INTRODUCTION TO REDUCTIVE GROUP SCHEMES OVER RINGS

In construction

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1. Introduction

The theory of reductive group schemes is due to Demazure and Grothendieck and was achieved fifty years ago in the seminar SGA 3. Roughly speaking it is the theory of reductive groups in family focusing to subgroups and classification issues. It occurs in several areas: representation theory, model theory, automorphic forms, arithmetic groups and buildings, infinite dimensional Lie theory, ...

The story started as follows. Demazure asked Serre whether there is a good reason for the map $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/d\mathbb{Z})$ to be surjective for all $d > 0$. Serre answered it is a question for Grothendieck ... Grothendieck answered it is not the right question !

The right question was the development of a theory of reductive groups over schemes and especially the classification of the "split" ones. The general underlying statement is now that the specialization map $G(\mathbb{Z}) \to G(\mathbb{Z}/d\mathbb{Z})$ is onto for each semisimple group split (or Chevalley) simply connected scheme $G/\mathbb{Z}$. It is a special case of strong approximation.

Demazure-Grothendieck's theory assume known the theory of reductive groups over an algebraically closed field due mainly to C. Chevalley ([Ch], see also [Bo], [Sp]) and we will do the same. In the meantime, Borel-Tits achieved the theory of reductive groups over an arbitrary field [BT65] and Tits classified the semisimple groups [Ti1]. In the general setting, Borel-Tits theory extends to the case of a local base.

Let us warn the reader by pointing out that we do not plan to prove all hard theorems of the theory, for example the unicity and existence theorem of split reductive groups. Our purpose is more to take the user viewpoint by explaining how such results permit to analyse and classify algebraic structures.

It is not possible to enter into that theory without some background on affine group schemes and strong technical tools of algebraic geometry (descent, Grothendieck topologies,...). Up to improve afterwards certain results, the first lectures avoid descent theory and general schemes.

The aim of the notes is to try to help the people attending the lectures. It is very far to be self-contained and quotes a lot in several references

Date: March 11, 2013.

starting with [SGA3], Demazure-Gabriel's book [DG], and also the material of the Luminy's summer school provided by Brochard [Br], Conrad [C] and Oesterlé [O].

Affine group schemes I

We shall work over a base ring R (commutative and unital).

2. Sorites

2.1. **R-Functors.** We denote by $\mathcal{A}ff_R$ the category of affine R-schemes. We are interested in R–functors, i.e. covariant functors from $\mathcal{A}ff_R$ to the category of sets. If $\mathfrak{X}$ an R-scheme, it defines a covariant functor

$$h_{\mathfrak{X}}: \ \mathcal{A}ff_R \ \to \ Sets, \ S \mapsto \mathfrak{X}(S).$$

Given a map $f : \mathfrak{Y} \to \mathfrak{X}$ of R-schemes, there is a natural morphism of functors $f_* : h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ of R-functors.

We recall now Yoneda's lemma. Let F be a R-functor. If $\mathfrak{X} = \mathrm{Spec}(R[\mathfrak{X}])$ is an affine R–scheme and $\zeta \in F(R[\mathfrak{X}])$, we define a morphism of R-functors

$$\widetilde{\zeta} : h_{\mathfrak{X}} \to F$$

by $\widetilde{\zeta}(S) : h_{\mathfrak{X}}(S) = \mathrm{Hom}_R(R[\mathfrak{X}], S) \to F(S)$, $f \mapsto F(f)(\zeta)$.

Each morphism $\varphi : h_{\mathfrak{X}} \to F$ is of this shape for a unique $\zeta \in F(R[\mathfrak{X}])$: ζ is the image of $Id_{R[X]}$ by the mapping $\varphi : h_X(R[\mathfrak{X}]) \to F(R[\mathfrak{X}])$.

In particular, each morphism of functors $h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ is of the shape h_v for a unique R–morphism $v : \mathfrak{Y} \to \mathfrak{X}$.

A R-functor F is representable by an affine scheme if there exists an affine scheme $\mathfrak{X}$ and an isomorphism of functors $h_X \to F$. We say that $\mathfrak{X}$ represents F. The isomorphism $h_X \to F$ comes from an element $\zeta \in F(R[\mathfrak{X}])$ which is called the universal element of $F(R[\mathfrak{X}])$. The pair $(\mathfrak{X}, \zeta)$ satisfies the following universal property:

For each affine R-scheme $\mathfrak{T}$ and for each $\eta \in F(R[\mathfrak{T}])$, there exists a unique morphism $u : \mathfrak{T} \to \mathfrak{X}$ such that $F(u^*)(\zeta) = \eta$.

2.2. **Definition.** An affine R–group scheme $\mathfrak{G}$ is a group object in the category of affine R-schemes. It means that $\mathfrak{G}/R$ is an affine scheme equipped with a section $\epsilon : \mathrm{Spec}(R) \to \mathfrak{G}$, an inverse $\sigma : \mathfrak{G} \to \mathfrak{G}$ and a multiplication $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ such that the three following diagrams commute:

Associativity:

$$\begin{tikzcd}
(\mathfrak{G} \times_R \mathfrak{G}) \times_R \mathfrak{G} \arrow[r, "m \times id"] \arrow[d, "can"', "\cong"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[r, "m"] & \mathfrak{G} \\
\mathfrak{G} \times_R (\mathfrak{G} \times_R \mathfrak{G}) \arrow[r, "id \times m"] & \mathfrak{G} \times_R \mathfrak{G} \arrow[ur, "m"'] &
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
\mathfrak{G} \times_R \mathrm{Spec}(R) \arrow[r, "id \times \epsilon"] \arrow[dr, "p_2"'] & \mathfrak{G} \times_R \mathfrak{G} \arrow[d, "m"] & \mathrm{Spec}(R) \times \mathfrak{G} \arrow[l, "\epsilon \times id"'] \arrow[dl, "p_1"] \\
& \mathfrak{G} &
\end{tikzcd}$$

Symetry:

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "id\times\sigma"] \arrow[d, "s_{\mathfrak{G}}"'] & \mathfrak{G}\times_R \mathfrak{G}. \arrow[d, "m"'] \\ \mathrm{Spec}(R) \arrow[r, "\epsilon"] & \mathfrak{G} \end{tikzcd}$$

We say that $\mathfrak{G}$ is commutative if furthermore the following diagram commutes

$$\begin{tikzcd} \mathfrak{G}\times_R \mathrm{Spec}(R) \arrow[r, "switch"] \arrow[d, "m"'] & \mathfrak{G}\times_R \mathfrak{G} \arrow[d, "m"'] \\ \mathfrak{G} \arrow[r, phantom, "="] & \mathfrak{G}. \end{tikzcd}$$

Let $R[\mathfrak{G}]$ be the coordinate ring of $\mathfrak{G}$. We call $\epsilon^* : R[\mathfrak{G}] \to \mathfrak{G}$ the counit (augmentation), $\sigma^* : R[\mathfrak{G}] \to R[G]$ the coinverse (antipode), and denote by $\Delta = m^* : R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]$ the comultiplication. They satisfies the following rules:

Co-associativity:

$$\begin{tikzcd} R[\mathfrak{G}] \arrow[r, "m^*"] \arrow[dr, "m^*"'] & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[r, "m^*\otimes id"] & (R[\mathfrak{G}]\otimes_R R[\mathfrak{G}])\otimes_R R[\mathfrak{G}] \\ & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] \arrow[r, "id\otimes m^*"] & R[\mathfrak{G}]\otimes_R (R[\mathfrak{G}]\otimes_R R[\mathfrak{G}]). \arrow[u, "can"', "\cong"] \end{tikzcd}$$

Counit:

$$\begin{tikzcd} R[\mathfrak{G}] \arrow[r, "id\otimes\epsilon^*"] & R[\mathfrak{G}]\otimes_R R[\mathfrak{G}] & R[\mathfrak{G}] \arrow[l, "\epsilon^*\times id"'] \\ & R[\mathfrak{G}] \arrow[ul, "id"] \arrow[u, "m^*"] \arrow[ur, "id"'] & \end{tikzcd}$$

Cosymmetry:

$$\begin{tikzcd} R[\mathfrak{G}]\otimes R[\mathfrak{G}] \arrow[r, "\sigma^*\otimes id"] & R[\mathfrak{G}]. \\ R[\mathfrak{G}] \arrow[r, "\epsilon^*"] \arrow[u, "m^*"] & R. \arrow[u, "s^*_{\mathfrak{G}}"] \end{tikzcd}$$

In other words, $(R[\mathfrak{G}], m^*, \sigma^*, \epsilon^*)$ is a commutative Hopf R–algebra[1]. Given an affine R–scheme $\mathfrak{X}$, there is then a one to one correspondence between group structures on $\mathfrak{X}$ and Hopf R–algebra structures on $R[\mathfrak{X}]$.

