Exceptional Jordan Algebras

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Examples

Example (Complexes)

- Consider the complexes $\mathbb{C}$ as an algebra over the reals $\mathbb{R}$.
- The norm form n, $n(z) := z\bar{z}$, is a quadratic form on $\mathbb{C}/\mathbb{R}$ which allows *composition*: $n(ab) = n(a)n(b)\ \forall a, b \in \mathbb{C}$.

Example (2 by 2 matrices)

- Let $M_2(F)$ be the 2 by 2 matrices with entries in a field F.
- The determinant det is a quadratic form on $M_2(F)/F$ which allows *composition*:
- $\det(AB) = \det(A)\det(B) \quad \forall A, B \in M_2(F)$.

Definitions

Definition (Algebra)

- A (*linear*) *algebra* A over a field F is an F-vector space with a product ab which is left and right distributive over addition.

Definition (Composition Algebra)

- A *composition algebra* C/F is a unital algebra with a regular quadratic form $n : A \to F$ which allows composition: $n(ab) = n(a)n(b)\ \forall a, b \in C$.
- The quadratic form n is (sometimes) called the *norm* of C.
- $\mathfrak{b}_n(a, b) := n(a + b) - n(a) - n(b)$. n is *regular* if $n(z) = 0$ and $\mathfrak{b}(z, C) = \{0\}$ imply $z = 0$.

Consequences

- $\mathfrak{b}_n(ca, cb) = n(c)\mathfrak{b}_n(a, b) = \mathfrak{b}_n(ac, bc)$.
- $\mathfrak{b}_n(ac, bd) + \mathfrak{b}_n(ad, bc) = \mathfrak{b}_n(a, b)\mathfrak{b}_n(c, d)$.
- $n(1_C) = 1_F$.
- Denoting $\mathfrak{b}_n(a, 1_C)$ by $t(a)$, the *trace* of a, we have
  $t(a^2) + \mathfrak{b}_n(a, a) = t(a)^2$, or
  $t(a^2) + 2n(a) = t(a)^2$.

Every $a \in C$ is of Degree 2

- Any $a \in C$ satisfies $a^2 - t(a)a + n(a)1_C = 0_C$.
- Letting $z = a^2 - t(a)a + n(a)1_C$,
  $\mathfrak{b}_n(z, c) = \mathfrak{b}_n(a^2, c) - t(a)\mathfrak{b}_n(a, c) + n(a)t(c)$.
- But $t(a)\mathfrak{b}_n(a, c) = \mathfrak{b}_n(a, 1)\mathfrak{b}_n(a, c) = \mathfrak{b}_n(a^2, c) + \mathfrak{b}_n(ac, 1a)$, and
  $\mathfrak{b}_n(z, c) = 0 \quad \forall c \in C$.
- $n(z) = n(a^2) + t(a)^2 n(a) + n(a)^2 - t(a)\mathfrak{b}_n(a^2, a) + n(a)t(a^2) - n(a)t(a)^2$
  $= 2n(a)^2 + n(a)t(a^2) - n(a)t(a)^2 = 0$.
- So $z = 0$.
- Linearize the first equation to get
  $ab + ba - t(b)a - t(a)b + \mathfrak{b}_n(a, b)1_C = 0_C \quad \forall a,\ b \in C$.

Standard Involution

Consider $a \mapsto \bar{a} := t(a)1_C - a$. One checks that $\bar{\ }$ is an involution, i.e.

- $\overline{ab} = \bar{b}\bar{a}$ and $\bar{\bar{a}} = a$.
- $a\bar{a} = n(a)1_c = \bar{a}a$. Linearizing, $a\bar{b} + b\bar{a} = \mathfrak{b}_n(a,b)1_C = \bar{a}b + \bar{b}a$.
- If $n(a) \neq 0$ then a is invertible, $a^{-1} = n(a)^{-1}\bar{a}$.
- Of course $t(a)1_C = a + \bar{a}$ and if $t(a) = 0$ then $\bar{a} = -a$.
- Moreover $n(\bar{a}) = n(a)$ and $\mathfrak{b}_n(ab, c) = \mathfrak{b}_n(b, \bar{a}c) = \mathfrak{b}_n(a, c\bar{b})$,
- $a(\bar{a}b) = n(a)b = (b\bar{a})a$.

From the definition of $\bar{a}$ and the last equations one gets

- $a(ab) = (aa)b, \qquad (ba)a = b(aa)$.

Alternative Algebras

Definition (Alternative Algebra)

- An algebra A/F is *alternative* if $a(ab) = (aa)b$, and $(ba)a = b(aa)$.

Proposition

Any composition algebra is alternative.

Definition (Associator)

- The *associator* $[a, b, c] := (ab)c - a(bc)$.

Proposition

An algebra A/F is alternative if and only if the associator is an alternating function.

Corollary

A is an alternative algebra if and only if any subalgebra of A generated by two elements is associative. In particular $(ab)a = a(ba)$.

The Radical of C

The *radical* of C, $R = \{r \mid \mathfrak{b}_n(r, c) = 0 \ \forall c \in C\}$. If $R \neq \{0\}$ then char $F = 2$. By the regularity of n, $n(r) \neq 0 \ \forall r \in R, r \neq 0$.

Since $\mathfrak{b}_n(ar, c) = \mathfrak{b}_n(r, \bar{a}c) = 0$ and $\mathfrak{b}_n(ra, c) = \mathfrak{b}_n(r, c\bar{a}) = 0$, R is an ideal of C. Since $n((ab)r) = n(a)n(b)n(r) = n(a(br))$, $(ab)r = a(br)$ $\forall a, b \in C, r \in r$. Any product of 3 elements of C, one of which is in R, is associative and any product of 2 elements, one of which is in R, is commutative. Therefore R is a purely inseparable field extension of F of degree 2; the norm is the square; C is an R-vector space. So $C = R$.

