I$. Then we find that

$$u = Iu = f(|a|)|a|u = f(|a|)a \in C^*(a)$$

Furthermore,

$$\|u - a\| = \|Iu - |a|u\| \leq \|I - |a|\| \leq \delta = \epsilon$$

Now, we find quite easily that

$$uu^* = f(|a|)aa^* f(|a|) = f(|a|)|a|^2 f(|a|) = I$$

Now we need to show that $u^*u = I$. Note that we have the identity $u^*a = u^*|a|u = a^*u$. First, we see that:

$$\|u^*u - a^*a\| = \|u^*u - u^*a + a^*u - a^*a\| \leq \|u^*\|\|u - a\| + \|a^*\|\|u - a\| \leq 2\delta$$

Therefore,

$$\|u^*u - I\| = \|u^*u - a^*a + a^*a - I\| \leq 3\delta.$$

Since u is a partial isometry, and $u^*uI = u^*u$, by lemma **??**, $I - u^*u$ is a projection, and since

$$\|I - u^*u\| \leq 3\delta \leq 3/4 < 1$$

$u^*u = I$, completing the proof. □

3.2. **Matrix units as types.**

3.3. **Finite dimensional C*-algebras as atomic models.** We wish to show that the C*-algebra of $n \times n$ matrices, $M_n(\mathbb{C})$, is an atomic model. Given any type p realized by $a \in M_n$, let $a = (\lambda_{ij})_{1 \le ij \le n}$ with $\lambda_{ij} \in \mathbb{C}$. We use

$$\psi_n(\bar{x}) = \| \sum_{1 \le i \le n} x_{ii} - 1\| + \sum_{1 \le i,j,k,l \le n} \|x_{ij}x_{kl} - \delta_{jk}x_{il}\| + \sum_{1 \le i,j \le n} \|x_{ij} - x^*_{ji}\|$$

from before where $\psi_n(\bar{y}) = 0 \iff \bar{y}$ are the matrix units of M_n

Let

$$\theta_a(x) = \inf_{\psi_n(\bar{y})=0} \|x - \sum \lambda_{ij}y_{ij}\|$$

Since ψ_n is stable, we can quantify over it, thus θ_a is a formula. We will show that $\theta_a(x) = d(x, \{x \mid x \text{ realizes } p\})$.

Lemma 3.10. *The zeroset of θ_a is the equivalence class of a under unitary equivalence.*

$$\{x \mid \theta_a(x) = 0\} = \{x \mid \exists u \text{ unitary } x = uau^*\}$$

Proof. Recall that given $\mathcal{A}$ a C*-algebra, $\exists \bar{x} \in \mathcal{A}\ \psi(\bar{x}) = 0$ implies that there exists unital *-homomorphism $F : M_n \longrightarrow \mathcal{A}$ where $F(e_{ij}) = x_{ij}$ where e_{ij} are the standard matrix units.

Letting $\mathcal{A} = M_n$, F becomes a unital *-automorphism. Recall that all unital *-automorphisms are of the form $x \mapsto uxu^*$ where u is unitary. Therefore, if $\bar{x} \in M_n$ satisfies $\psi(\bar{x}) = 0$, then there exists u unitary such that $\forall 1 \le i,j \le n \quad x_{ij} = ue_{ij}u^*$. Thus, given b

$$\begin{aligned} \theta_a(b) = 0 &\iff \exists \bar{x} \quad \psi(\bar{x}) = 0 \quad b = \lambda_{ij}x_{ij} \\ &\iff b = \lambda_{ij}ue_{ij}u^* = u\lambda_{ij}e_{ij}u^* = uau^* \end{aligned}$$

thus the zeroset of θ_a is precisely the equivalence class of a. □

We can observe that the value of θ_a is the distance from its zeroset since the zeroset of ψ_n also respects unitary equivalence.

Lemma 3.11. *Given any unitary u, given any type t, a realizes type $t \implies uau^*$ realizes t. More specifically, for all formula ϕ, $\phi(a) = \phi(uau^*)$.*

Proof. We will show that $F(x) := uxu^*$ is an elementary equivalence, by induction on the complexity of ϕ, using

$$\begin{aligned} u0u^* &= 0 \\ u1u^* = uu^* &= 1 \\ ux^*u^* &= (uxu^*)^* \\ u(x+y)u^* &= uxu^* + uyu^* \\ uxyu^* &= uxu^*uyu^* \end{aligned}$$

and since $x \mapsto uxu^*$ is a bijection

$$\begin{aligned} \sup_x P(x,y) &= \sup_x P(uxu^*, y) \\ \inf_x P(x,y) &= \inf_x P(uxu^*, y) \end{aligned}$$

Finally, unitary equivalence preserves the operator norm

$$\|x\| = \|uxu^*\|$$

Therefore, $M_n \equiv F(M_n)$, thus, for all ϕ, $\phi(a) = \phi(F(a)) = \phi(uau^*)$. □

Proposition 3.12. *M_n is an atomic model.*

Proof. Let p be a complete type realized in M_n by a, ie. $p = \mathrm{tp}(a)$. As shown by Lemma 3.10, θ_a is a formula and $\theta_a(y) = d(y, \{x \mid \exists u \text{ unitary } x = uau^*\})$. By Lemma 3.11, since a realizes p and every uau^* realizes p,

$$\{x \mid \exists u \text{ unitary } x = uau^*\} \subseteq \{x \mid x \text{ realizes } p\}$$

To show that the sets are equal, consider any x that realizes p. Since $p = \mathrm{tp}(a)$, $\theta_a \in p$, thus x realizes $p \implies \theta_a(x) = 0 \implies \exists u$ unitary $\quad x = uau^*$ by Lemma 3.10. Therefore,

$$\{x \mid \exists u \text{ unitary } x = uau^*\} = \{x \mid x \text{ realizes } p\}$$

$$\theta_a(y) = d(y, \{x \mid \exists u \text{ unitary } x = uau^*\}) = d(y, \{x \mid x \text{ realizes } p\}$$

Thus p is principal via θ_a, and M_n is an atomic model. □

Now we consider finite dimensional C*-algebras, and we claim that they are also atomic models. By the Wedderburn-Artin theorem, every finite dimensional C*-algebra is isomorphic to a direct sum of finitely many full matrix algebras. Given finite dimensional C*-algebra A, we can write

$$A \cong M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$$

The following Ψ_A defines "matrix units" of $M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$

$$\Psi_A(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^k) = \|\sum_{\xi=1}^{k} \sum_{i=1}^{n_\xi} x_{ii}^\xi - 1\| + \Psi_1(\bar{x}^1) + \Psi_2(\bar{x}^2) + \cdots + \Psi_k(\bar{x}^k)$$

with

$$\Psi_\xi(\bar{x}^\xi) = \sum_{1 \le i,j,k,l \le n_\xi} \|x_{ij}^\xi x_{kl}^\xi - \delta_{jk} x_{il}^\xi\| + \sum_{1 \le i,j \le n} \|x_{ij}^\xi - (x_{ji}^\xi)^*\|$$

Notice that Ψ_A is essentially the generalization of ψ_n defined above to direct sums of full matrix algebras.