If $\mathfrak{G}/R$ is an affine R–group scheme, then for each R–algebra S the abtract group $\mathfrak{G}(S)$ is equipped with a natural group structure. The multiplication is $m(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to \mathfrak{G}(S)$, the unit element is $1_S = (\epsilon \times_R S) \in \mathfrak{G}(S)$ and the inverse is $\sigma(S) : \mathfrak{G}(S) \to \mathfrak{G}(S)$. It means that the functor $h_{\mathfrak{G}}$ is actually a group functor.

[1] This is Waterhouse definition [Wa, §I.4], other people talk abour cocommutative coassociative Hopf algebra.

2.2.1. **Lemma.** *Let $\mathfrak{X}/R$ be an affine scheme. Then the Yoneda lemma induces a one to one correspondence between group structures on $\mathfrak{X}$ and group structures on $h_{\mathfrak{X}}$.*

In other words, defining a group law on $\mathfrak{X}$ is the same that do define compatible group laws on each $\mathfrak{G}(S)$ for S running over the R-algebras.

2.2.2. **Remark.** We shall encounter certain non-affine group R-schemes. A group scheme $\mathfrak{G}/R$ is a group object in the category of R-schemes.

3. Examples

3.1. **Constant group schemes.** Let Γ be an abstract group. We consider the R–scheme $\mathfrak{G} = \bigsqcup_{\gamma \in \Gamma} \mathrm{Spec}(R)$. Then the group structure on Γ induces a group scheme structure on $\mathfrak{G}$ with multiplication

$$\mathfrak{G} \times_R \mathfrak{G} = \bigsqcup_{(\gamma,\gamma') \in \Gamma^2} \mathrm{Spec}(R) \to \mathfrak{G} = \bigsqcup_{\gamma \in \Gamma} \mathrm{Spec}(R)$$

applying the component (γ, γ') to $\gamma\gamma'$; This group scheme is affine iff Γ is finite.

There usual notation for such an object is Γ_R. This group scheme occurs as solution of the following universal problem.

3.2. **Vector groups.** Let N be a R–module. We consider the commutative group functors

$$V_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto \mathrm{Hom}_S(N \otimes_R S, S) = (N \otimes_R S)^\vee,$$
$$W_N : \mathcal{A}ff_R \to Ab, \;\; S \mapsto N \otimes_R S.$$

3.2.1. **Lemma.** *The R–group functor V_N is representable by the affine R–scheme $\mathfrak{V}(N) = \mathrm{Spec}(S^*(N))$ which is then a commutative R–group scheme. Furthermore N is of finite presentation if and only if $\mathfrak{V}(N)$ is of finite presentation.*

Proof. It follows readily of the universal property of the symmetric algebra

$$\mathrm{Hom}_{R'-mod}(N \otimes_R R', R') \xleftarrow{\sim} \mathrm{Hom}_{R-mod}(N, R') \xrightarrow{\sim} \mathrm{Hom}_{R-alg}(S^*(N), R')$$

for each R-algebra R'. □

The commutative group scheme $\mathfrak{V}(N)$ is called the vector group-scheme associated to N. We note that $N = \mathfrak{V}(N)(R)$.

Its group law on the R–group scheme $\mathfrak{V}(N)$ is given by $m^* : S^*(N) \to S^*(N) \otimes_R S^*(N)$, applying each $X \in N$ to $X \otimes 1 + 1 \otimes X$. The counit is $\sigma^* : S^*(N) \to S^*(N)$, $X \mapsto -X$.

3.2.2. **Remarks.** (1) If $N = R$, we get the affine line over R. Given a map $f : N \to N'$ of R–modules, there is a natural map $f^* : \mathfrak{V}(N') \to \mathfrak{V}(N)$.
(2) If N is projective and finitely generated, we have $W(N) = V(N^\vee)$ so that $\mathfrak{W}(N)$ is representable by an affine group scheme.

(3) If R is noetherian, Nitsure showed the converse holds [Ni04]. If N is finitely generated projective, then $\mathfrak{W}(N)$ is representable iff N is locally free.

3.2.3. **Lemma.** *The construction of (1) provides an antiequivalence of categories between the category of R-modules and that of vector group R-schemes.*

3.3. **Group of invertible elements, linear groups.** Let A/R be an algebra (unital, associative). We consider the R-functor

$$S \mapsto \mathrm{GL}_1(A)(S) = (A \otimes_R S)^\times.$$

3.3.1. **Lemma.** *If A/R is finitely generated projective, then* $\mathrm{GL}_1(A)$ *is representable by an affine group scheme. Furthermore,* $\mathrm{GL}_1(A)$ *is of finite presentation.*

Proof. We shall use the norm map $N : A \to R$ defined by $a \mapsto \det(L_a)$ constructed by glueing. We have $A^\times = N^{-1}(R^\times)$ since the inverse of L_a can be written L_b by using the characteristic polynomial of L_a. The same is true after tensoring by S, so that

$$\mathrm{GL}_1(A)(S) = \Big\{ a \in (A \otimes_R S) = \mathfrak{W}(A)(S) \quad | \quad N(a) \in R^\times \Big\}.$$

We conclude that $\mathrm{GL}_1(A)$ is representable by the fibered product

$$\begin{tikzcd} \mathfrak{G} \arrow[r] \arrow[d] & \mathfrak{W}(A) \arrow[d, "N"'] \\ \mathfrak{G}_{m,R} \arrow[r] & \mathfrak{W}(R). \end{tikzcd}$$

□

Given a R–module N, we consider the R–group functor

$$S \mapsto \mathrm{GL}_1(N)(S) = \mathrm{Aut}_{S-mod}(N \otimes_R S).$$

So if N is finitely generated projective. then $\mathrm{GL}_1(N)$ is representable by an affine R–group scheme. Furthermore $\mathrm{GL}_1(N)$ is of finite presentation.

3.3.2. **Remark.** If R is noetherian, Nitsure has proven that $\mathrm{GL}_1(N)$ is representable if and only if N is projective [Ni04].

3.4. **Diagonalisable group schemes.** Let A be a commutative abelian (abstract) group. We denote by $R[A]$ the group R–algebra of A. As R-module, we have

$$R[A] = \bigoplus_{a \in A} R\, e_a$$

and the multiplication is given by $e_a\, e_b = e_{a+b}$ for all $a, b \in A$.

For $A = \mathbb{Z}$, $R[\mathbb{Z}] = R[T, T^{-1}]$ is the Laurent polynomial ring over R. We have an isomorphism $R[A] \otimes_R R[B] \xrightarrow{\sim} R[A \times B]$. The R-algebra $R[A]$ carries the following Hopf algebra structure:

Comultiplication: $\Delta : R[A] \to R[A] \otimes R[A]$, $\Delta(e_a) = e_a \otimes e_a$, *Antipode:* $\sigma^* : R[A] \to R[A]$, $\sigma^*(e_a) = e_{-a}$;

Augmentation: $\epsilon^* : R[A] \to R$, $\epsilon(e_a) = 1$.

3.4.1. **Definition.** *We denote by* $\mathfrak{D}(A)/R$ *(or* $\widehat{A}$*) the affine commutative group scheme* $\mathrm{Spec}(R[A])$*. It is called the diagonalizable* R*–group scheme of base* A*. An affine* R*–group scheme is diagonalizable if it is isomorphic to some* $\mathfrak{D}(B)$*.*

We denote by $\mathfrak{G}_m = \mathfrak{D}(\mathbb{Z}) = \mathrm{Spec}(R[T, T^{-1}])$, it is called the multiplicative group scheme. We note also that there is a natural group scheme isomorphism $\mathfrak{D}(A \oplus B) \xrightarrow{\sim} \mathfrak{D}(A) \times_R \mathfrak{D}(B)$. We let in exercise the following fact.