Until further notice we will assume that $R = \{0\}$, so $\mathfrak{b}_n$ is non-degenerate.

Moufang Identities

$$(ab)(ca) = a((bc)a) = (a(bc))a, \tag{1}$$

$$(aba)c = a(b(ac)), \tag{2}$$

$$c(aba) = ((ca)b)a. \tag{3}$$

$$\mathfrak{b}_n((ab)(ca), d) = \mathfrak{b}_n(ca, (\bar{b}\bar{a})d) = \mathfrak{b}_n(c, \bar{b}\bar{a})\mathfrak{b}_n(a, d) - \mathfrak{b}_n(cd, (\bar{b}\bar{a})a)$$
$$= \mathfrak{b}_n(bc, \bar{a})\mathfrak{b}_n(a, d) - n(a)n(cd, \bar{b}),$$

$$\mathfrak{b}_n(a((bc)a), d) = \mathfrak{b}_n((bc)a, \bar{a}d) = \mathfrak{b}_n(bc, \bar{a})\mathfrak{b}_n(a, d) - \mathfrak{b}_n((bc)d, \bar{a}a)$$
$$= \mathfrak{b}_n(bc, \bar{a})\mathfrak{b}_n(a, d) - n(a)\mathfrak{b}_n(bc, \bar{d}).$$

The Structure of Composition Algebras

Let B be a finite dimensional subalgebra of C **on which the bilinear form** $\mathfrak{b}_n(\ ,\ )$ **is non-degenerate**. Then $C = B \oplus B^\perp$ as vector spaces.

If $B^\perp \neq \{0\}$, $\exists v \in B^\perp$ with $n(v) = -\lambda \neq 0$.

Lemma

$B \oplus Bv$ is a subalgebra of C.

- $\mathfrak{b}_n(v, B) = \{0\}$. In particular $t(v) = 0$ and $v^2 = \lambda$.

$$\begin{aligned}(a + bv)(c + dv) &= ac + a(dv) + (bv)c + (bv)(dv) \\ &= ac + (da)v + (b\bar{c})v + \lambda \bar{d}b \\ &= (ac + \lambda \bar{d}b) + (da + b\bar{c})v.\end{aligned}$$

Is $B \oplus Bv$ a Composition Algebra?

- $n(a + bv) = n(a) - \lambda n(b)$.
- $\overline{a + bv} = \bar{a} - bv$; thus $t(a + bv) = t(a)$.
- For $B \oplus Bv$ to be a composition algebra we need: $n((a + bv)(c + dv)) = n(a + bv)n(c + dv)$, i.e., $n(ac + \lambda \bar{d} b) - \lambda n(da + b\bar{c}) = (n(a) - \lambda n(b))(n(c) - \lambda n(d))$?
- In other words $\mathfrak{b}_n(ac, \bar{d} b) = \mathfrak{b}_n(da, b\bar{c})$? Equivalently $\mathfrak{b}_n(d(ac) - (da)c, b) = 0$, $\forall b \in B$.
- If $(ab)v = a(bv) = (ba)v$ then $ab = ba$.

Lemma

For $B \oplus Bv$ to be a composition algebra, B **must** be associative. For $B \oplus Bv$ to be associative, B **must** be commutative.

The Cayley-Dickson Process

Recall $(a + bv)(c + dv) = (ac + \lambda \bar{d} b) + (da + b\bar{c})v$.

If F is of characteristic not 2, start with $B_1 = K1$ to construct
$B_2 = B_1 \oplus B_1 v_1, \quad B_4 = B_2 \oplus B_2 v_2 \quad \text{and} \quad B_8 = B_4 \oplus B_4 v_4$.
If F is of characteristic 2, start with B_2 a separable quadratic field extension or two copies of F with the exchange involution.

B_1 and B_2 are commutative so B_4 is associative. In B_4,
$[a, dv_2] = (da - d\bar{a})v_2 = d(a - \bar{a})v_2 \neq 0$ if $\bar{a} \neq a$.
Since $\bar{\ }$ is not the identity on B_2, B_4 is not commutative and the process stops at B_8. The possible dimensions are therefore 1, 2, 4, and 8.
Composition algebras of dimension 4 are *quaternion algebras*, those of dimension 8, *octonion algebras*.

The Norm Form

Theorem

If A is a simple alternative algebra then A is an associative algebra or an octonion algebra.

Theorem

Two composition algebras C, n and C', n' are isomorphic if and only if C, n and C', n' are isometric.

- If $v_i^2 = \lambda_i$ then the norm form is a Pfister form $<< \lambda_1, \lambda_2, \lambda_3 >>$. See [EKM] for Pfister forms in characteristic 2.
- Arason Invariant.

The Norm Form

- If $n(a) \neq 0$ then $a^{-1} = n(a)^{-1}\bar{a}$. So C is a division algebra if and only if n is anisotropic.
- If n is isotropic then C contains a hyperbolic pair, say (u, v). Then $C = uC \oplus vC$; uC and vC are totally isotropic subspaces and n has maximal Witt index.
- In fact, $1 = x_0 + y_0$, (x_0, y_0) a hyperbolic pair. $\overline{x_0} = y_0$, $\overline{y_0} = x_0$.
- Hence x_0, y_0 are orthogonal idempotents and $C = Fx_0 \oplus x_0 C y_0 \oplus F y_0 \oplus y_0 C x_0$.
- If dim $C = 2$ then $C = Fx_0 \oplus Fy_0$. If dim $C = 4$ then $C = M_2(F)$.