Consider any complete type $p = \mathrm{tp}(a)$ with $a \in M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$. Let $a =< (\lambda_{ij}^1)_{1 \le i,j \le n_1}, (\lambda_{ij}^2)_{1 \le i,j \le n_2}, \dots, (\lambda_{ij}^k)_{1 \le i,j \le n_k} >$. We claim that the following formula Θ_a shows that p is a definable type

$$\Theta_a(x) = \inf_{P(\bar{e})=0} \|x - \sum_{1 \le \xi \le k} \sum_{1 \le i,j \le n_\xi} \lambda_{ij} e_{ij}\|$$

where $\hat{\Psi}_A$ is a formula based on Ψ_A that will be defined in the next few paragraphs. Recall that we can quantify over the zeroset of a stable formula to produce a formula.

For simplicity sake, we will consider $A = M_n \oplus M_m$, then

$$\Psi_A(\bar{x}, \bar{y}) = \|\sum_{i=1}^{n} x_{ii} + \sum_{i=1}^{m} y_{ii} - 1\| + \sum_{1 \le i,j,k,l \le n} \|x_{ij} x_{kl} - \delta_{jk} x_{il}\| + \sum_{1 \le i,j \le n} \|x_{ij} - x_{ji}^*\|$$
$$+ \sum_{1 \le i,j,k,l \le m} \|y_{ij} y_{kl} - \delta_{jk} y_{il}\| + \sum_{1 \le i,j \le m} \|y_{ij} - y_{ji}^*\|$$

Lemma 3.13. *Given $\bar{b}, \bar{c} \in M_n \oplus M_m$ such that $\Psi_A(\bar{b}, \bar{c}) = 0$, a unital *-endomorphism is induced by taking the standard matrix units in A to $\bar{b}$ and $\bar{c}$*

$$\begin{aligned} F : M_n \oplus M_m &\longrightarrow M_n \oplus M_m \\ \langle e_{ij}, 0\rangle &\longmapsto b_{ij} \\ \langle 0, \varepsilon_{ij}\rangle &\longmapsto c_{ij} \end{aligned}$$

*where e_{ij} are the standard matrix units in M_n and ε_{ij} are the standard matrix units M_m. If $\bar{b} \neq \bar{0}$ and $\bar{b} \neq \bar{0}$ then F is a unital *-automorphism.*

Proof. Clearly, F is a *-homomorphism because any element of $M_n \oplus M_m$ has a unique decomposition into matrix units, and by definition of Psi_A the behavior of sums, products and adjoints of the images of matrix units is the same as those of the matrix units. F is unital by definition of Ψ_A

$$\begin{aligned} F(1) = F(e_{11} + \cdots + e_{nn} + \varepsilon_{11} + \cdots + \varepsilon_{mm}) &= F(e_{11}) + \cdots + F(e_{nn}) + F(\varepsilon_{11}) + \cdots + F(\varepsilon_{mm}) \\ &= b_{11} + \cdots + b_{nn} + c_{11} \cdots + c_{mm} = 1 \end{aligned}$$

Thus F is a unital *-endomorphism.

Suppose that $\bar{a} \neq \bar{0}$ and $\bar{b} \neq \bar{0}$. The ideals of $M_n \oplus M_m$ are $\{\langle 0,0\rangle\}$, $M_n \oplus M_m$, $M_n \oplus \{0\}$ and $\{0\} \oplus M_m$. We know that the kernel of F cannot be all of $M_n \oplus M_m$ because $F(\langle 1,1\rangle) = \langle 1,1\rangle$. On the other hand, $\bar{b} \neq \bar{0} \implies \exists i,j \quad F(\langle e_{ij}, 0\rangle) \neq b_{ij} \implies$ the kernel is not $M_n \oplus \{0\}$. Similarly $\bar{a} \neq \bar{0} \implies$ the kernel is not $\{0\} \oplus M_m$. Therefore the kernel can only be $\{\langle 0,0\rangle\}$. Thus F is injective, thus a unital *-automorphism. □

In order to force $\bar{a}, \bar{b}$ to be nonzero, thus making F into a untial *-automorphism, we will modify our matrix units formula Ψ_A to admit only nonzero x_{ij} and y_{ij}.

Note that $\forall i,j,k,l \quad x_{ij} = 0 \implies x_{kl} = x_{ki}x_{ij}x_{jl} = 0$, thus if we have $x_{11} \neq 0$, then $x_{ij} \neq 0$, therefore we only have to check x_{11} and y_{11} are nonzero.

Furthermore, notice that since e_{11} is a projection,

$$\begin{aligned} {e_{11}}^2 = e_{11} = {e_{11}}^* &\implies F(\langle e_{11}, 0\rangle^2) = F(\langle e_{11}, 0\rangle) = F(\langle e_{11}, 0\rangle^*) \\ &\implies F(\langle e_{11}, 0\rangle)^2 = F(\langle e_{11}, 0\rangle) = F(\langle e_{11}, 0\rangle)^* \implies x_{11}^2 = x_{11} = x_{11}^* \end{aligned}$$

thus x_{11} is also a projection. This is also obvious from the definition of Ψ_A. Using the C*-identity axiom

$$\begin{aligned} \|{x_{11}}^* x_{11}\| &= \|{x_{11}}^*\| \|x_{11}\| \\ \implies \|x_{11}\| &= \|x_{11}\|^2 \end{aligned}$$

thus $\|x_{11}\|$ is either 0 or 1. Therefore saying $\|x_{11}\| \neq 0$ is the same as saying $\|x_{11}\| = 1$, given $\Psi_A(\bar{x}, \bar{y}) = 0$.

We can define a new formula $\hat{\Psi}_A$ by modifying Ψ_A. Let

$$\hat{\Psi}_A(\bar{x}, \bar{y}) := \Psi_A(\bar{x}, \bar{y}) + (1 \dot{-} \|x_{11}\|) + (1 \dot{-} \|y_{11}\|)$$

Therefore

$$\hat{\Psi}_A(\bar{x}, \bar{y}) = 0 \iff x_{11} \neq 0 \quad y_{11} \neq 0 \quad \Psi_A(\bar{x}, \bar{y}) = 0 \iff \bar{x}, \bar{y} \text{ induces a unital *-automorphism}$$

Thus consider any complete type $p = \mathrm{tp}(\langle a, b\rangle)$ in $M_n \oplus M_m$ with $a = (a_{ij})_{1 \le i,j \le n} \in M_n$ and $b = (b_{ij})_{1 \le i,j \le m} \in M_m$ (a_{ij} and b_{ij} are complex numbers)

We will use the following formula to show that p is a principal type.

$$\Theta_{\langle a,b\rangle}(x) = \inf_{\hat{\Psi}_A(\bar{e},\bar{\varepsilon})} \|x - \sum_{1\le i,j\le n} a_{ij}e_{ij} - \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij}\|$$