3.4.2. **Lemma.** *The following are equivalent:*

(i) A *is finitely genereated;*

(ii) $\mathfrak{D}(A)/R$ *is finite presentation;*

(iii) $\mathfrak{D}(A)/R$ *is of finite type.*

If $f : B \to A$ is a morphism of abelian groups, it induces a group homomorphism $f^* : \mathfrak{D}(A) \to \mathfrak{D}(B)$. In particular, when taking $B = \mathbb{Z}$, we have a natural mapping

$$\eta_A : A \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{G}_m).$$

3.4.3. **Lemma.** *If* R *is connected,* η_A *is bijective.*

Proof. Let $f : \mathfrak{D}(A) \to \mathfrak{G}_m$ be a group R–morphisms. Equivalently it is given by the map $f^* : R[T, T^{-1}] \to R[A]$ of Hopf algebra. In other words, it is determined by the function $X = f(T) \in R[A]^\times$ satisfying $\Delta(X) = X \otimes X$. Writing $X = \sum_{a \in A} r_a e_a$, the relation reads as follows $r_a r_b = 0$ if $a \neq b$ and $r_a \, r_a = r_a$. Since the ring is connected, 0 and 1 are the only idempotents so that $r_a = 0$ or $r_a = 1$. Then there exists a unique a such that $r_a = 1$ and $r_b = 0$ for $b \neq a$. This shows that the map η_A is surjective. It is obviously injective so we conclude that η_A is bijective. $\square$

3.4.4. **Proposition.** *Assume that* R *is connected. The above construction induces an anti-equivalence of categories between the category of abelian groups and that of diagonalizable* R*–group schemes.*

Proof. It is enough to contruct the inverse map $\mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{D}(B)) \to \mathrm{Hom}(A, B)$ for abelian groups A, B. We are given a group homomorphism $f : \mathfrak{D}(A) \to \mathfrak{D}(B)$. It induces a map

$$f^* : \mathrm{Hom}_{R-gp}(\mathfrak{D}(B), \mathfrak{G}_m) \to \mathrm{Hom}_{R-gp}(\mathfrak{D}(A), \mathfrak{G}_m),$$

hence a map $B \to A$. $\square$

4. Sequences of group functors

4.1. **Exactness.** We say that a sequence of R–group functors

$$1 \to F_1 \xrightarrow{u} F_2 \xrightarrow{v} F_3 \to 1$$

is exact if for each R–algebra S, the sequence of abstract groups

$$1 \to F_1(S) \xrightarrow{u(S)} F_2(S) \xrightarrow{v(S)} F_3(S) \to 1$$

is exact.

If $w : F \to F'$ is a map of R–group functors, we denote by $\ker(w)$ the R–group functor defined by $\ker(w)(S) = \ker(F(S) \to F'(S))$ for each R–algebra S.

If $w(S)$ is onto for each R–algebra S/R, it follows that

$$1 \to \ker(w) \to F \xrightarrow{w} F' \to 1$$

is an exact sequence of R–group functors.

4.1.1. **Lemma.** *Let $f : \mathfrak{G} \to \mathfrak{G}'$ be a morphism of R–group schemes. Then the R–functor $\ker(f)$ is representable by a closed subgroup scheme of $\mathfrak{G}$.*

Proof. Indeed the carthesian product

$$\begin{tikzcd} \mathfrak{N} \arrow[r] \arrow[d] & \mathfrak{G} \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r, "\epsilon'"] & \mathfrak{G}' \end{tikzcd}$$

does the job. $\square$

We can define also the cokernel of a R–group functor. But it is very rarely representable. The simplest example is the Kummer morphism $f_n : \mathfrak{G}_{m,R} \to \mathfrak{G}_{m,R}$, $x \mapsto x^n$ for $n \geq 2$ for $R = \mathbb{C}$, the field of complex numbers. Assume that there exists an affine $\mathbb{C}$–group scheme $\mathfrak{G}$ such that there is a four terms exact sequence of $\mathbb{C}$–functors

$$1 \to h_{\mu_n} \to h_{\mathfrak{G}_m} \xrightarrow{h_{f_n}} h_{\mathfrak{G}_m} \to h_{\mathfrak{G}} \to 1$$

We denote by T' the parameter for the first $\mathfrak{G}_m$ and by $T = (T')^n$ the parameter of the second one. Then $T \in \mathfrak{G}_m(R[T, T^{-1}])$ defines a non trivial element of $\mathfrak{G}(R[T, T^{-1}])$ which is trivial in $\mathfrak{G}(R[T', T'^{-1}])$ It is a contradiction.

4.2. **Semi-direct product.** Let $\mathfrak{G}/R$ be an affine group scheme acting on another affine group scheme $\mathfrak{H}/R$, that is we are given a morphism of R–functors

$$\theta : h_{\mathfrak{G}} \to \mathrm{Aut}(h_{\mathfrak{H}}).$$

The semi-direct product $h_{\mathfrak{H}} \rtimes^\theta h_{\mathfrak{G}}$ is well defined as R–functor.

4.2.1. **Lemma.** *$h_{\mathfrak{Y}} \rtimes^\theta h_{\mathfrak{X}}$ is representable by an affine R-scheme.*

Proof. We consider the affine R-scheme $\mathfrak{X} = \mathfrak{H} \times_R \mathfrak{G}$. Then $h_{\mathfrak{X}} = h_{\mathfrak{H}} \rtimes^{\theta} h_{\mathfrak{G}}$ has a group structure so defines a group scheme structure on $\mathfrak{X}$. □

4.3. **Monomorphisms of group schemes.** A morphism of R–functors $f : F \to F'$ is a monomorphism if $f(S) : F(S) \to F'(S)$ is injective for each R–algebra S/R. We say that a morphism $f : \mathfrak{Y} \to \mathfrak{X}$ of affine R-group schemes is a monomorphism if h_f is a monomorphism.

Over a field F, we know that a monomorphism of algebraic groups is a closed immersion [SGA3, VI_B.1.4.2].

Over a DVR, it is not true in general that an open immersion (and a fortiori a monomorphism) of group schemes of finite type is a closed immersion. We consider the following example [SGA3, VIII.7]. Assume that R is a DVR and consider the constant group scheme $\mathfrak{H} = (\mathbb{Z}/2\mathbb{Z})_R$. Now let $\mathfrak{G}$ be the open subgroup scheme of $\mathfrak{H}$ which is the complement of the closed point 1 in the closed fiber. By construction $\mathfrak{G}$ is dense in $\mathfrak{H}$, so that the immersion $\mathfrak{G} \to \mathfrak{H}$ is not closed.

However diagonalizable groups have a wonderful behaviour with that respect.

4.3.1. **Proposition.** *Let $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ be a group homomorphism of diagonalizable R–group schemes. Then the following are equivalent:*

(i) $f^* : A \to B$ is onto;

(ii) f is a closed immersion;

(iii) f is a monomorphism.

Proof. $(i) \Longrightarrow (ii)$: Then $R[\mathfrak{B}]$ is a quotient of $\mathfrak{R}[A]$ so that $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is a closed immersion.

$(ii) \Longrightarrow (iii)$: obvious.

$(iii) \Longrightarrow (i)$: We denote by $B_0 \subset B$ the image of f^*. The compositum of monomorphisms

$$\mathfrak{D}(B/B_0) \to \mathfrak{D}(B) \to \mathfrak{D}(B_0) \to \mathfrak{D}(A)$$

is a monomorphism and is zero. It follows that $\mathfrak{D}(B/B_0) = \mathrm{Spec}(R)$ and we conclude that $B_0 = B$. □

Of the same flavour, the kernel of a map $f : \mathfrak{D}(B) \to \mathfrak{D}(A)$ is isomorphic to $\mathfrak{D}(f(A))$. The case of vector groups is more subtle.

4.3.2. **Proposition.** *Let $f : N_1 \to N_2$ be a morphism of finitely generated projective R-modules. Then the morphism of functors $f_* : W(N_1) \to W(N_2)$ is a mononomorphism if and only if f identifies locally N_1 as a direct summand of N_2. If it the case, $f_* : \mathfrak{W}(N_1) \to \mathfrak{W}(N_2)$ is a closed immersion.*

Proof. We can assume that R is local with maximal ideal M, so that N_1 and N_2 are free. If f identifies locally N_1 as a direct summand of N_2, then f_* identifies locally $W(N_1)$ as a direct summand of $W(N_2)$ hence f is a monomorphism. Conversely suppose that f_* is a monomorphism. Then the map $f_*(R/M) : N_1 \otimes_R R/M \to N_2 \otimes_R R/M$ is injective and there exists a R/M-base $(\overline{w}_1, ..., \overline{w}_r, \overline{w}_{r+1}, \dots, \overline{w}_n)$ of $N_2 \otimes_R R/M$ such that $(\overline{w}_1, ..., \overline{w}_r)$ is a base of $f(N_1 \otimes_R R/M)$. We have $\overline{w}_i = f(\overline{v}_i)$ for $i = 1, .., r$. We lift the $\overline{v}_i$ in an arbitrary way in N_1 and the $\overline{w}_{r+1}, \dots, \overline{w}_n$ in N_2. Then $(v_1, ..., v_r)$ is a R-base of N_1 and $(f(v_1), \dots, f(v_r), w_{r+1}, \dots, w_n)$ is a R–base of N_2. We conclude that f identifies N_1 as a direct summand of N_2. □

This shows that an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of f.g. modules with N_1, N_2 projective induces an exact sequence of R–functors

$$0 \to W(N_1) \to W(N_2) \to W(N_3) \to 0$$

if and only if the starting sequence splits locally.