Split Octonions

- Can pick $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ bases of $x_0 C y_0$ and $y_0 C x_0$ respectively, such that $(x_i,\ y_i)$ are mutually orthogonal hyperbolic pairs.
- $\mathfrak{b}_n(x_1x_2, x_0) = -\mathfrak{b}_n(x_1, x_0x_2) = -\mathfrak{b}_n(x_1, x_2) = 0,$
- $\mathfrak{b}_n(x_1x_2, y_0) = -\mathfrak{b}_n(x_1, y_0x_2) = 0,$
- $\mathfrak{b}_n(x_1x_2, x_1) = 0 = \mathfrak{b}_n(x_1x_2, x_2),$
- $x_0(x_1x_2) = (x_0x_1)x_2 - x_1(x_0x_2) + (x_1x_0)x_2 = x_1x_2 - x_1x_2 = 0.$
- So $x_1x_2 = \mu y_3$, where $\mu = \mathfrak{b}_n(x_1x_2, x_3)$.

Split Octonions

- $\mathfrak{b}_n(x_1x_2, x_3) = -\mathfrak{b}_n(x_2, x_1x_3) = -\mathfrak{b}_n(x_1x_3, x_2)$.
- $\mathfrak{b}_n(x_1x_2, x_3) = -\mathfrak{b}_n(x_1, x_3x_2) = -\mathfrak{b}_n(x_3x_2, x_1) = \mathfrak{b}_n(x_3x_1, x_2)$.
- So $\mathfrak{b}_n(x_1x_2, x_3)$ is alternating. $x_ix_{i+1} = \mu y_{i+2}$.
- Replacing $\{x_3, y_3\}$ by $\{\mu^{-1}x_3, \mu y_3\}$,
  we get $x_ix_{i+1} = y_{i+2}$ and $y_iy_{i+1} = x_{i+2}$.
- $x_iy_j = -\delta_{ij}x_0, \qquad y_ix_j = -\delta_{ij}y_0$.

Zorn Vector Matrices

Example (Zorn Vector Matrices)

- Consider $\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}$, α, $\beta \in F$, u, $v \in V$ a vector space of dimension 3 over F .
- Define $\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \begin{pmatrix} \gamma & w \\ z & \delta \end{pmatrix} := \begin{pmatrix} \alpha\gamma - u \cdot z & \alpha w + \delta u + v \times z \\ \gamma v + \beta z + u \times w & \beta\delta - v \cdot w \end{pmatrix}$, where $u \cdot z$ and $u \times z$ are the usual dot and cross products in a 3 dimensional space.
- $n \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} := \alpha\beta + u \cdot v, \quad \overline{\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -u \\ -v & \alpha \end{pmatrix}$.

Local Triality

- Let C be an octonion algebra and $C_0 = \{a \in C | t(a) = 0\}$.
- Denote by L_a and R_a, the left and right multiplication maps i.e., $\mathrm{L}_a x := ax$, $\mathrm{R}_a x := xa$, a, $x \in C$, and by $\mathrm{V}_a := \mathrm{L}_a + \mathrm{R}_a$.
- Rewriting $[x, y, c] = [c, x, y]$, $(xy)c + c(xy) = x(yc) + (cx)y$ we have $\mathrm{V}_c(xy) = (\mathrm{L}_c x)y + x\mathrm{R}_c y$.
- For $a \in C_0$, $\mathfrak{b}_n(\mathrm{L}_\mathrm{a} x, y) = \mathfrak{b}_n(ax, y) = \mathfrak{b}_n(x, \bar{a}y) = -\mathfrak{b}_n(x, \mathrm{L}_a y)$. Similarly $\mathfrak{b}_n(\mathrm{R}_a x, y) = -\mathfrak{b}(x, \mathrm{R}_a y)$ and $\mathfrak{b}_n(\mathrm{V}_a x, y) = -\mathfrak{b}(x, \mathrm{V}_a y)$.

Local Triality

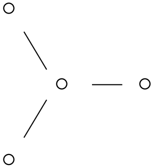
Theorem(Principle of Local Triality)

Let C be an octonion algebra over a field F of characteristic not 2 and L the Lie algebra of skew linear transformations with respect to $\mathfrak{b}_n$ then for every $t_1 \in L$ there exist unique $t_2,\ t_3 \in L$ such that
$t_1(xy) = t_2(x)y + xt_3(y) \quad \forall x, y \in C$.
The mappings $\phi_2 : t_1 \mapsto t_2$, and $\phi_3 : t_1 \mapsto t_3$ are Lie algebra automorphisms of L. They are inequivalent and generate the symmetric group S_3.

See [SV] for the result in characteristic 2 as well as the group version of triality.

Triality

D_4



Derivation Algebras

Definition (Derivation)

If A is an algebra over a field F, $D \in \mathrm{End}_F A$ is a *derivation* of A if $D(ab) = (Da)b + a(Db), \quad \forall a, b \in A$.

- $\mathrm{Der}(A) = \{D \mid D \text{ a derivation of } A\}$ is a Lie algebra. $D1 = 0$.
- If A is alternative, for any a, $b \in A$,

$$\begin{aligned} \mathrm{D}_{a,b} :&= [\mathrm{L}_a, \mathrm{L}_b] + [\mathrm{L}_a, \mathrm{R}_b] + [\mathrm{R}_a, \mathrm{R}_b] \\ &= \mathrm{L}_{[a,b]} - \mathrm{R}_{[a,b]} - 3[\mathrm{L}_a, \mathrm{R}_b] \in \mathrm{Der}(A), \end{aligned}$$

a so-called *standard derivation*.