Note that $\Theta_{\langle a,b\rangle}$ is a formula because we can quantify over the zeroset of stable formulas, and we can show that $\hat{\Psi}_A$ is a stable formula from knowing that Ψ_A is stable. Also, from the format of $\Theta_{\langle a,b\rangle}$ it is clear that

$$\Theta_{\langle a,b\rangle}(y) = d(y, \left\{x \;\middle|\; x = \sum_{1\le i,j\le n} a_{ij}e_{ij} + \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij} \text{ and } \hat{\Psi}_A(\bar{e},\bar{\varepsilon}) = 0\right\})$$

Lemma 3.14. *The operator norm on $M_n \oplus M_m$ can be obtained from the operator norm on each component*

$$\|\langle x,y\rangle\| = \max(\|x\|,\|y\|)$$

Proof. Recall that by definition of the operator norm,

$$\|x\|_{op} = \min\{c \mid \|xu\| \le c\|u\| \quad \forall u \in \mathbb{C}^n\}$$
$$\|y\|_{op} = \min\{c \mid \|yv\| \le c\|v\| \quad \forall v \in \mathbb{C}^m\}$$
$$\|\langle x,y\rangle\|_{op} = \min\{c \mid \|\langle xu,yv\rangle\| \le c\|\langle u,v\rangle\| \quad \forall\langle u,v\rangle \in \mathbb{C}^n \oplus \mathbb{C}^m\}$$

Given any $x \in M_n,\ y \in M_m$ we assume without loss of generality that $\|x\|_{op} \ge \|y\|_{op}$.

Consider

$$\{\langle u,0\rangle \mid u \in M_n\} \subset C^n \oplus C^m$$

So

$$\{c \mid \|\langle xu,0\rangle\| \le c\|\langle u,0\rangle\| \quad \forall\langle u,0\rangle \in C^n \oplus C^m\} \supseteq \{c \mid \|\langle xu,yv\rangle\| \le c\|\langle u,v\rangle\| \quad \forall\langle u,v\rangle \in \mathbb{C}^n \oplus \mathbb{C}^m\}$$

Recall that we are using the Euclidean norm, thus

$$\|\langle u,v\rangle\| = \sqrt{\|u\|^2 + \|v\|^2} \quad \forall\langle u,v\rangle \in \mathbb{C}^n \oplus \mathbb{C}^m$$

Thus, $\|\langle u,0\rangle\| = \|u\|$ and $\|\langle xu,0\rangle\| = \|xu\|$, therefore

$$\begin{aligned}\{c \mid \|xu\| \le c\|u\| \quad \forall u \in \mathbb{C}^n\} &= \{c \mid \|\langle xu,0\rangle\| \le c\|\langle u,0\rangle\| \quad \forall\langle u,0\rangle \in \mathbb{C}^n \oplus \mathbb{C}^m\}\\ &\supseteq \{c \mid \|\langle xu,yv\rangle\| \le c\|\langle u,v\rangle\| \quad \forall\langle u,v\rangle \in \mathbb{C}^n \oplus \mathbb{C}^m\}\end{aligned}$$

Therefore,

$$\min\{c \mid \|xu\| \le c\|u\| \quad \forall u \in \mathbb{C}^n\} \le \min\{c \mid \|\langle xu,yv\rangle\| \le c\|\langle u,v\rangle\| \quad \forall\langle u,v\rangle \in \mathbb{C}^n \oplus \mathbb{C}^m\}$$

Thus,

$$\|x\|_{op} \le \|\langle x,y\rangle\|_{op}$$

Suppose $\|x\|_{op} < \|\langle x,y\rangle\|_{op}$, then there exists $\langle u,v\rangle \in \mathbb{C}^n \oplus \mathbb{C}^m$ such that

$$\|\langle xu,yv\rangle\| > \|x\|_{op}\|\langle u,v\rangle\|$$

Thus,

$$\begin{aligned}\sqrt{\|xu\|^2 + \|yv\|^2} &> \|x\|_{op}\sqrt{\|u\|^2 + \|v\|^2}\\ &\ge \sqrt{\|x\|_{op}^2\|u\|^2 + \|y\|_{op}^2\|v\|^2} && \text{since } \|x\|_{op} \ge \|y\|_{op}\\ &\ge \sqrt{\|xu\|^2 + \|yv\|^2} && \text{by definition of operator norm}\end{aligned}$$

Which is a contradiction. Therefore

$$\|x\|_{op} = \|\langle x, y\rangle\|_{op}$$

□

Lemma 3.15.

$$\left\{x \;\middle|\; x = \sum_{1\le i,j\le n} a_{ij}e_{ij} + \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij} \textit{ and } \hat{\Psi}_A(\bar{e},\bar{\varepsilon}) = 0\right\} = \{x \mid x \textit{ realizes } \mathrm{tp}(\langle a,b\rangle)\}$$

Proof. Suppose $x \in \{x \mid x = \sum_{1\le i,j\le n} a_{ij}e_{ij} + \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij}$ and $\hat{\Psi}_A(\bar{e},\bar{\varepsilon}) = 0\}$, then

$$\exists \bar{\alpha},\bar{\beta} \quad \hat{\Psi}_A(\bar{\alpha},\bar{\beta}) = 0 \text{ and } x = \sum_{1\le i,j\le n} a_{ij}\alpha_{ij} + \sum_{1\le i,j\le m} b_{ij}\beta_{ij}$$

Recall the unital *-automorphism F induced by taking the standard matrix units of $M_n \oplus M_m$ to $\bar{\alpha},\bar{\beta}$ in Lemma 3.13:

$$\begin{aligned} F : M_n \oplus M_m &\longrightarrow M_n \oplus M_m \\ \langle e_{ij}, 0\rangle &\longmapsto \alpha_{ij} \\ \langle 0, \varepsilon_{ij}\rangle &\longmapsto \beta_{ij} \end{aligned}$$

Then

$$\begin{aligned} x = \sum_{1\le i,j\le n} a_{ij}\alpha_{ij} + \sum_{1\le i,j\le m} b_{ij}\beta_{ij} &= \sum_{1\le i,j\le n} a_{ij}F(\langle e_{ij},0\rangle) + \sum_{1\le i,j\le m} b_{ij}F(\langle 0,\varepsilon_{ij}\rangle) \\ &= F(\sum_{1\le i,j\le n} a_{ij}\langle e_{ij},0\rangle + \sum_{1\le i,j\le m} b_{ij}\langle 0,\varepsilon_{ij}\rangle) = F(\langle a,b\rangle) \end{aligned}$$

thus $x = F(\langle a,b\rangle)$. We will show that $F(\langle a,b\rangle)$ realizes $\mathrm{tp}(\langle a,b\rangle)$, or more generally, given any formula ϕ and s we have $\phi(F(s)) = \phi(s)$

We will use induction on the complexity of ϕ. Since F is a unital *-automorphism, the following are obvious:

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(s+t) &= F(s) + F(t) \\ F(st) &= F(s)F(t) \\ F(s^*) &= F(s)^* \\ \sup_s Q(F(s),\bar{t}) &= \sup_x Q(s,\bar{t}) \\ \inf_s Q(F(s),\bar{t}) &= \inf_x Q(s,\bar{t}) \end{aligned}$$

The only other property we have to look for is $\|F(s)\| = \|s\|$. To show this, consider again the ideals of of $M_n \oplus M_m$ which are $\{\langle 0,0\rangle\}$, $M_n \oplus M_m$, $M_n \oplus \{0\}$ and $\{0\} \oplus M_m$. Because F is onto, the image of ideals under F are ideals. Clearly $F(\{\langle 0,0\rangle\}) = \{\langle 0,0\rangle\}$ and $F(M_n \oplus M_m) = M_n \oplus M_m$.