Affine group schemes II

5. Flatness

5.1. **The DVR case.** Assume that R is a DVR with uniformizing parameter π and denote by K its field of fractions. We know that an affine scheme $\mathfrak{X}/R$ is flat iff it is torsion free, that is $R[\mathfrak{X}]$ embeds in $K[\mathfrak{X}]$ [Ma, Exercise 10.2]. If $\mathfrak{X}/R$ is flat, there is a correspondence between the flat closed R-subschemes of $\mathfrak{X}$ and the closed K–subschemes of the generic fiber $\mathfrak{X}_K$ [EGA4, 2.8.1]. In one way we take the generic fiber and in the way around we take the schematic adherence. Let us explain the construction in terms of rings. If Y/K is a closed K–subscheme of X/K, it is defined by an ideal $I(Y) = \mathrm{Ker}(K[\mathfrak{X}] \to K[Y])$ of $K[\mathfrak{X}]$. The the schematic closure $\mathfrak{Y}$ of Y in $\mathfrak{X}$ is defined by the ideal $I(\mathfrak{Y}) = I \cap R[\mathfrak{X}]$. Since $I(\mathfrak{Y}) \otimes_R K = I(Y)$, we have $R(\mathfrak{Y}) \otimes_R K = K[Y]$, that is $\mathfrak{Y} \times_R K = Y_K$. Also the map $R[\mathfrak{Y}] \to K[Y]$ is injective, i.e $\mathfrak{Y}$ is a flat affine R–scheme.

This correspondence commutes with fibered products over R. In particular, if $\mathfrak{G}/R$ is a flat group scheme, it induces a one to one correspondence between flat closed R-subgroup schemes of $\mathfrak{G}$ and closed K–subgroup schemes of ${\mathfrak{G}_K}$[2].

We can consider the centralizer closed subgroup scheme of GL_2

$$\mathfrak{Z} = \Big\{ g \in \mathrm{GL}_{2,R} \quad | \quad g\,A = A\,g \Big\}$$

of the element $A = \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$. Then $\mathfrak{Z} \times_R R/\pi R \xrightarrow{\sim} \mathrm{GL}_{2,R}$ and

$$\mathfrak{Z} \times_R K = \mathfrak{G}_{m,K} \times_K \mathfrak{G}_{a,K} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \right\}$$

Then the adherence of $\mathfrak{Z}_K$ in $\mathrm{GL}_{2,R}$ is $\mathfrak{G}_{m,R} \times_R \mathfrak{G}_{a,R}$, so does not contain the special fiber of $\mathfrak{Z}$. We conclude that $\mathfrak{Z}$ is not flat.

5.2. **A necessary condition.** In the above example, the geometrical fibers were of dimension 4 and 2 respectively. It illustrates then the following general result.

5.2.1. **Theorem.** [SGA3, VI_B.4.3] *Let $\mathfrak{G}/R$ be a flat group scheme of finite presentation. Then the dimension of the geometrical fibers is locally constant.*

5.3. **Examples.** Constant group schemes and diagonalizable groups schemes are flat. If N is a finitely generated projective R-module, the affine groups schemes $\mathfrak{V}(N)$ and $\mathfrak{W}(N)$ are flat.

5.3.1. **Remark.** What about the converse ? If N is of finite presentation, is is true that N is flat iff $\mathfrak{W}(N)$ is flat (?).

[2]Warning: the fact that the schematic closure of a group scheme is a group scheme is specific to Dedekind rings.

The group scheme of invertible elements of an algebra A/R f.g. projective is flat since it is open in $\mathfrak{V}(A)$.

6. Representations

Let $\mathfrak{G}/R$ be an a affine group scheme.

6.0.2. **Definition.** *A (left) $R-\mathfrak{G}$-module (or $\mathfrak{G}$-module for short) is a R–module M equipped with a morphism of group functors*

$$\rho : h_{\mathfrak{G}} \to \mathrm{Aut}(W(M)).$$

We say that the $\mathfrak{G}$–module M is faithful is ρ is a monomorphism.

It means that for each algebra S/R, we are given an action of $\mathfrak{G}(S)$ on $W(M)(S) = M \otimes_R S$. We use again Yoneda lemma. The mapping ρ is defined by the image of the universal point $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ provides an element called the coaction

$$c_\rho \in \mathrm{Hom}_R\Big(M, M \otimes_R R[\mathfrak{G}]\Big) \xrightarrow{\sim} \mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big).$$

We denote $\widetilde{c}_\rho$ its image in $\mathrm{Hom}_{R[\mathfrak{G}]}\Big(M \otimes_R R[\mathfrak{G}], M \otimes_R R[\mathfrak{G}]\Big)$.

6.0.3. **Proposition.** *(1) Both diagrams*

$$\begin{array}{ccc}
M & \xrightarrow{c_\rho} & M \otimes_R R[\mathfrak{G}] \\
{\scriptstyle c_\rho}\downarrow & & \downarrow{\scriptstyle id\times\Delta_{\mathfrak{G}}} \\
M \otimes_R \mathfrak{R}[\mathfrak{G}] & \xrightarrow{c_\rho\times id} & M \otimes_R R[\mathfrak{G}] \otimes R[\mathfrak{G}],
\end{array}$$

$$\begin{array}{ccc}
M & \xrightarrow{c_\rho} & M \otimes_R R[\mathfrak{G}] \\
\downarrow & \swarrow{\scriptstyle id\times\epsilon^*} & \\
M & &
\end{array}$$

commute.

(2) Conversely, if a R–map $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above, there is a unique representation $\rho_c : h_{\mathfrak{G}} \to \mathrm{GL}(W(M))$ such that $c_{\rho_c} = c$.

A module M equipped with a R–map $c : M \to M \otimes_R R[\mathfrak{G}]$ satisfying the two rules above is called a $\mathfrak{G}$-module (and also a comodule over the Hopf algebra $R[\mathfrak{G}]$). The proposition shows that it is the same to talk about representations of $\mathfrak{G}$ or about $\mathfrak{G}$-modules.

In particular, the comultiplication $R[\mathfrak{G}] \to R[\mathfrak{G}] \otimes R[\mathfrak{G}]$ defines a $\mathfrak{G}$-structure on the R–module $R[\mathfrak{G}]$. It is called the regular representation.

Proof. (1) We double the notation by putting $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$. We consider the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{G}(R[\mathfrak{G}_1]) \times \mathfrak{G}(R[\mathfrak{G}_2]) & \xrightarrow{\rho\times\rho} & \mathrm{Aut}(V(M))(R[\mathfrak{G}_1]) \times \mathrm{Aut}(V(M))(R[\mathfrak{G}_2]) \\
\downarrow & & \downarrow \\
\mathfrak{G}(R[\mathfrak{G}_1\times\mathfrak{G}_2]) \times \mathfrak{G}(R[\mathfrak{G}_1\times\mathfrak{G}_2]) & \xrightarrow{\rho\times\rho} & \mathrm{Aut}(V(M))(R[\mathfrak{G}_1\times\mathfrak{G}_2]) \times \mathrm{Aut}(V(M))(R[\mathfrak{G}_1\times\mathfrak{G}_2]) \\
{\scriptstyle m}\downarrow & & {\scriptstyle m}\downarrow \\
\mathfrak{G}(R[\mathfrak{G}_1\times\mathfrak{G}_2]) & \xrightarrow{\rho} & \mathrm{Aut}(W(M))(R[\mathfrak{G}_1\times\mathfrak{G}_2])
\end{array}$$

and consider the image $\eta \in \mathfrak{G}(R[\mathfrak{G}_1 \times \mathfrak{G}_2])$ of the couple (ζ_1, ζ_2) of universal elements. Then η is defined by the ring homomorphism $\eta^* : R[G] \overset{\Delta_{\mathfrak{G}}}{\to} R[\mathfrak{G}\times\mathfrak{G}] \overset{\sim}{\longrightarrow} R[\mathfrak{G}_1\times\mathfrak{G}_2]$ It provides the commutative diagram

$$\begin{array}{ccc}
M \otimes R[\mathfrak{G}_1\times\mathfrak{G}_2] & \xrightarrow{\widetilde{c}_{\rho(\eta)}} & M \otimes R[\mathfrak{G}_1\times\mathfrak{G}_2] \\
{\scriptstyle \widetilde{c}_{\rho,2}}\downarrow & & \nearrow {\scriptstyle \widetilde{c}_{\rho,1}} \\
M \otimes R[\mathfrak{G}_1\times\mathfrak{G}_2] & &
\end{array}$$

But the map $M \to M \otimes_R R[\mathfrak{G}_1\times\mathfrak{G}_2] \overset{\widetilde{c}_{\rho(\eta)}}{\to} M \otimes R[\mathfrak{G}_1\times\mathfrak{G}_2]$ is the compositum $M \overset{c}{\to} R[\mathfrak{G}] \overset{\Delta_{\mathfrak{G}}}{\to} M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \cong M \otimes_R R[\mathfrak{G}_1] \otimes_R R[\mathfrak{G}_2]$. Hence we get the commutative square

$$\begin{array}{ccc}
M & \xrightarrow{c_\rho} & M \otimes R[\mathfrak{G}] \\
{\scriptstyle c_{\rho,1}}\downarrow & & {\scriptstyle id\times\Delta_{\mathfrak{G}}}\downarrow \\
M \otimes R[\mathfrak{G}_1] & \xrightarrow{c_{\rho,2}} & M \otimes R[\mathfrak{G}_1\times\mathfrak{G}_2].
\end{array}$$

as desired. The other rule comes from the fact that $1 \in G(R)$ acts trivially on M.