Derivation Algebras

- Let C be a split octonion algebra, D a derivation. So $Dx_0 = -Dy_0$.
- $Dx_0 = (Dx_0)x_0 + x_0(Dx_0) = (Dx_0)y_0 + y_0(Dx_0)$. So $Dx_0 \in x_0 C y_0 \oplus y_0 C x_0$.
- For $1 \leq i \leq 3$, $\mathrm{D}_{x_0,x_i} x_0 = -x_i$ and $\mathrm{D}_{y_0,y_i} x_0 = -y_i$.
- Assume $Dx_0 = 0$ then $D(x_0 C y_0) = x_0(DC)y_0$. Let $\mathrm{Der}(C)_0 = \{D \in \mathrm{Der}(C) \mid Dx_0 = 0\}$.
- For $D \in \mathrm{Der}(C)_0$, $D : x_0 C y_0 \to x_0 C y_0$ and $D : y_0 C x_0 \to y_0 C x_0$.
- Denote these actions by D' and D'' respectively. In the Zorn vector matrix context, letting D act on $u_{12} z_{21} = -(u \cdot z)_{11}$ yields $(D' u_{12}) z_{21} + u_{12} D'' z_{21} = 0$.
- So $\mathfrak{b}_n(D'u, z) + \mathfrak{b}_n(u, D''z) = 0$ and $D'' = -D^*$, where * is the adjoint with respect to $\mathfrak{b}_n$.

G_2

One can check that we further need that the trace of D' be 0 and that these two conditions are sufficient for $D = (D', D'') \in \mathrm{Der}(C)_0$. Therefore the dimension of Der(C) is $6 + 8 = 14$.

Theorem

If the characteristic of F is not 3 then Der(C) is a simple Lie algebra of type G_2. In characteristic 3, Der(C) has an ideal of dimension 7.

Associative Algebras with Involution

- An associative algebra with involution $(A, *)$ is *simple* (as an algebra with involution) if it contains no non-trivial *-stable ideal.
- In that case, either A is simple or $A = B \oplus B^{op}$, B simple, B^{op} the *opposite algebra* has the same additive structure as B and the product $a^{\mathrm{op}}b := ba$, and * is the exchange involution $(a, b)^* = (b, a)$.
- Denote by $\mathcal{C}(A)$ the centre of A. The *centre* of the associative algebra with involution $(A, *)$, $\mathcal{C}(A, *) = \{c \in \mathcal{C}(A) \mid c^* = c\}$.
- The involution * is of the *first kind* if $\mathcal{C}(A, *) = \mathcal{C}(A)$, e.g. $(\mathcal{M}_n(F), t)$, t the transpose involution. Otherwise it is of the *second kind*, e.g if B is a central algebra over F then $\mathcal{C}(B \oplus B^{op}) = F \oplus F$ while $\mathcal{C}(B \oplus B^{op}, *) = \{(\alpha, \alpha) \mid \alpha \in F\} \cong F$.

Hermitian Elements

Example (Hermitian Elements)

- Let A/F be an asssociative an algebra over a field F and $*$ an involution of A which fixes F elementwise.
- Denote by $\mathcal{H}(A,*) = \{a \in A \mid a^* = a\}$ the *hermitian elements* of A and by $\mathcal{S}(A,*) = \{a \in A \mid a^* = -a\}$ the *skew-symmetric elements*.
- The Lie bracket $[a,b] := ab - ba$ gives A a Lie algebra structure, denoted A^-; $\mathcal{S}(A,*)$ is a Lie subalgebra of A^-.
- The subspace $\mathcal{H}(A,*) = \{a \in A \mid a^* = a\}$ is closed under $a \mapsto a^2$ and hence under $\mathrm{V}_a b = a \circ b := ab + ba$.

Linear Jordan Algebras

- Assume that $\frac{1}{2} \in F$. Denote $\frac{1}{2}(a \circ b)$ by $a \cdot b$. Note that $a^{\cdot 2} = a \cdot a = a^2$.
- One checks that the above product in an associative algebra satisfies:

$$a \cdot b = b \cdot a, \tag{4}$$

$$((a \cdot a) \cdot b) \cdot a = (a \cdot a) \cdot (b \cdot a). \tag{5}$$

- An algebra over a field of characteristic not 2 whose product satisfies (4) and (5) is called a *linear Jordan algebra*.
- $\mathcal{H}(A, *)$ is a Jordan subalgebra of A^+ the Jordan structure on an associative algebra given by $a \cdot b = \frac{1}{2} a \circ b$.

Quadratic Jordan Algebras

- In a linear Jordan algebra, consider $\mathrm{U}_a b := 2a \cdot (a \cdot b) - a^{\cdot 2} \cdot b$.
- In A^+, $\mathrm{U}_a b = \frac{1}{2}(aab + aba + aba + baa) - \frac{1}{2}(a^2 b + ba^2) = aba$.
- Consider $\mathrm{U}_{a,c} := \mathrm{U}_{a+c} - \mathrm{U}_a - \mathrm{U}_c$.
- In A^+, $\mathrm{U}_{a,c} b = abc + cba$.
- $\mathrm{V}_{a,b} c := \mathrm{U}_{a,c} b := \{abc\}$.

Quadratic Jordan Algebras

Definition (Quadratic Jordan Algebra)

- A *unital quadratic Jordan algebra* $\mathcal{J}$ over a field F is an F-vector space $\mathcal{J}$, a *unit element* $1_{\mathcal{J}} \in \mathcal{J}$ and a quadratic map U of $\mathcal{J}$ into $\mathrm{End}_F \mathcal{J}$ satisfying

$$\mathrm{U}_1 = \mathrm{I}_{\mathcal{J}}, \tag{6}$$

$$\mathrm{U}_{\mathrm{U}_a b} = \mathrm{U}_a \mathrm{U}_b \mathrm{U}_a, \tag{7}$$

$$\mathrm{U}_a \mathrm{V}_{b,a} = \mathrm{V}_{a,b} \mathrm{U}_a, \tag{8}$$

  and (6), (7) and (8) remain valid under field extensions,

- where $\mathrm{U}_{a,b} := \mathrm{U}_{a+b} - \mathrm{U}_a - \mathrm{U}_b$, $\mathrm{V}_{a,b} c := \mathrm{U}_{a,c} b = \{abc\}$.
- Equation (7) is sometimes refered to as the fundamental formula.