By simply looking at the dimensions of M_n and M_m, we see that when $n \ne m$, $F(M_n \oplus \{0\}) = M_n \oplus \{0\}$ and $F(\{0\} \oplus M_m) = \{0\} \oplus M_m$.

Therefore there exists unital *-automorphisms $f : M_n \to M_n$ and $g : M_m \to M_m$ such that $F(\langle s,t\rangle) = \langle f(s), g(t)\rangle$.

When $n = m$, the only other possibility to consider is when $F(M_n \oplus \{0\}) = \{0\} \oplus M_n$ and $F(\{0\} \oplus M_n) = M_n \oplus \{0\}$. Therefore, either $F(\langle s,t\rangle) = \langle f(s), g(t)\rangle$ or $F(\langle s,t\rangle) = \langle g(t), f(s)\rangle$

In both cases, recall that all unital *-automorphisms on full matrix algebras are inner, i.e.

$$\exists u, v \text{ unitary, such that } f(x) = usu^* \quad g(y) = utu^*$$

and recall that unitary equivalence preserves the operator norm, thus

$$\|f(s)\| = \|s\| \text{ and } \|g(t)\| = \|t\|$$

Now, using Lemma 3.14 we know that in the case where $F(\langle s,t\rangle) = \langle f(s), g(t)\rangle$

$$\|F(\langle s,t\rangle)\| = \|\langle f(s), g(t)\rangle\| = \max(\|f(s)\|, \|g(t)\|) = \max(\|s\|, \|t\|) = \|\langle s,t\rangle\|$$

Similarly for $F(\langle s,t\rangle) = \langle g(t), f(s)\rangle$

$$\|F(\langle s,t\rangle)\| = \|\langle g(t), f(s)\rangle\| = \max(\|g(t)\|, \|f(s)\|) = \max(\|t\|, \|s\|) = \|\langle s,t\rangle\|$$

Therefore,

$$\|F(s)\| = \|s\|$$

thus, given any formula ϕ

$$\phi(F(s)) = \phi(s)$$

so we get

$$\mathrm{tp}(x) = \mathrm{tp}(\langle a,b\rangle) \implies x \text{ realizes } \mathrm{tp}(x)$$

Therefore,

$$\left\{x \;\middle|\; x = \sum_{1\le i,j\le n} a_{ij}e_{ij} + \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij} \text{ and } \hat{\Psi}_A(\bar{e},\bar{\varepsilon}) = 0\right\} \subseteq \{x \mid x \text{ realizes } \mathrm{tp}(\langle a,b\rangle)\}$$

Notice that the above method works for direct sums of more matrix algebras as well, but the number of possible swaps is higher, but they all preserve the operator norm.

The other direction is easy:

$$\Theta_{\langle a,b\rangle}(\langle a,b\rangle) = 0 \implies \Theta_{\langle a,b\rangle}(\langle x,y\rangle) = 0 \in \mathrm{tp}(\langle a,b\rangle)$$

Therefore, given any x,

$$x \text{ realizes } \mathrm{tp}(\langle a,b\rangle) \implies \Theta_{\langle a,b\rangle}(x) = 0$$

$$\implies x \in \left\{x \;\middle|\; x = \sum_{1\le i,j\le n} a_{ij}e_{ij} + \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij} \text{ and } \hat{\Psi}_A(\bar{e},\bar{\varepsilon}) = 0\right\}$$

Therefore, we have obtained inclusion in the other direction as well

$$\left\{x \;\middle|\; x = \sum_{1\le i,j\le n} a_{ij}e_{ij} + \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij} \text{ and } \hat{\Psi}_A(\bar{e},\bar{\varepsilon}) = 0\right\} = \{x \mid x \text{ realizes } \mathrm{tp}(\langle a,b\rangle)\}$$

□

Theorem 3.16. *Finite dimensional C*-algebras are atomic models.*

Proof. Consider any complete type p on finite dimensional C*-aglebra A. Recall that $A \cong M_{n_1} \oplus \cdots \oplus M_{n_1}$, and again, we will consider $A \cong M_n \oplus M_m$ for simplicity of notation. Since p is a complete type, $p = \mathrm{tp}(\langle a, b\rangle)$ for some a, b. Considering from before

$$\Theta_{\langle a,b\rangle}(x) = \inf_{\hat{\Psi}_A(\bar{e},\bar{\varepsilon})} \|x - \sum_{1\le i,j\le n} a_{ij}e_{ij} - \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij}\|$$

we noticed from the syntax of this formula that

$$\Theta_{\langle a,b\rangle}(y) = d(y, \left\{x \;\middle|\; x = \sum_{1\le i,j\le n} a_{ij}e_{ij} + \sum_{1\le i,j\le m} b_{ij}\varepsilon_{ij} \text{ and } \hat{\Psi}_A(\bar{e},\bar{\varepsilon}) = 0\right\})$$

By Lemma 3.15

$$\Theta_{\langle a,b\rangle}(y) = d(y, \{x \mid x \text{ realizes } \mathrm{tp}(\langle a,b\rangle)\})$$

Therefore, the formula $\Theta_{\langle a,b\rangle}$ witnesses that $p = \mathrm{tp}(\langle a, b\rangle)$ is a principal type. Thus, by definition, A is an atomic model. □

3.4. **Characterisation of UHF Algebras.** We will now use the following formula to proposition about UHF algebras:

$$\psi_n(\bar{x}) = \|\sum_i^n x_{ii} - I\| + \sum_{i,j,k,l}^n \|x_{ij}x_{kl} - \delta_{jk}x_{il}\| + \sum_{i,j}^n \|x_{ij}^* - x_{ji}\|$$

Note that $\psi_n(\bar{e}) = 0$ if and only if $\bar{e}$ is a copy of a set of matrix units in $M_n(\mathbb{C})$. In section 3.1 we show that ψ_n is stable, which we be assuming in what follows.