(2) This follows again from Yoneda.

□

A morphism of $\mathfrak{G}$-modules is a R–morphism $f : M \to M'$ such that $f(S) \circ \rho(g) = \rho'(g) \circ f(S) \in \mathrm{Hom}_S(M \otimes_R S, M' \otimes_R S)$ for each S/R. It is clear that the R–module $\mathrm{coker}(f)$ is equipped then with a natural structure of $\mathfrak{G}$-module. For the kernel $\ker(f)$, we cannot proceed similarly because the mapping $\ker(f) \otimes_R S \to \ker(M \otimes_R S \overset{f(S)}{\to} M' \otimes_R S)$ is not necessarily injective. One tries to use the module viewpoint by considering the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] & M \arrow[r, "f"] \arrow[d, "c_\rho"] & M' \arrow[d, "c_{\rho'}"] \\
& \ker(f)\otimes_R R[\mathfrak{G}] \arrow[r] & M\otimes_R R[\mathfrak{G}] \arrow[r, "f\otimes id"] & M'\otimes_R R[\mathfrak{G}].
\end{tikzcd}$$

If $\mathfrak{G}$ is flat, then the left bottom map is injective, and the diagram defines a map $c : \ker(f) \to \ker(f) \otimes_R R[\mathfrak{G}]$. This map c satisfies the two compatibilities and define then a $\mathfrak{G}$-module structure on $\ker(f)$. We have proven the important fact.

6.0.4. **Proposition.** *Assume that $\mathfrak{G}/R$ is flat. Then the category of $\mathfrak{G}$-modules is an abelian category.*

6.1. **Representations of diagonalizable group schemes.** Let $\mathfrak{G} = \mathfrak{D}(A)/R$ be a diagonalizable group scheme. For each $a \in A$, we can attach a character $\chi_a = \eta_A(a) : \mathfrak{D}(A) \to \mathfrak{G}_m = \mathrm{GL}_1(R)$. It defines then a $\mathfrak{G}$–structure of the R–module R. If $M = \oplus_{a\in A} M_a$ is a A-graded R–module, the group scheme $\mathfrak{D}(A)$ acts diagonally on it by χ_a on each piece M_a.

6.1.1. **Proposition.** *The category of A–graded R-modules is equivalent to the category of $R - \mathfrak{D}(A)$-modules.*

Proof. Let M be a $R - \mathfrak{D}(A)$-module and consider the underlying map $c : M \to M \otimes_R R[A]$. We write $c(m) = \sum\limits_{m\in M} \varphi_a(m) \otimes e_a$ and the first (resp. second) condition reads

$$\varphi_a \circ \varphi_b = \delta_{a,b}\varphi_a \quad (\text{resp. } \sum_{a\in A} \varphi_a = id_M).$$

Hence the φ_a's are pairwise orthogonal projectors whose sum is the identity. Thus $M = \bigoplus_{a\in A} \varphi_a(M)$ which decomposes a direct summand of eigenspaces. □

6.1.2. **Corollary.** *Each exact sequence* $0 \to M_1 \to M_2 \to M_3 \to 1$ *of $R - \mathfrak{D}(A)$–modules splits.*

6.2. **Existence of faithful representations.** Over a field, we know that an affine algebraic group admits a faithful finite dimensional representation. It is a special case of the following

6.2.1. **Theorem.** *Assume that R is noetherian. Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a faithful $\mathfrak{G}$-module which is f.g. free as R–module.*

The key thing is the following fact due to Serre [Se3, §1.5, prop. 2].

6.2.2. **Proposition.** *Under the assumptions of the theorem, let M be a $\mathfrak{G}$-module which is flat as R-module. Let N be a R-submodule of M of finite type. Then there exists a $R-\mathfrak{G}$-submodule $\widetilde{N}$ of M which contains N and is f.g. as R-module.*

We can now proceed to the proof of Theorem 6.2.1. We take $M = R[\mathfrak{G}]$ seen as the regular representation $\mathfrak{G}$–module. The proposition shows that M is the direct limit of the family of $\mathfrak{G}$-submodules $(M_i)_{i\in I}$ which are f.g. as R-modules. Since M is flat over R, the M_i's are flat as well and are then projective. We look at the kernel $\mathfrak{N}_i/R$ of the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$. The regular representation is faithful and its kernel is the intersection of the $\mathfrak{N}_i$. Since $\mathfrak{G}$ is a noetherian scheme, there is an index i such that $\mathfrak{N}_i = 1$. In other words, the representation $\mathfrak{G} \to \mathrm{GL}(M_i)$ is faithful. Now M_i is a direct summand of a free module R^n, i.e. $R^n = M_i \oplus M_i'$. It provides a representation $\mathfrak{G} \to \mathrm{GL}(M_i) =\!\!\to \mathrm{GL}(M_i \oplus M_i')$ which is faithful and such that the underlying module is free.

In the Dedekind ring case, Raynaud proved a stronger statement which was extended by Gabber in dimension two.

6.2.3. **Theorem.** *(Raynaud-Gabber* [SGA3, VI_B.13.2]*) Assume that R is a regular ring of dimension ≤ 2. Let $\mathfrak{G}/R$ be a flat affine group scheme of finite type. Then there exists a $\mathfrak{G}$-module M isomorphic to R^n as R–module such that $\rho_M : \mathfrak{G} \to \mathrm{GL}_n(R)$ is a closed immersion.*

In the Dedekind case, an alternative proof is §1.4.5 of [BT2].

6.3. **Hochschild cohomology.** We assume that $\mathfrak{G}$ is flat. If M is a $\mathfrak{G}$–module, we consider the R–module of invariants $M^{\mathfrak{G}}$ defined by

$$M^{\mathfrak{G}} = \Big\{ m \in M \ \mid m \otimes 1 = c(m) \in M \otimes_R R[\mathfrak{G}] \Big\}.$$

It is the largest trivial $\mathfrak{G}$-submodules of M and we have also $M^{\mathfrak{G}} = \mathrm{Hom}_{\mathfrak{G}}(R, M)$. We can then mimick the theory of cohomology of groups.

6.3.1. **Lemma.** *The category of $R-\mathfrak{G}$-modules has enough injective.*

We shall use the following extrem case of induction, see [J, §2, 3] for the general theory.

6.3.2. **Lemma.** *Let N be a R–module. Then for each $\mathfrak{G}$-module M the mapping*

$$\psi : \mathrm{Hom}_{\mathfrak{G}}(M, N \otimes_R R[\mathfrak{G}]) \to \mathrm{Hom}_R(M, N),$$

given by taking the composition with the augmentation map, is an isomorphism.

Proof. We define first the converse map. We are given a R-map $f_0 : M \to N$ and consider the following diagram

$$\begin{array}{ccccc} M & \xrightarrow{c_M} & M \otimes_R R[\mathfrak{G}] & \xrightarrow{f_0 \otimes id} & N \otimes_R R[\mathfrak{G}] \\ {\scriptstyle c_M}\downarrow & & {\scriptstyle id \times \Delta_G}\downarrow & & {\scriptstyle id \times \Delta_G}\downarrow \\ M \otimes_R R[\mathfrak{G}] & \xrightarrow{c_M \otimes id} & M \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] & \xrightarrow{f_0 \otimes id} & N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}]. \end{array}$$

The right square commutes obviously and the left square commutes since M is a $\mathfrak{G}$-module. It defines then a map $f : M \to N \otimes_R R[\mathfrak{G}]$ of $\mathfrak{G}$-modules. By construction we have $\psi(f) = f_0$. In the way around we are given a $\mathfrak{G}$–map $h : M \to N \otimes_R R[\mathfrak{G}]$ and denote by $h_0 : M \to N \otimes_R R[\mathfrak{G}] \to N \to 0$. We consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & N \otimes_R R[\mathfrak{G}] \\ {\scriptstyle c_M}\downarrow & & {\scriptstyle id \times \Delta_{\mathfrak{G}}}\downarrow \\ M \otimes_R R[\mathfrak{G}] & \xrightarrow{h \otimes id} & N \otimes_R R[\mathfrak{G}] \otimes_R R[\mathfrak{G}] \\ {\scriptstyle \epsilon^*}\downarrow & & {\scriptstyle id \times \epsilon^*}\downarrow \\ M & \xrightarrow{h_0 \otimes id} & N \otimes_R \otimes_R R[\mathfrak{G}]. \end{array}$$

The vertical maps are the identities so we conclude that $h = h_0 \otimes id$ as desired. $\square$

We can proceed to the proof of Lemma 6.3.1.