Powers

- Let $\mathrm{V}_a := \mathrm{U}_{a,1}$. In A^+, $\mathrm{U}_{a,1}b = ab1 + ba1 = a \circ b$.
- If $\frac{1}{2} \in F$, one checks the $a \cdot b := \frac{1}{2}\mathrm{V}_a b$ defines a linear Jordan algebra structure on $\mathcal{J}$.
- Powers are defined inductively: $a^0 = 1_{\mathcal{J}}$, $a^1 = a$, $a^{n+2} = \mathrm{U}_a a^n$. Writing $b^2 = \mathrm{U}_b 1$, we have $\mathrm{U}_{b^2} = \mathrm{U}_{\mathrm{U}_b 1} = \mathrm{U}_b \mathrm{U}_1 \mathrm{U}_b = \mathrm{U}_b^2$.

Special Jordan Algebras

Example (A^+)

- Let A/F be a unital asssociative an algebra over a field F and $\mathrm{U}_a x := axa$. $\mathrm{U}_{1_A} = \mathrm{I}_{\mathcal{J}}$.
- $\mathrm{U}_{\mathrm{U}_a b} x = abaxaba = \mathrm{U}_a \mathrm{U}_b \mathrm{U}_a x$.
- $\mathrm{U}_a \mathrm{V}_{b,a} x = abaxa + axaba = \mathrm{V}_{a,b} \mathrm{U}_a x$.
- Denote also by A^+ the quadratic Jordan algebra structure on A.

Hermitian Algebras

Definition (Special Jordan Algebras)

- A quadratic Jordan algebra is *special* if it can be embedded in an A^+, otherwise we say it is *exceptional*.

- If $(A, *)$ is an associative algebra with involution then the *Hermitian Jordan algebra* $\mathcal{H}(A, *)$ is a subalgebra of A^+ and hence special.
- Let $B = A \oplus A^{\mathrm{op}}$, * the exchange involution $(a, b)^* = (b, a)$. Then $\mathcal{H}(B, *) \cong A^+$, which is therefore a Hermitian Jordan algebra.
- Let $\mathcal{D}$ be an associative division algebra with involution $\bar{\ }$, V a left $\mathcal{D}$ vector space and $h : V \to \mathcal{D}$ a non-degenerate hermitian form on V, i.e., for $d \in \mathcal{D}$, u, $v \in V$, $h(du, v) = dh(u, v)$, $h(u, dv) = h(u, v)\overline{d}$, $h(v, u) = \overline{h(u, v)}$. The form h induces an involution * on $\mathrm{End}_{\mathcal{D}}(V)$: $h(uM, v) = h(u, vM^*), \forall u, v \in V, M \in \mathrm{End}_{\mathcal{D}}(V)$. The involutions $\bar{\ }$ and $*$ allow us to define a right vector space structure on V and a left action of $\mathrm{End}_{\mathcal{D}}(V)$ on V.

In Matrix Form

- Assume $\mathcal{D}$ is a quaternion algebra and $\bar{\ }$ the standard involution.
- If V is of dimension n we may assume that with respect to a suitable basis $\{v_1, v_2, \dots, v_n\}$ the matrix of h is diagonal say $\mathrm{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$, $\gamma_i \in F^\times$. Then with respect to this basis, $\mathcal{H}(\mathrm{End}_{\mathcal{D}}(V), *)$ are matrices $\sum_{i=1}^n \alpha_i[ii] + \sum_{1=i<j}^n \gamma_i\gamma_j^{-1}a_{ij}[ij] + \gamma_j\gamma_i^{-1}\overline{a_{ij}}[ji]$, $\alpha_i \in F$, $a_{ij} \in \mathcal{D}$. (this is not quite right in characteristic 2).
- For example, if $n = 3$ we have $\begin{bmatrix} \alpha_1 & a_{12} & a_{13} \\ \gamma_2\gamma_1^{-1}\overline{a_{12}} & \alpha_2 & a_{23} \\ \gamma_3\gamma_1^{-1}\overline{a_{13}} & \gamma_3\gamma_2^{-1}\overline{a_{23}} & \alpha_3 \end{bmatrix}$.

Jordan Algebras of a Quadratic Form

Let V/F be a vector space, Q a quadratic form on V with *base point* $c \in V$, i.e., $Q(c) = 1_F$. Let

$$\begin{aligned} T(v) &:= \mathfrak{b}_Q(x, c), \\ \bar{v} &:= T(v)c - v, \\ U_a b &:= \mathfrak{b}_Q(a, \bar{b})a - Q(a)\bar{b}. \end{aligned}$$

- This yields a quadratic Jordan algebra $J(V, Q, c)$ with $1_J = c$, the quadratic *Jordan algebra of the quadratic form* Q with base point c.
- $a^2 = \mathrm{U}_a 1_J = \mathfrak{b}_Q(a, 1_J)a - Q(a)1_J$ or
- $a^2 - T(a)a + Q(a)1_J = 0$, where the *trace* $T(a) = \mathfrak{b}_Q(a, 1_J)$.
- $a \circ b - T(a)b - T(b)a + \mathfrak{b}_Q(a, b)1_J = 0$.

The Clifford Algebra of (V, Q, c)

Definition (Clifford Algebra of (V, Q, c))

Let $\mathcal{T}(V)$ be the tensor algebra of V and $\mathcal{I}$ the ideal of $\mathcal{T}(V)$ generated by $c - 1_{\mathcal{T}}$, $v \otimes v - T(v)v + Q(v)c$. The *Clifford algebra* of (V, Q, c), $\mathcal{C}(V, Q, c) = \mathcal{T}(V)/\mathcal{I}$. V embeds as a vector space in $\mathcal{C}(V, Q, c)$.