Lemma 3.17. *Let A be a unital C*-algebra. Then for all $\delta > 0, n \in \mathbb{N}$ there exists a $\kappa > 0$ such that if we have $\bar{a}$ and $\bar{e}$ such that $\psi_n^A(\bar{e}) = 0$ and $\|\bar{a} - \bar{e}\| \le \kappa$, then $\psi_n^A(\bar{a}) \le \delta$.*

Proof. Fix $\delta > 0$ and let $\kappa = \frac{1}{3n^4}(\delta/3)$. We recall that

$$\psi_n(\bar{a}) = \|\sum_i^n a_{ii} - I\| + \sum_{i,j,k,l}^n \|a_{ij}a_{kl} - \delta_{jk}a_{il}\| + \sum_{i,j}^n \|a_{ij}^* - a_{ji}\|$$

We will first look at each of the three summands.

$$\|\sum_i^n a_{ii} - I\| = \|\sum_i^n a_{ii} - \sum_i^n e_{ii}\| \le n\delta \le 3n^4\kappa$$

$$\sum_{i,j}^n \|a_{ij}^* - a_{ji}\| = \sum_{i,j}^n \|a_{ij}^* + e_{ij}^* - e_{ji} - a_{ji}\| \le 2n^2\kappa \le 3n^4\kappa$$

$$\sum_{i,j,k,l}^n \|a_{ij}a_{kl} - \delta_{jk}a_{il}\| \le \sum_{i,j,k,l}^n \|a_{ij}a_{kl} - a_{ij}e_{kl} + a_{ij}e_{kl} - e_{ij}e_{kl} + \delta_{jk}e_{il} - \delta_{jk}a_{il}\| \le 3n^4\kappa$$

Therefore we get that

$$\psi_n(\bar{a}) = \|\sum_i^n a_{ii} - I\| + \sum_{i,j,k,l}^n \|a_{ij}a_{kl} - \delta_{jk}a_{il}\| + \sum_{i,j}^n \|a_{ij}^* - a_{ji}\| \le 3(3n^4\delta) \le 3(\delta/3) = \delta$$

□

The following is a very important lemma we will use when we repeatedly when we characterize UHF algebras. We say that, for subsets B and C of a normed space A, $B \subseteq_\epsilon C$ if for all $a \in A$, $\inf_{b \in B} \|a - b\| < \epsilon$.

Lemma 3.18. *Let A be a unital C*-algebra with unital subalgebra B of A such that $B \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Let $\bar{e} = \{e_{ij}\}_{i,j=1}^n \subset B$ be such that $\psi_n^A(\bar{e}) = 0$. Furthermore, suppose that for every $\delta > 0$ there exists an $N(\delta) \in \mathbb{N}$ such that $C(\delta) \cong M_{N(\delta)}(\mathbb{C})$ is a unital subalgebra of A such that $\bar{e} \subseteq_\delta C(\delta)$. Then there exists $\kappa > 0$ such that there is a $\bar{f} = \{f_{ij}\}_{i,j=1}^n \subset C(\kappa)$ which has that $\psi_n^A(\bar{f}) = 0$. Furthermore we then get a unital *-homomorphism $\phi : B \to C(\kappa) \Rightarrow n|N(\kappa)$.*

Proof. Fix some $\epsilon > 1$, and let $\delta = \delta(\epsilon, n)$ where δ has that if $\bar{a} \in D$, where D is a unital subalgebra of A, and $\psi_n^A(\bar{a}) \le \delta$, then there exists a $\bar{f} \in D$ such that $\psi_n^A(\bar{f}) = 0$. We know such a δ exists by the stability of $\psi_n(\bar{x}) = 0$. Now let $\kappa = \kappa(\delta, n)$ be as in the last proof, so that if $\|\bar{e} - \bar{a}\| \le \kappa \Rightarrow \psi_n(\bar{a}) \le \delta$. Then let $N(\kappa) \in \mathbb{N}$ be such that $C(\kappa) \cong M_{N(\kappa)}(\mathbb{C})$ is a unital subalgebra of A such that $\bar{e} \subseteq_\kappa C(\kappa)$. Then there exists a $\bar{a} \in C$ such that $\|\bar{a} - \bar{e}\| \le \kappa \Rightarrow \psi_n^A(\bar{a}) \le \delta$. Therefore, $\bar{a}$ satisfies the hypotheses of the proof of the stability of matrix units, so we can find $\bar{f} = \{f_{ij}\}_{i,j=1}^n \subset C(\kappa)$ such that $\psi_n^A(\bar{f}) = 0$. We can then set up a unital *-homomorphism $\phi : B \to C(\kappa)$ in the obvious way, by sending matrix units $\bar{e}$ in B to matrix units $\bar{f}$ in $C(\kappa)$.

□

Proposition 3.19. *Let A be a UHF algebra with generalized integer κ_A, and define the following sentence:*

$$\mu_n = \inf_{\substack{\bar{x} \in A^{n^2} \\ \|x_{ij}\| \le 1}} \psi_n(\bar{x})$$

Then $\mu_n^A = 0$ if and only if $n|\kappa_A$.

Proof. Without loss of generality, we let $A = \overline{\bigcup_{i \in \mathbb{N}} M_{n_i}(\mathbb{C})}$. Then due to the stability of $\psi_n(\bar{x}) = 0$, we have that if $\mu_n^A = 0$, there exists a set $\bar{e} = \{e_{ij}\}_{i,j=1}^n$ which is a copy of matrix units for $M_k(\mathbb{C})$ in our UHF algebra A. If $\bar{e} \subset \bigcup_{i \in \mathbb{N}} M_{n_i}(\mathbb{C})$, then let $N = \max_{1 \le i,j \le n} \{n_i : e_{ij} \in M_{n_i}(\mathbb{C})\}$. Using the unital *-homorphisms from the UHF algebra, we can then get that $\bar{e} \subset M_N(\mathbb{C})$. Since the composition of unital *-homomorphisms is a unital *-homomorphism, we then get such a map from $M_k(\mathbb{C})$ into $M_N(\mathbb{C})$ which implies $k|N \Rightarrow k|\kappa_A$.

Now suppose that we have that, for some i and j, $e_{ij} \notin \bigcup_{i \in \mathbb{N}} M_{n_i}(\mathbb{C})$. Then we have that for every $\delta > 0$ and e_{ij}, there exists an $l_\delta(ij) \in \mathbb{N}$ such that for some $a \in M_{l_\delta(ij)}(\mathbb{C})$, $\|e_{ij} - a\| \le \delta$. Let $N(\delta) = \max_{1 \le i,j \le n} \{l_\delta(ij)\}$. Then it follows that $\bar{e} \subseteq_\delta M_{N(\delta)}(\mathbb{C})$. Since δ was arbitrary, we see that our C*-algebra A satisfies the hypotheses lemma 3.18, and so there exists a $\kappa > 0$ such that $k|N(\kappa) \Rightarrow k|\kappa_A$.