Proof. The argument is similar as Godement's one in the case of sheaves. Let M be a $\mathfrak{G}$–module and let us embed the R–module M in some injective module I. Then we consider the following injective $\mathfrak{G}$-map

$$M \overset{c_M}{\to} M \otimes_R R[\mathfrak{G}] \to I \otimes_R R[\mathfrak{G}].$$

We claim that $I \otimes_R R[\mathfrak{G}]$ is an injective $\mathfrak{G}$-module. We consider a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{i} & N' \\ & & {\scriptstyle f}\downarrow & & \\ & & I \otimes_R R[\mathfrak{G}]. & & \end{array}$$

From Frobenius reciprocity, we have the following

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{G}}(N', I \otimes_R R[\mathfrak{G}]) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathfrak{G}}(N, I \otimes_R R[\mathfrak{G}]) \\ {\scriptstyle \cong}\downarrow & & {\scriptstyle \cong}\downarrow \\ \mathrm{Hom}_R(N', I) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathfrak{G}}(N, I). \end{array}$$

Since I is injective, the bottom map is onto. Thus f extends to a $\mathfrak{G}$-map $N' \to I \otimes_R R[\mathfrak{G}]$. $\square$

We can then take the right derived functors of the left exact functor $R-\mathfrak{G}-mod \to R-Mod$, $M \to M^{\mathfrak{G}} = H_0^0(\mathfrak{G}, M)$. It defines the Hochschild cohomology groups $H_0^i(\mathfrak{G}, M)$. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of $\mathfrak{G}$–modules, we have the long exact sequence

$$\dots \to H_0^i(\mathfrak{G}, M_1) \to H_0^i(\mathfrak{G}, M_2) \to H_0^i(\mathfrak{G}, M_3) \xrightarrow{\delta_i} H_0^{i+1}(\mathfrak{G}, M_1) \to \dots$$

6.3.3. **Lemma.** *Let M be a $R[\mathfrak{G}]$-module. Then $M \otimes_R R[\mathfrak{G}]$ is acyclic, i.e. satisfies*

$$H_0^i(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0 \qquad \forall i \geq 1.$$

Proof. We embed the M in an injective R-module I and put $Q = I/M$. The sequence of $\mathbf{G}$-modules

$$0 \to M \otimes_R R[\mathfrak{G}] \to I \otimes_R R[\mathfrak{G}] \to Q \otimes_R R[\mathfrak{G}] \to 0$$

is exact. We have seen that $I \otimes_R R[\mathfrak{G}]$ is injective, so that $H_0^i(\mathfrak{G}, I \otimes_R R[\mathfrak{G}]) = 0$ for each $i > 0$. The long exact sequence induces an exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathfrak{G}}(R, I \otimes_R R[\mathfrak{G}]) & \longrightarrow & \mathrm{Hom}_{\mathfrak{G}}(R, Q \otimes_R R[\mathfrak{G}]) & \longrightarrow & H_0^1(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) \to 0 \\ \cong\downarrow & & \cong\downarrow & & \\ I & \longrightarrow & Q & \longrightarrow & 0. \end{array}$$

Therefore $H_0^1(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$. The isomorphisms $H_0^i(\mathfrak{G}, Q \otimes_R R[\mathfrak{G}]) \xrightarrow{\sim} H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}])$ permits to use the standard shifting argument to conclude that $H^{i+1}(\mathfrak{G}, M \otimes_R R[\mathfrak{G}]) = 0$ for each $i \geq 0$. $\square$

As in the usual group cohomology, these groups can be computed by the bar resolution

$$\dots \to R[\mathfrak{G}^3] \to R[\mathfrak{G}^2] \to R[\mathfrak{G}] \to R \to 0.$$

This provides a description of $H_0^i(\mathfrak{G}, M)$ in terms of cocycles, see [DG, II.3] for details. A n–cocycle (resp. a boundary) in this setting is the data of a n-cocycle $c(S) \in Z^n(\mathfrak{G}(S), M \otimes_R S)$ in the usual sense and which agree with base changes.

6.3.4. **Remark.** In particular, there is a natural map $Z^n(\mathfrak{G}, M) \to Z^n(\mathfrak{G}(R), M)$. If $\mathfrak{G} = \Gamma_R$ is finite constant, this map induces an isomophism $H_0^*(\Gamma, M) \xrightarrow{\sim} H_0^*(\Gamma, M)$ (see [DG, II.3.4]).

We can state an important vanishing statement.

6.3.5. **Theorem.** *Let $\mathfrak{G} = \mathfrak{D}(A)$ be a diagonalizable group scheme. Then for each $\mathfrak{G}$-module M, we have $H^i(\mathfrak{G}, M) = 0$ for each $i \geq 1$.*

Proof. The argument is the same as before, the point being that M is a direct summand of an injective $\mathfrak{G}$–module. $\square$

6.4. **First Hochschild cohomology group.** We just focus on H^1 and H^2. Then

$$H^1_0(\mathfrak{G}, M) = Z^1_0(\mathfrak{G}, M)/B^1_0(\mathfrak{G}, M)$$

are given by equivalence of Hochshild 1-cocycles. More precisely, a 1–cocycle (or crossed homomorphism) is a R–functor

$$z : h_{\mathfrak{G}} \to W(M)$$

which satisfies the following rule for each algebra S/R

$$z(g_1 g_2) = z(g_1) + g_1 \,.\, z(g_2) \quad \forall \;\, g_1, g_2 \in \mathfrak{G}(S).$$

The coboundaries are of the shape $g \,.\, m \otimes 1 \,-\, m \otimes 1$ for $m \in M$. As in the classical case, crossed homomorphisms arise from sections of the morphism of R–group functors $V(M) \rtimes \mathfrak{G} \to \mathfrak{G}$ and $H^1_0(\mathfrak{G}, M)$ is nothing but the set of M–conjugacy classes of those sections.

6.5. **H^2 and group extensions.** A 2-cocycle for $\mathfrak{G}$ and M is the data for each S/R of a 2-cocycle $f(S) : \mathfrak{G}(S) \times \mathfrak{G}(S) \to M \otimes_R S$ in a compatible way. It satisfies the rule

$$g_1 \,.\, f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0$$

for each S/R and each $g_1, g_2, g_3 \in \mathfrak{G}(S)$. The 2-cocycle c is normalized if it satifies furthermore the rule

$$f(g, 1) = f(1, g) = 0.$$

each S/R and each $g \in \mathfrak{G}(S)$. Up to add a coboundary, we can always deal with normalized cocycles. The link in the usual theory between normalized classes and group extensions [We, §6.6] extends mecanically. Given a normalized Hochshild cocycle $c \in Z^2(\mathfrak{G}, M)$, we can define the following group law on the R–functor $V(M) \times \mathfrak{G}$ by

$$(m_1, g_1) \,.\, m_2, g_2) = \Big(m_1 + g_1 \,.\, m_2 + c(g_1, g_2), \, g_1 \, g_2 \Big)$$

for each S/R and each $m \in M \otimes_R S$ and $g \in \mathfrak{G}(S)$. In other words, we defined a group extension E_f of R-functors in groups of $h_{\mathfrak{G}}$ by $V(M)$. Now we denote by $\mathrm{Ext}_{R-functor}(\mathfrak{G}, V(M))$ the abelian group of classes of extensions (equipped with the Baer sum) of R–group functors of $h_{\mathfrak{G}}$ by $V(M)$ with the given action $h_{\mathfrak{G}} \to \mathrm{GL}(V(M))$.

The O is the class of the semi-direct product $V(M) \rtimes h_{\mathfrak{G}}$. As in the classical case, it provides a nice description of the H^2.

6.5.1. **Theorem.** [DG, II.3.2] *The construction above induces a group isomorphism $H^2_0(\mathfrak{G}, M) \xrightarrow{\sim} \mathrm{Ext}_{R-functor}(\mathfrak{G}, V(M))$.*

As consequence of the vanishing theorem 6.3.5, we get the following

6.5.2. **Corollary.** *Let A be an abelian group and let M be a $\mathfrak{D}(A)$–module. Let $0 \to V(M) \to F \to \mathfrak{D}(A) \to 1$ be a group R-functor extension. Then F is the semi-direct product of $\mathfrak{D}(A)$ by $V(M)$ and all sections are M–conjugated.*

6.6. **Linearly reductive algebraic groups.** Let k be a field and let G/k be an affine algebraic group. Recall that a $k-G$–module V is simple if 0 and V are its only G-submodules. Note that simple $k-G$–module are finite dimensional according to Proposition 6.2.2. A $k-G$–module is semisimple if it is a direct summand of its simple submodule.