- In $\mathcal{C}(V, Q, c)$, for a, $b \in V$,

$$\begin{aligned} aba &= -baa + T(a)ba + T(b)a^2 - \mathfrak{b}_Q(a, b)a \\ &= -b(a^2 - T(a)a) + T(b)(T(a)a - Q(a)1) - \mathfrak{b}_Q(a, b)a \\ &= Q(a)b + T(a)T(b)a - Q(a)T(b)1 - \mathfrak{b}_Q(a, b)a \\ &= \mathfrak{b}_Q(a, T(b)1 - b)a - Q(a)(T(b)1 - b) \\ &= \mathfrak{b}_Q(a, \bar{b})a - Q(a)\bar{b}. \end{aligned}$$

$J(V, Q, c)$

Proposition

The quadratic Jordan algebra $J(V, Q, c)$ is special.

- $T(1_J) = 2$, $\bar{\bar{a}} = a$.
- For $a \in V \subset \mathcal{C}(V, Q, c)$, $a\bar{a} = a(T(a)c - a) = T(a)a - a^2 = Q(a)1_{\mathcal{C}}$.
- If $Q(a) = 0$ then $a\bar{a} = 0$.
- If $Q(a) \neq 0$ then $Q(a)^{-1}\bar{a} = a^{-1}$ and a is invertible in $\mathcal{C}$ iff $Q(a) \neq 0$.

Jordan Division Algebras

- An element $a \in \mathcal{J}$ is *invertible* with inverse b if $\mathrm{U}_a b = a$ and $\mathrm{U}_a b^2 = 1_{\mathcal{J}}$.
- If $\mathcal{J} = A^+$ then $ab^2a = 1$ implies a is invertible in A and $b = a^{-1}abaa^{-1} = a^{-1}aa^{-1} = a^{-1}$.

Lemma

The element $a \in \mathcal{J}$ is invertible if and only if U_a is invertible in $\mathrm{End}_F\mathcal{J}$. In that case $(\mathrm{U}_a)^{-1} = \mathrm{U}_{a^{-1}}$.

- If $\mathrm{U}_a b = a$ and $\mathrm{U}_a b^2 = 1_{\mathcal{J}}$ then $\mathrm{U}_{\mathrm{U}_a b} = \mathrm{U}_a$ or $\mathrm{U}_a\mathrm{U}_b\mathrm{U}_a = \mathrm{U}_a$.
- Similarly $\mathrm{U}_a\mathrm{U}_b^2\mathrm{U}_a = \mathrm{I}_{\mathcal{J}}$. So U_a is invertible in $\mathrm{End}_F\mathcal{J}$.

Definition (Jordan Division Algebra)

A Jordan algebra $\mathcal{J}$ is a *division algebra* if every $0 \neq a \in \mathcal{J}$ is invertible.

Examples of Jordan Division Algebras

Example ($J(V, Q, c)$)

$J(V, Q, c)$ is a Jordan division algebra if and only if Q is anisotropic.

Example (A^+)

A^+ is a Jordan division algebra if and only if A is division algebra.

Example ($\mathcal{H}(A, *)$)

If A is a simple associative algebra and $*$ an involution of A then $\mathcal{H}(A, *)$ is a Jordan division algebra if and only if A is division algebra.

Matrix Units

- The associative algebra $A = \mathcal{M}_n(\mathcal{D})$, $\mathcal{D}$ a division algebra, contains *matrix units* $\{e_{ij}, 1 \leq i, j \leq n\}$.
- Conversely if an associative algebra A contains a set of matrix units $\{e_{ij}, 1 \leq i, j \leq n\}$ such that $\sum e_{ii} = 1$ and $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$ then $A \cong \mathcal{M}_n(\mathcal{D})$, where $\mathcal{D}$ is the centralizer in A of the e_{ij}. If A is simple then $\mathcal{D}$ is a division algebra.

Idempotents in Jordan Algebras

- $e \neq 0 \in \mathcal{J}$ is an *idempotent* if $e^2 = e$ (recall $e^2 = \mathrm{U}_e 1_{\mathcal{J}}$).
- Two idempotents e, f are *orthogonal* if $e \circ f = 0$. One can show this implies $\mathrm{U}_e f = \mathrm{U}_f e = 0$.
- If $e \in \mathcal{J}$ then $f = 1_{\mathcal{J}} - e$ is an idempotent orthogonal to e and $\mathcal{J} = \mathrm{U}_e\mathcal{J} \oplus \mathrm{U}_{e,f}\mathcal{J} \oplus \mathrm{U}_f\mathcal{J}$. We write this $\mathcal{J}_2(e) \oplus \mathcal{J}_1(e) \oplus \mathcal{J}_0(e)$. This is the *Peirce decomposition* of $\mathcal{J}$ with respect to e. If the characteristic is not 2, $\mathcal{J}_i(e) = \{a \in \mathcal{J} \mid \mathrm{V}_e a = ia\}$.
- If $(\mathrm{U}_e\mathcal{J}, \mathrm{U}, e)$ is a Jordan division algebra we say that e is a *division idempotent*.
- Two orthogonal idempotents e_1, $e_2 \in \mathcal{J}$ are *connected* if there exists an element $u_{12} \in \mathrm{U}_{e_1,e_2}\mathcal{J}$ which is invertible in the Jordan algebra $\mathrm{U}_e\mathcal{J}$, where $e = e_1 + e_2$.
- A set of pairwise orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$ is *supplementary* if their sum $e_1 + e_2 + \cdots + e_n = 1_{\mathcal{J}}$.