For the converse, suppose we have $n \in \mathbb{N}$ such that $n|\kappa_A$. Then it follows from theorem 2.34 that there exists a unital *-homomorphism $\Phi : M_n(\mathbb{C}) \to A$. We immediately see that if $\bar{e}$ is a set of matrix units from $M_n(\mathbb{C})$, then $\psi_n^A(\Phi(\bar{e})) = 0 \Rightarrow \mu_n^A = 0$. □

Lemma 3.20. *Let A and B be UHF algebras. Then $A \cong B$ if and ony if $\kappa_A = \kappa_B$.*

Proof. Suppose that $\kappa_A \neq \kappa_B$. Then let p be a prime such that $p^k | \kappa_A$ but $p^k \not| \kappa_B$. Then by proposition 3.19, $\mu^A_{p^k} = 0 \neq \mu^B_{p^k} \Rightarrow A \not\equiv B \Rightarrow A \not\cong B$, proving the contrapositive of our first direction.

Now suppose that $\kappa_A = \kappa_B$ and let $A = \varinjlim A_i$ and $B = \varinjlim B_i$, where $B_i \cong M_{k(i)}(\mathbb{C})$ and $A_i \cong M_{n(i)}(\mathbb{C})$ for all i. Then we have that $n(1)|k(i)$ for some i, and then we have that $k(i)|n(j)$ for some j, and if we continue on as such inductively, we see that we can pass the sequences $\{n(i)\}$ and $\{k(i)\}$ off to subsequences $\{a(i)\}$ and $\{b(i)\}$ respectively such that $a(i)|b(i)$ and $b(i)|a(i+1)$ for all $i \in \mathbb{N}$. We can then use the theorem from matrix algebras as well as the definition of UHF algebras to get sets of unital *-homomorphisms $\{\varphi_i : M_{a(i)}(\mathbb{C}) \to M_{b(i)}(\mathbb{C})\}$ and $\{\psi : M_{b(i)}(\mathbb{C}) \to M_{a(i+1)}(\mathbb{C})\}$ along with the direct limit maps for A and B respectively such that every triangle in the following diagram commutes:

$$\begin{tikzcd}
A_1 \arrow[r, hook] \arrow[d, "\phi_1"'] & A_2 \arrow[r, hook] \arrow[d, "\phi_2"'] & A_3 \arrow[r, hook] \arrow[d, "\phi_3"'] & A_4 \arrow[r, hook, dotted] \arrow[d, "\phi_4"'] & A \\
B_1 \arrow[r, hook] \arrow[ur, "\psi_1"] & B_2 \arrow[r, hook] \arrow[ur, "\psi_2"] & B_3 \arrow[r, hook] \arrow[ur, "\psi_3"] & B_4 \arrow[r, hook, dotted] & B
\end{tikzcd}$$

We note that for all $i > 1$, $\phi_i = \phi_{i-1}$ when restricted to A_{i-1}, and $\psi_i = \psi_{i-1}$ when restricted to B_{i-1}. This allows us to define the well-defined maps Φ and Ψ on $A' = \cup_n A_n$ and $B' = \cup_n B_n$ respectively, where for $a \in A_n$, $\Phi(a) = \phi_n(a)$, and likewise for Ψ. This gives us that Φ and Ψ can both be extended to all of A and B respectively,, and that they are furthermore each unital *-homomorphism.

We now want to show they must also be inverses, i.e. $\Phi \circ \Psi = \Psi \circ \Phi = I$, and we will do this by showing they agree on their dense subsets A' and B' respectively, which is sufficient.

Choose an arbitrary $a \in A'$, so that $a \in A_i$. Then by the commutativity of the above diagram, we get that $(\Psi \circ \Phi)a = \Psi(\phi_i(a)) = (\psi_i \circ \phi_i)(a) = a$. So $\Psi \circ \Phi = I$ and likewise we can find that $\Phi \circ \Psi = I$ so that Φ and Ψ are *-isomorphisms implying that $A \cong B$. □

Theorem 3.21. *Let A and B be unital, separable UHF algebras. Then $A \cong B$ if and only if $A \equiv B$.*

Proof. If $A \cong B$, then $A \equiv B$.

Now suppose $A \equiv B$. By lemma 3.19, $\mu^A_n = 0$ if and only if $n|\kappa_A$, and $\mu^A_n = 0$ if and only if $n|\kappa_B$. But $\mu^A_n = \mu^B_n$ for all $n \in \mathbb{N}$, so if p is any prime, $p^i|\kappa_A$ if and only if $p^i|\kappa_B$, and so we get that $\kappa_A = \kappa_B$. By lemma 3.20, this implies that $A \cong B$. □

We now define another class of C*-algebras, which will turn out to be very closely related to UHF algebras.

Definition 3.22. Let A be a C*-algebra. Then we say that A is *Locally Matricial*, or *LM*, if for all $\epsilon > 0$ and every finite subset F of A, there exists a natural number n and a *-homomorphism $\Phi : M_n(\mathbb{C}) \to A$ such that $F \subseteq_\epsilon \Phi(M_n(\mathbb{C}))$.

The following is a important theorem:

Theorem 3.23. *Let A be a unital, separable C^*-algebra. Then A is UHF if and only if A is LM.*

Proof. If A is UHF, then A begin LM follows immediately.

Now suppose A is LM, and denote a countable dense subset of A by $\{a_k\}$. Then we have that for some $\epsilon_1 > 0$ there exists an $n(1) \in \mathbb{N}$ and a *-homomorphism $\Phi_1 : M_{n(1)}(\mathbb{C}) \to A$ such that $\{a_{k(1)}\} := \{a_1\} \subseteq_{\epsilon_1} \Phi(M_{n(1)}(\mathbb{C}))$. This also implies that there is a unital copy of $M_{n(1)}(\mathbb{C})$ in A. We let $A_1 = \Phi_1(M_{n(1)}(\mathbb{C}))$.

We now continue inductively, supposing that we have sets $A_1, A_2, \dots, A_m \subseteq A$ and unital *-homomorphisms $\phi_{ij} : A_i \to A_j$, with $1 \le i \le j \le m$ such that each A_k is isomorphic to full matrix algebra $M_n(k)(\mathbb{C})$. Also, we suppose we have a decreasing sequence of epsilons, $\{\epsilon_1, \epsilon_2, \dots, \epsilon_m\}$ such that $\epsilon_i < \epsilon_{i-1}/2^i$, and a set of $\{a_{k(i)}\}_{i=1}^m$ where $k(i) < k(j)$ for $i < j$. We then check if there exists an $0 < \epsilon'_{m+1} < \epsilon_m/2^{(m+1)}$ such that $\{a_{k(m)+1}\} \notin_{\epsilon'_{m+1}} A_n$. If such an ϵ'_{m+1} does not exist, we do the same for $a_{k(m)+2}, a_{k(m)+3}, \dots$ until we find an $a_{k(m+1)} := a_{k(m)+i}$ such that $\{a_{k(m+1)}\} \notin_{\epsilon'_{m+1}} A_1$. If such an element of $\{a_k\}$ does not exist, it follows that $\{a_k\} \subseteq_\epsilon A_m$ for all $\epsilon > 0$, and we let $A_n = A_m$ for all $m \ge n$, and $\phi_{ij} = id_{A_m}$ for all $i, j \ge n$.