6.6.1. **Definition.** *The k–group $\mathfrak{G}$ is linearly reductive if each finite dimensional representation of $\mathfrak{G}$ is semisimple.*

We have seen that diagonalizable groups are linearly reductive. An important point is that this notion is stable by base change and is geometrical, namely G is linearly reductive iff $G \times_k \overline{k}$ is linearly reductive (see [Mg, prop. 3.2]). As in for the case of diagonalizable groups, we have the following vanishing statement.

6.6.2. **Theorem.** *Assume that the affine algebraic group G/k is linearly reductive. Then for each representation V of G, we have $H^i_0(G,V)$ for each $i > 0$.*

6.6.3. **Corollary.** *Each extension of group functors of G by a vector algebraic group splits. Furthermore $G(k)$ acts transitively on the sections.*

By using a similar method (involving sheaves), Demarche gave a proof of the following classical result [De].

6.6.4. **Theorem.** *(Mostow* [Mo]*) Assume that* $\mathrm{char}(k) = 0$ *and let G/k be a linearly reductive group and let U/k be a split unipotent k-group. Then each extension of algebraic groups of G by U is split and the sections are conjugated under $U(k)$.*

The smooth connected linearly reductive groups are the reductive groups in characteristic zero and only the tori in positive characteristic (Nagata, see [DG, IV.3.3.6]).

Lie algebras, lifting tori

7. Weil restriction

We are given the following equation $z^2 = 1 + 2i$ in $\mathbb{C}$. A standard way to solve it is to write $z = x + iy$ with $x, y \in \mathbb{R}$. It provides then two real equations $x^2 - y^2 = 1$ and $xy = 1$. We can abstract this method for affine schemes and for functors.

We are given a ring extension S/R or $j : R \to S$. Since a S-algebra is a R–algebra, a R-functor F defines a S-functor denoted by F_S and called the scalar extension of F to S. For each S–algebra S', we have $F_S(S') = F(S')$. If X is a R-scheme, we have $(h_X)_S = h_{X \times_R S}$.

Now we consider a S–functor E and define its Weil restriction to S/R denoted by $\prod\limits_{S/R} E$ by

$$\Big(\prod_{S/R} E\Big)(R') = E(R' \otimes_R S)$$

for each R–algebra R'. We note the two following functorial facts.

(I) For a R-map or rings $u : S \to T$, we have a natural map

$$u_* : \prod_{S/R} E \to \prod_{T/R} E_T.$$

(II) For each R'/R, there is natural isomorphism of R'–functors

$$\Big(\prod_{S/R} E\Big)_{R'} \xrightarrow{\sim} \prod_{S \otimes_R R'/R'} E_{S \otimes_R R'}.$$

For other functorial properties, see appendix A.5 of [CGP].

At this stage, it is of interest to discuss the example of vector group functors. Let N be a R–module. We denote by j_*N the scalar restriction of N from S to R [Bbk1, §II.1.13]. The module j_*N is N equipped with the R–module structure induced by the map $j : R \to S$. It satifies the adjunction property $\mathrm{Hom}_R(M, j_*N) \xrightarrow{\sim} \mathrm{Hom}_S(M \otimes_R S, N)$ (*ibid*, §III.5.2).

7.0.5. **Lemma.** *(1)* $\prod\limits_{S/R} V(N) \xrightarrow{\sim} V(j_*N)$.

(2) If N is f.g. projective and S/R is finite and locally free, then $\prod\limits_{S/R} W(N)$ *is representable by the vector group scheme* $\mathfrak{W}(j_*N)$.

For a more general statement, see [SGA3, I.6.6].

Proof. (1) For each R-algebra R', we have

$$\Big(\prod_{S/R} W(N)\Big)(R') = W(N)(R' \otimes_R S) = N \otimes_S (R' \otimes_R S) = j_*N \otimes_R R' = W(j_*N)(R').$$

(2) The assumptions implies that j_*N is f.g. over R, hence $W(j_*N)$ is representable by the vector R–group scheme $\mathfrak{W}(j_*N)$. □

If F is a R-functor, we have for each R'/R a natural map

$$\eta_F(R') : F(R') \to F(R' \otimes_R S) = F_S(R' \otimes_R S) = \Bigl(\prod_{S/R} F_S\Bigr)(R');$$

it defines a natural mapping of R–functor $\eta_F : F \to \prod\limits_{S/R} F_S$. For each S–functor E, it permits to defines a map

$$\phi : \mathrm{Hom}_{S-functor}(F_S, E) \to \mathrm{Hom}_{R-functor}\bigl(F, \prod_{S/R} E\bigr)$$

by applying a S–functor map $g : F_S \to E$ to the composition

$$F \overset{\eta_F}{\to} \prod_{S/R} F_S \xrightarrow{\prod_{S/R} g} \prod_{S/R} E.$$

7.0.6. **Lemma.** *The map ϕ is bijective.*

Proof. We apply the compatibility with $R' = S_2 = S$. The map $S \to S \otimes_R S_2$ is split by the codiagonal map $\nabla : S \otimes_R S_2 \to S, s_1 \otimes s_2 \to s_1 s_2$. Then we can consider the map

$$\theta_E : \Bigl(\prod_{S/R} E\Bigr)_{S_2} \xrightarrow{\sim} \prod_{S \otimes_R S_2/S_2} E_{S \otimes_R S_2} \overset{\nabla_*}{\to} \prod_{S/S} E = E.$$

This map permits to construct the inverse map ψ of ϕ as follows

$$\psi(h) :\, F_S \overset{l_S}{\to} \Bigl(\prod_{S/R} E\Bigr)_{S_2} \overset{\theta_E}{\to} E$$

for each $l \in \mathrm{Hom}_{R-functor}\bigl(F, \prod\limits_{S/R} E\bigr)$. By construction, the maps ϕ and ψ are inverse of each other. □

In conclusion, the Weil restriction from S to R is then right adjoint to the functor of scalar extension from R to S.

7.0.7. **Proposition.** *Let $\mathfrak{Y}/S$ be an affine scheme of finite type (resp. of finite presentation). Then the functor $\prod_{S/R} h_{\mathfrak{Y}}$ is representable by an affine scheme of finite type (finite representation).*

Again, it is a special case of a much more general statement, see [BLR, §7.6].

Proof. Up to localize for the Zariski topology, we can assume that S is free over R, namely $S = \oplus_{i=1,\dots,d} R\,\omega_i$. We see $\mathfrak{Y}$ as a closed subscheme of some affine space $\mathbb{A}^n_S$, that is given by a system of equations $(P_\alpha)_{\alpha \in I}$

with $P_\alpha \in S[t_1, \dots, t_n]$. Then $\prod_{S/R} h_{\mathfrak{Y}}$ is a subfunctor of $\prod_{S/R} W(S^n) \xrightarrow{\sim} W(j_*(S^n)) \xrightarrow{\sim} W(R^{nd})$ by Lemma 7.0.5. For each I, we write

$$P_\alpha\Big(\sum_{i=1,..,d} y_{1,i}\omega_i, \sum_{i=1,..,d} y_{2,i}\omega_i, \dots, \sum_{i=1,..,d} y_{n,i}\Big) = Q_{\alpha,1}\,\omega_1 + \cdots + Q_{\alpha,r}\,\omega_r$$

with $Q_{\alpha,i} \in R\big[y_{k,i}; i = 1,..,d; k = 1,...,n\big]$. Then for each R'/R, $\Big(\prod_{S/R} h_{\mathfrak{Y}}\Big)(R')$ inside R'^{nd} is the locus of the zeros of the polynomials $Q_{\alpha,j}$. Hence this functor is representable by an affine R-scheme $\mathfrak{X}$ of finite type. Furthermore, if $\mathfrak{Y}$ is of finite presentation, we can take I finite so that $\mathfrak{X}$ is of finite presentation too. $\square$

In conclusion, if $\mathfrak{H}/S$ is an affine group scheme of finite type, then the R–group functor $\prod_{S/R} h_{\mathfrak{H}}$ is representable by an R-affine group scheme of finite type. There are two basic examples of Weil restrictions.