Jordan Matrix Algebras

- Let $\mathcal{D}$ be a unital algebra with involution $^{-}$ and Γ a subspace of $\mathcal{H}(\mathcal{D}, {}^{-})$, containing all norms $a\bar{a}$, $a \in \mathcal{D}$. In particular $1_{\mathcal{D}} \in \Gamma$.
- Since traces $a + \bar{a} \in \Gamma$, if $\frac{1}{2} \in F$ then $\mathcal{H}(\mathcal{D}.^{-}) = \Gamma$. In characteristic 2, we can have $\mathcal{H}(\mathcal{D}.^{-}) \neq \Gamma$.
- Let $* : \mathcal{M}_n(\mathcal{D}) \to \mathcal{M}_n(\mathcal{D})$ given by $M^* = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)\overline{M}^t \mathrm{diag}(\gamma_1^{-1}, \ldots, \gamma_n^{-1})$ and $\mathcal{H}(\mathcal{M}_n(\mathcal{D}), \Gamma, *)$ the matrices of $\mathcal{H}(\mathcal{M}_n(\mathcal{D}), *)$ having elements of Γ along the diagonal.

Coordinatization Theorem

The *nucleus* of an algebra A is the set
$\{z \in a \mid [z, a, b] = [a, z, b] = [a, b, z] = 0 \ \forall a, b \in A\}$.

Coordinatization Theorem

Any unital Jordan algebra $\mathcal{J}/F$ containing a set of $n \geq 3$ supplementary orthogonal connected idempotents is isomorphic to an algebra $\mathcal{H}(\mathcal{M}_n(\mathcal{D}), \Gamma, *)$, where $\mathcal{D}$ is an alternative algebra with involution $\bar{\ }$ satisfying $\mathcal{H}(\mathcal{D}, \bar{\ })$ is contained in the nucleus of $\mathcal{D}$. If $n \geq 4$ then $\mathcal{D}$ must be associative.

Nondegenerate Jordan Algebras

Definition (Nondegenerate Jordan Algebra)

An element $z \neq 0 \in \mathcal{J}$ is an *absolute zero divisor* if $\mathrm{U}_z = 0$. A Jordan algebra $\mathcal{J}$ is *nondegenerate* if it has no absolute zero divisor.

Definition (Capacity)

A Jordan algebra $\mathcal{J}$ has *capacity* n if $1_{\mathcal{J}} = e_1 + \cdots + e_n$, e_i mutually orthogonal division idempotents.

Structure Theorem

Structure Theorem

Any simple nondegenerate unital Jordan algebra $\mathcal{J}/F$ with a capacity is isomorphic to
1) a Jordan division algebra,
2) $J(V, Q, c)$ the Jordan algebra of a regular quadratic form,
3) $\mathcal{H}(\mathcal{M}_n(\mathcal{D}), \Gamma, *)$, $\mathcal{D}$ an associative division algebra with involution, or the sum of two copies of an associative division algebra with the exchange involution,
4) $\mathcal{H}(\mathcal{M}_3(C), *)$, C an octonion algebra.

Algebras in 2), 3) are special; 4) are exceptional and (up to now) 1) is a ?. Stated this way the classes are not exclusive e.g. $J(V, Q, c)$ is a division algebra if Q is anisotropic.

Algebras of Degree 3

Example

Let $A = \mathcal{M}_3(F)$. Any $a \in A$ satisfies the characteristic polynomial

$$x^3 - T(x)x^2 + S(x)x - N(x)1_A,$$

where the trace T is a linear form, S a quadratic form, sometimes called the quadratic trace, and the determinant N is a cubic form. If $a^{\#}$ is the classical adjoint then $aa^{\#} = N(a)1_A = a^{\#}a$ and $(a^{\#})^{\#} = N(a)a$.
$T(1_A) = 3, \quad a^{\#} = a^2 - T(a)a + S(a)1_A, \quad S(a) = T(a^{\#})$.

Cubic Forms

- A *cubic form* is a map $f : V \to F$ such that $f(\alpha v) = \alpha^3 f(v)$ $\forall \alpha \in F, v \in V$ and for which this remains true for all field extensions.
- Over $F(\omega_1, \omega_2, \dots)$ the rational field extension over the indeterminates ω_i, $f(\sum \omega_i v_i) = \sum \omega_i^3 f(v_i) + \sum_{i \neq j} \omega_i^2 \omega_j f(v_i; v_j) + \sum_{i \neq j \neq k} \omega_i \omega_j \omega_k f(v_i; v_j; v_k)$, where $f(x; y)$ is quadratic in x and linear in y and $f(x, y, z)$ is symmetric and trilinear.

Cubic Norm Structure

Definition (Cubic Norm Structure)

A *cubic norm structure* consists of a vector space V/F containing a *base point* $1 = 1_V \in V$ together with a quadratic map $\# : V \to V$, $v \mapsto v^\#$ the *adjoint* and a cubic form $N : V \to F$, the *norm*, satisfying for all a, $b \in V$ and all field extensions

$$N(1) = 1, \qquad 1^\# = 1, \tag{9}$$

$$(a^\#)^\# = N(a)a, \tag{10}$$

$$N(a; b) = T(a^\#, b), \tag{11}$$

$$1 \times a = T(a)1 - a, \tag{12}$$

Cubic Norm Structure

where

$$T(a) := N(1; a), \tag{13}$$

$$T(a, b) := T(a)T(b) - N(1, a, b), \tag{14}$$

$$a \times b := (a + b)^{\#} - a^{\#} - b^{\#}. \tag{15}$$

The Jordan Algebra of a Cubic Norm Structure

Theorem

Given a cubic norm structure $(V, N, \#, 1)$, the following U operator
$\mathrm{U}_a b := T(a,b)a - a^\# \times b$
defines a unital quadratic Jordan algebra structure on V, $J(V, N, \#, 1)$ the *Jordan algebra of the cubic norm structure*.
For all $a \in J(V, N, \#, 1)$,

$$a^3 - T(a)a^2 + S(a)a + N(a)1 = 0,$$

where the quadratic form $S(a) := T(a^\#)$, $x^\# = x^2 - T(x)x + S(x)1$.
N allows Jordan composition $N(\mathrm{U}_x y) = N(x)^2 N(y)$.
$x \in J$ is invertible if and only if $N(x) \neq 0$ in which case $x^{-1} = N(x)^{-1}x^\#$.