Otherwise, we consider $E_m = \{\Phi_m(e_{ij})\}_{i,j=1}^{n(m)} \subset A_m$, where $\{e_{ij}\}_{i,j=1}^{n(m)}$ is a set of matrix units in $M_{n(m)}(\mathbb{C})$. Since A is LM, we see then that it satisfies the hypotheses of lemma 3.18 again with $n = n(m)$ and $E_m = \bar{e}$, and so there exists a $\kappa_{m+1} > 0$ and a unital subalgebra $A_{m+1} = C(\kappa_{m+1}) \subseteq A$, with $\{a_{k(m+1)}\} \subset_{\epsilon'_{m+1}} A_{m+1}$, such that there is a unital *-homomorphism $\phi_{m(m+1)} : A_m \to A_{m+1} \cong M_{n(m+1)}(\mathbb{C})$. Let $\epsilon_{m+1} = \min\{\epsilon'_{m+1}, \kappa_{m+1}\}$.

By the principle of mathematical induction, the above construction gives us a direct system (A_n, ϕ_{ij}). We let B be the norm closure of the direct limit of the system, i.e.

$$B = \overline{\bigcup_n A_n}$$

Since we clearly have that $B \subseteq A$, we want to show that the countable dense subset $\{a_k\} \subset B$, and the since B is closed $A = B$ will follow. Consider a_k, the kth element of our countable dense subset. By the construction of our epsilons, it follows that we can find a sequence $\{x_n\} \subset B$, with $\|a_k - x_n\| \le 1/2^n$ for all $n \in \mathbb{C}$. Therefore $\lim_{n\to\infty} \|a_k - x_n\| = 0$, and since it is not hard to check that $\{x_n\}$ is Cauchy, we get that $a_k = \lim_{n\to\infty} x_n \in B$ and so $\{a_k\} \subset B \Rightarrow A = B$. Since B is UHF, we get that A is UHF, completing the proof. □

We now will construct a set of types $t^{(U)}_{\epsilon,h}$ such that an algebra A will be UHF if and only if A omits all types $t^{(U)}_{\epsilon,h}$. Let $\psi_n(\bar{x}) = \|\sum_i^n x_{ii} - I\| + \sum_{i,j,k,l}^n \|x_{ij}x_{kl} - \delta_{jk}x_{il}\| + \sum_{i,j}^n \|x^*_{ij} - x_{ji}\|$ be the previously used formula which gives that $\psi^A_n(\bar{e}) = 0$ if and only if A contains a unital copy of $M_n(\mathbb{C})$ in it. Then we define the formula

$$\Delta_{n,h}(\bar{a}) = \inf_{\substack{\psi_n(\bar{x})=0 \\ \bar{\lambda}\in\mathbb{C}^{hn^2} \\ |\lambda^(_{ij}k)|\le 1}} \max\{d(a_1, \sum_{i,j=1}^n \lambda^(_{ij}1)e_{ij}), d(a_2, \sum_{i,j=1}^n \lambda^(_{ij}2)e_{ij}), \dots, d(a_h, \sum_{i,j=1}^n \lambda^(_{ij}h)e_{ij})\}$$

where $h, n \in \mathbb{N}$ and $\bar{a} \in A^h$. If we have that, for some $\epsilon > 0$ and $\bar{a} \in A^h$ that $\Delta_{n,h}(\bar{a}) \ge \epsilon$, it follows that either no unital copy of $M_n(\mathbb{C})$ in A ϵ-includes the h-tuple $\bar{a}$, or there doesn't exist a unital copy of $M_n(\mathbb{C})$ in A, and so the infimum over the empty set was infinite. We now use this formula to define the types to be omitted:

For every $\epsilon > 0$, $n \in \mathbb{N}$, define the type $t^{(U)}_{\epsilon,h}$ by

$$t^{(U)}_{\epsilon,h} = \{\Delta_{n,h}(\bar{a}) \geq \epsilon : n \in \mathbb{N}\}$$

Theorem 3.24. *A unital, separable C*-algebra A omits all types $t^{(U)}_{\epsilon,h}$ if and only if A is UHF.*

Proof. If A is UHF, then by theorem 3.23, A is LM, and so it immediately follows that A omits all types $t^{(U)}_{\epsilon,h}$.

Suppose A omits all types $t^{(U)}_{\epsilon,h}$, and let $F = \bar{a} \in A^h$ be an arbitrary finite subset of A. Then $\forall \epsilon > 0$ $\bar{a}$ does not realize the type $t_{\epsilon,h}$. This impies that there exists an $n \in \mathbb{N}, \bar{e} \in A^h$ such that $\psi^A_n(e) = 0$ and the span of $\bar{e}$ in A ϵ-includes F. Let $\{f_{ij}\}^n_{i,j=1}$ be matrix units in $M_n(\mathbb{C})$. Then we let $\Phi : M_n(\mathbb{C}) \to A$ take $f_{ij} \longmapsto e_{ij}$. We see immediately that this is a unital *-homomorphism, which means all the conditions of A being LM are satisfied, and so A must be UHF.

□

We now provide a counterexample to the class of elementary equivalences of UHF algebras being closed.

Proposition 3.25. *Let M_{2^∞} be the CAR algebra as previously described. Then there exists a unital separable C*-algebra A such that $A \equiv M_{2^\infty}$ but A is not UHF.*

Proof. Consider the 1-type $t^{(U)}_{\epsilon,1}$ for some $\epsilon > 0$. It's not hard to see that $t^{(U)}_{\epsilon,1}$ is consistent, so by the compactness theorem there exists an ultrafilter $\mathcal{D}$ such that for some $a \in (M_{2^\infty})^{\mathcal{D}}$, a realizes $t^{(U)}_{\epsilon,1}$.

Let $F = \{a, I_{(M_{2^\infty})^{\mathcal{D}}}\}$ Then by the Downward Lowenheim-Skolem Theorem, we can find a elementary substructure $A \preceq (M_{2^\infty})^{\mathcal{D}}$ such that $F \subset A$ and A is separable. Furthermore, $A \equiv (M_{2^\infty})^{\mathcal{D}} \equiv M_{2^\infty}$. Since $a \in A$ realizes $t^{(U)}_{\epsilon,1}$, it does not omit all types $t^{(U)}_{\epsilon,h}$, and so by theorem 3.24, A is not UHF.

□

3.5. **Characterisation of AF Algebras.** We first show that AF algebras have an analogue for locally matricially in the case of UHF algebras, and then use it in a similar way to show it omits a certain set of types.

Definition 3.26. A C*-algebra A is *locally finite*, or *LF*, if for all $\epsilon > 0$ and for every finite $F \subset A$, there exists a finite dimensional subalgebra B of A such that $F \subseteq_\epsilon B$.