(a) The case of a finite separable field extension k'/k (or more generally an étale k-alegbra). Given an affine algebraic k'-group G'/k', we associate the affine algebraic k–group $G = \prod_{k'/k} G'$ which is often denoted by $R_{k'/k}(G)$, see [Vo, §3. 12]. In that case, $R_{k'/k}(G) \times_k k_s \xrightarrow{\sim} (G'_{k_s})^d$. In particular, the dimension of G is $[k' : k] \dim_{k'}(G')$; the Weil restriction of a finite algebraic group is a finite group.

(b) The case where $S = k[\epsilon]$ is the ring of dual numbers which is of very different nature. For example if $p = \mathrm{char}(k) > 0$, $\prod_{k[\epsilon]/k} \mu_{p,k[\epsilon]}$ is of dimension 1. Also the quotient k-group $\prod_{k[\epsilon]/k} (\mathfrak{G}_m)/\mathfrak{G}_m$ is the additive k–group.

7.0.8. **Remark.** It is natural to ask whether the functor of scalar extension from R to S admits a left adjoint. It is the case and we denote by $\bigsqcup_{S/R} E$ this lefat adjoint functor, see [DG, §I.1.6]. It is called the Grothendieck restriction.

If $\rho : S \to R$ is a R–ring section of j, it defines a structure R^ρ of S–ring. We have $\bigsqcup_{S/R} E = \bigsqcup_{\rho: S \to R} E(R^\rho)$. If $E = h_{\mathfrak{Y}}$ for a S–scheme $\mathfrak{Y}$, $\bigsqcup_{S/R} \mathfrak{Y}$ is representable by the R–scheme $\mathfrak{Y}$. This is simply the following R-scheme $\mathfrak{Y} \to \mathrm{Spec}(S) \xrightarrow{j^*} \mathrm{Spec}(R)$.

8. Tangent spaces and Lie algebras

8.1. **Tangent spaces.** We are given an affine R–scheme $\mathfrak{X} = \mathrm{Spec}(A)$. Given a point $x \in \mathfrak{X}(R)$, it defines an ideal $I(x) = \ker(A \xrightarrow{s_x} R)$ and defines an A–structure on R denoted R^x. We denote by $R[\epsilon] = R[t]/t^2$ the ring of R-dual numbers. We claim that we have a natural exact sequence of pointed set

$$\begin{array}{ccccccc} 1 \longrightarrow & \mathrm{Der}_A(A,R^x) & \xrightarrow{i_x} & \mathfrak{X}(R[\epsilon]) & \longrightarrow & \mathfrak{X}(R) \to 1 \\ & & & || & & \\ & & & \mathrm{Hom}_R\big(A, R[\epsilon]\big). & & \end{array}$$

where the base points are $x \in \mathfrak{X}(R) \subset \mathfrak{X}(R[\epsilon])$. The map i_x applies a derivation D to the map $f \mapsto s_x(f) + \epsilon\, D(f)$. It is a ring homomorphism since for $f, g \in A$ we have

$$\begin{aligned} i_x(fg) &= s_x(fg) + \epsilon\, D(fg) \\ &= s_x(f)\, s_x(g) + \epsilon\, D(f)\, s_x(g) + \epsilon s_x(f)\, D(g) \qquad \text{[derivation rule]} \\ &= (s_x(f) + \epsilon\, D(f))\, .\, (s_x(g) + \epsilon\, D(g)) \qquad [\epsilon^2 = 0]. \end{aligned}$$

Conversely, one sees that a map $u : A \to R[\epsilon]$, $f \mapsto u(f) = s_x(f) + \epsilon\, v(f)$ is a ring homomorphism iff $v \in \mathrm{Der}_A(A, R^x)$.

8.1.1. **Remark.** The geometric interpretation of $\mathrm{Der}_A(A, R^x)$ is the tangent space at x of the scheme $\mathfrak{X}/R$ (see [Sp, 4.1.3]). Note there is no need of smoothness assumption to deal with that.

We have a natural A-map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \;\to\; \mathrm{Der}_A(A, R^x);$$

it maps a A–map $l : I(x)/I^2(x) \to R$ to the derivation $D_l : A \to R$, $f \mapsto D_l(f) = l(f - f(x))$. This map is clearly injective but is split by mapping a derivation $D \in \mathrm{Der}_A(A, R^x)$ to its restriction on $I(x)$. Hence the map above is an isomorphism. Furthermore $I(x)/I^2(x)$ is a R^x–module hence the forgetful map

$$\mathrm{Hom}_{A-mod}(I(x)/I^2(x), R^x) \xrightarrow{\sim} \mathrm{Hom}_{R-mod}(I(x)/I^2(x), R)$$

is an isomorphism. We conclude that we have the fundamental exact sequence of pointed sets

$$1 \longrightarrow \mathrm{Hom}_{R-mod}(I(x)/I^2(x), R) \xrightarrow{i_x} \mathfrak{X}(R[\epsilon]) \longrightarrow \mathfrak{X}(R) \to 1.$$

We record that the R–module structure on $I(x)/I(x)^2$ is induced by the change of variable $\epsilon \mapsto \lambda\, \epsilon$.

8.2. **Lie algebras.** To be written.

9. Fixed points of diagonalizable groups

9.1. **Representatibility.**

9.1.1. **Proposition.** *Let $\mathfrak{X}$ be an affine R-scheme equipped with an action of a diagonalizable group scheme $\mathfrak{G}/R = \mathfrak{D}(A)$. Then the R–functor of fixed points F defined by*

$$F(S) = \Big\{ x \in \mathfrak{X}(S) \mid G(S') \,.\, x_{S'} = x_{S'} \;\; \forall S'/S \Big\}$$

is representable by a closed subscheme of $\mathfrak{X}$.

It is denoted by $\mathfrak{X}^{\mathfrak{G}}/R$. The proof below is inspired by [CGP, Lemma 2.1.4].

Proof. The R–module $R[\mathfrak{X}]$ decomposes in eigenspaces $\bigoplus_{a \in A} R[\mathfrak{X}]_a$. We denote by $J \subset R[\mathfrak{X}]$ the ideal generated by the $R[\mathfrak{X}]_a$ for a running over $A \setminus \{0\}$. We denote by $\mathfrak{Y}$ the closed subscheme of $\mathfrak{X}$ defined by J. Since J is a $\mathfrak{D}(A)$-submodule of $R[\mathfrak{X}]$, $R[\mathfrak{Y}]$ is $\mathfrak{D}(A)$-module with trivial structure. Hence the R-map $h_{\mathfrak{Y}} \to h_{\mathfrak{X}}$ factorises by F, and we have a monomorphism $h_{\mathfrak{Y}} \to F$. Again by Yoneda, we have

$$F(R) = \Big\{ x \in \mathfrak{X}(R) \mid \zeta \, x_{R[\mathfrak{G}]} = x_{R[\mathfrak{G}]} \Big\}$$

where $\zeta \in \mathfrak{G}(R[\mathfrak{G}])$ stands for the universal element of $\mathfrak{G}$. Let $x \in F(R)$ and denote by $s_x : R[\mathfrak{G}] \to R$ the underlying map. Then the fact $\zeta \, x_{R[\mathfrak{G}]} = x_{R[\mathfrak{G}]} \in \mathfrak{X}(R[\mathfrak{G}])$ translates as follows

$$\begin{array}{ccc}
R[\mathfrak{X}] & \xrightarrow{\;c\;} & R[\mathfrak{X}] \otimes_R R[A] \\
{\scriptstyle s_x}\big\downarrow & & \big\downarrow{\scriptstyle s_x \otimes id} \\
R & \longrightarrow & R[A] \\
 & & \\
r & \mapsto & r.
\end{array}$$

If $f \in R[\mathfrak{X}]_a$, $a \neq 0$, we have $c(f) = f \otimes e_a$ which maps then to $f(x) \otimes e_a = f(x)$. Therefore $f(x) = 0$. It follows that $J \subset \ker(s_x)$, that is x defines a R–point of $\mathfrak{Y}(R)$. The same holds for any S/R, hence we conclude that $h_{\mathfrak{Y}} = F$. □

9.2. **Smoothness of the fixed point locus.**

9.2.1. **Theorem.** *Assume that R is noetherian. Let $\mathfrak{X}/R$ be an affine smooth R–scheme equipped with an action of the diagonalizable group scheme $\mathfrak{G} = \mathfrak{D}(A)$. Then the scheme of fixed points $\mathfrak{X}^{\mathfrak{G}}$ is smooth.*

For more general statements, see [SGA3, XII.9.6], [CGP, A.8.10] and [De, th. 5.4.4].

9.2.2. **Corollary.** *Assume that R is noetherian. Let $\mathfrak{H}/R$ be an affine smooth group scheme equipped with an action of the diagonalizable group scheme $\mathfrak{G} = \mathfrak{D}(A)$. Then the centralizer subgroup scheme $\mathfrak{H}^{\mathfrak{G}}$ is smooth.*

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