Remark 3.27. Since we have that every finite dimensional C*-algebra is isomorphic to a direct sum over matrix algebras, the above definition could be restated as follows: A C*-algebra A is LF, if for all $\epsilon > 0$ and for every finite $F \subset A$, there exists a $\bar{n} = (n(1), \ldots, n(m)) \in \mathbb{N}^m$ for some $m \in \mathbb{N}$ and a unital *-homomorphism $\Phi : \bigoplus^m_{i=1} M_{n(i)}(\mathbb{C}) \to A$ such that that $F \subseteq_\epsilon \Phi(\bigoplus^m_{i=1} M_{n(i)}(\mathbb{C}))$.

We now recall the following formula, which, when set equal to 0, states that there is a unital copy of $\bigoplus_{i=1}^{m} M_{n(i)}(\mathbb{C})$ in our C*-algebra.

$$\Psi_{\bar{n}}(\bar{x}) = \sum_{i=1}^{m} \varphi_{n(i)}(\bar{x}^{(i)}) + \| \sum_{i=1}^{m} \sum_{j=1}^{n(i)} x_{jj}^{(i)} - I \|$$

where

$$\varphi_{n(i)}(\bar{x}^{(i)}) = \sum_{h,j,k,l=1}^{n(i)} \| x_{hj}^{(i)} x_{kl}^{(i)} - \delta_{jk} x_{hl}^{(i)} \| + \sum_{h,j=1}^{n(i)} \| x_{hj}^{(i)*} - x_{jh}^{(i)} \|$$

It should be clear that all unital, separable AF algebras are also LF algebras. We now seek to show the converse. The proof is very similar to the steps to show that LM algebras are UHF. We outline the lemmas needed and leave it to the reader to fill in the details.

Lemma 3.28. *Let A be a unital C*-algebra. Then for all $\delta > 0, \bar{n} \in \mathbb{N}^m$ for some $m \in \mathbb{N}$, there exists a $\kappa > 0$ such that if we have $\bar{a}$ and $\bar{e}$ such that $\Psi_{\bar{n}}^{A}(\bar{e}) = 0$ and $\|\bar{a} - \bar{e}\| \le \kappa$, then $\Psi_{\bar{n}}^{A}(\bar{a}) \le \delta$.*

Proof. Proceed in the same manner as for ψ_n in lemma 3.17. □

Lemma 3.29. *Let A be a unital C*-algebra with unital subalgebra B of A such that $B \cong \bigoplus_{i=1}^{m} M_{n(i)}(\mathbb{C})$ for some $\bar{n} \in \mathbb{N}^m, m \in \mathbb{N}$. Let $\bar{e} = \{e_{jk}^{(i)}\}_{\substack{1 \le i \le m \\ 1 \le j,k \le n(i)}} \subset B$ be such that $\Psi_{\bar{n}}^{A}(\bar{e}) = 0$. Furthermore, suppose that for every $\delta > 0$ there exists an $\bar{N}(\delta) = (n_\delta(1), \ldots, n_\delta(m_\delta)) \in \mathbb{N}^{m_\delta}$ such that $C(\delta) \cong \bigoplus_{i=1}^{m_\delta} M_{n_\delta(i)}(\mathbb{C})$ is a unital subalgebra of A such that $\bar{e} \subseteq_\delta C(\delta)$. Then there exists $\kappa > 0$ such that there is a $\bar{f} = \{f_{jk}^{(i)}\}_{\substack{1 \le i \le m \\ 1 \le j,k \le n(i)}} \subset C(\kappa)$ which has that $\Psi_{\bar{n}}^{A}(\bar{f}) = 0$. Furthermore we then get a unital *-homomorphism $\phi : B \to C(\kappa)$.*

Proof. Again, the proof is very similar to lemma 3.18. The unital *-homomorphism results from sending direct sum matrix units to direct sum matrix units.

□

Theorem 3.30. *Let A be a unital, separable C*-algebra. Then A is LF if and only if A is AF.*

Proof. Again, if A is AF, being LF follows almost immediately.

The proof of the converse is the same as in theorem 3.23. One again must be careful to get a countable dense subset of A in an inductively constructed direct limit of finite dimensional C*-algebras. □

We now move on to constructing a set of types whose omission in C*-algberas will allow us to characterize AF algebras. The main formula of our eventual type is as follows, given a $\bar{a} \in A^h$ and$\bar{n} \in \mathbb{N}^m$ for some $m \in \mathbb{N}$ from the type:

$$\Theta_{\bar{n}}(\bar{a}) = \inf_{\substack{\Psi_{\bar{n}}(\bar{x})=0 \\ \bar{\lambda} \in \mathbb{C}^{h\eta}, \\ |\lambda_i| \le 1}} \max\{d(a_1, \sum_{i,j,k}^{\eta} \lambda_{i_{jk}}^{1} x_{i_{jk}}), \ldots, d(a_h, \sum_{i,j,k}^{\eta} \lambda_{i_{jk}}^{h} x_{i_{jk}})\}$$

The above formula, when set with the condition, say, that $\Theta_{\bar{n}}(\bar{a}) \ge \epsilon$, says that there is some $i, 1 \le i \le k$, such that copy of $\bigoplus_{i=0}^{m} M_{n_i}(\mathbb{C})$ in A does not ϵ-include a_i.

Furthermore, we note that if, for some $\bar{n} \in \mathbb{N}^m$, there does not exist $\bar{x} \in A^\eta$ such that $\Psi_{\bar{n}}(\bar{x}) = 0$, the infinum over the empty set will be $\infty > \epsilon$, so the condition will be satisfied by any $\bar{a}$, i.e. the non-existent copy of $\bigoplus_{i=0}^{m} M_{n_i}(\mathbb{C})$ will not ϵ-include any elements from A.

Now that we have defined the above condition, we can define the principal type to be omitted. Let $\epsilon > 0, n \in \mathbb{N}$. Then we define the type $t^{(A)}_{\epsilon,n}$ as follows:

$$t^{(A)}_{\epsilon,n} = \{\Theta_{\bar{n},}(\bar{a}) \geq \epsilon : \bar{n} \in \mathbb{N}^m \text{ for some } m \in \mathbb{N},\ \bar{a} \in A^n\}$$

Theorem 3.31. *Let A be a unital, separable C^*-algebra. Then A is AF if and only if A omits all types $t^{(A)}_{\epsilon,n}$.*

Proof. If A is AF, then by our previous theorem it is also LF which clearly implies it omits all types $t^{(A)}_{\epsilon,n}$. Now suppose A omits all types $t^{(A)}_{\epsilon,n}$. Let $F = \bar{a} \in A^m$ for some $m \in \mathbb{N}$, and let $\epsilon > 0$. Then since $\bar{a}$ does not realize the type $t^{(A)}_{\epsilon,m}$, there exists a set of direct sum matrix units which ϵ-include F. We immediately see then that A is LF, and so by our last theorem it is also AF, completing the proof. □

References

[1] Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson, Alexander Usvyatsov *Model theory for metric structures.*