Examples

Example ($\mathcal{M}_3(F)$)

$N(a) = \det(a)$, $\#$ is the classical adjoint and $1 = \mathrm{I}$. One can check that the above definition yields $\mathcal{M}_3(F)^+$. But $\mathcal{M}_3(F)^+ \cong \mathcal{H}(\mathcal{M}_3(F \oplus F), *)$, where $b^* := \bar{b}^t$, $^-$ the exchange involution of $F \oplus F$ and t the transpose.

Example ($F \oplus J(V, Q, c)$)

Let $X = F \oplus J(V, Q, c)$, $J(V, Q, c)$, the Jordan algebra of a quadratic form with base point. Let the base point $1_X = 1_F \oplus c$, the adjoint $(\alpha \oplus v)^\# := Q(v) \oplus \alpha\bar{v}$ and the norm $N_x(\alpha \oplus v) := \alpha Q(v)$.

Examples

Example ($\mathcal{H}_3(C, J_\gamma)$)

Consider $\begin{bmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3 \overline{a_2} \\ \gamma_1 \overline{a_3} & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2 \overline{a_1} & \alpha_3 \end{bmatrix}$, $\alpha_i \in F$, $a_i \in C$ and $\gamma_i \in F^\times$. Denoting

$\gamma_k a_i e_{jk} + \gamma_j \overline{a_i} e_{kj}$ by $a_i[jk]$, we can write the above matrix $\sum_{(123)}(\alpha_i e_{ii} + a_i[jk])$, where the sum $\sum_{(123)}$ is over cyclic permutations of $\{1, 2, 3\}$.

$\mathcal{H}_3(C, J_\gamma)$

Theorem

Let C be a composition algebra. Denote by $\mathcal{H}(C_3, J_\gamma)$ the matrices of the form $\sum \alpha_i e_{ii} + \sum_{(123)} a_i[jk]$, $\alpha_i \in F$, $a_i \in C$. Then the unit element, cubic form and adjoint

$$1 = e_{11} + e_{22} + e_{33},$$

$$N(x) := \alpha_1\alpha_2\alpha_3 - \sum_{(123)} \alpha_i\gamma_j\gamma_k n(a_i) - \gamma_1\gamma_2\gamma_3 t(a_1a_2a_3)$$

$$x^\# := \sum_{(123)} ((\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i))e_{ii} + (\gamma_i\overline{a_j a_k} - \alpha_i a_i)[jk])$$

define a cubic norm structure on $\mathcal{H}_3(C, J_\gamma)$. The Jordan algebras obtained from this cubic norm structure are simple.

$\mathcal{H}_3(C, J_\gamma)$

- The quadratic trace $S(x) = T(x^\#) = \sum_{(123)}((\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i))$.
- The trace bilinear form $T(x,y) = \sum_{(123)}(\alpha_i\beta_i + \gamma_j\gamma_k \mathfrak{b}_n(a_i, b_i))$, where $y = \sum_{(123)}(\beta_i e_{ii} + b_i[jk])$.
- If C is associative, the above Jordan algebra structure coincides with that induced by C_3^+.
- The involution J_γ is induced by the hermitian form $\mathrm{diag}(\gamma_1, \gamma_2, \gamma_3)$. Multiplying this form by a non-zero scalar yields a form which induces the same involution. In fact, replacing each γ_i by $\mu_i^2\gamma_i$, $\mu_i \in F^\times$, yields an isomorphic algebra. The same holds when C is an octonion algebra!

Albert Algebras

Theorem

If C is an octonion algebra then $\mathcal{H}(C_3, J_\gamma)$ is exceptional. In fact it is not even a homomorphic image of a special algebra.

Definition (Albert Algebra)

An *Albert algebra* is an algebra $J(V, N, \#, 1)$ of dimension 27. A Jordan algebra is *reduced* if it contains a proper idempotent.

Theorem

A reduced Albert algebra is isomorphic to $\mathcal{H}(C_3, J_\gamma)$, C an octonion algebra.

Isomorphism of Reduced Albert Algebras

Theorem

Two reduced Albert algebras $\mathcal{H}(C_3, J_\gamma) \cong \mathcal{H}(C'_3, J'_\gamma)$, C and C' octonion algebras, if and only if $C \cong C'$ and their quadratic traces are equivalent.

Example ($\mathcal{H}_3(\mathbb{O}, J_\gamma)$)

Let $\mathbb{O}$ be Cayley-Graves numbers (unique division octonion algebra over the reals $\mathbb{R}$). $\mathcal{H}(\mathbb{O}_3, J_{\{1,1,1\}}) \ncong \mathcal{H}(\mathbb{O}_3, J_{\{1,1,-1\}})$. Moreover if C is the split octonion algebra, $\mathcal{H}(C_3, J_{\{1,1,1\}})$ is not isomorphic to the previous 2. One can show that these are exactly the three non isomorphic Albert algebras over the reals $\mathbb{R}$.

An Albert algebra is said to be *split* if it is reduced and its coefficient octonion algebra is split.

First Tits Construction

- Let A/F be a central simple associative algebra of degree 3. Every $a \in A$ satisfies the reduced characteristic polynomial $a^3 - T_A(a)a^2 + S_A(a)a - N_A(a)1_A$. N_A the reduced norm, T_A the reduced trace. For $a \in A$, define $a^\# := a^2 - T_A(a)a + S_A(a)1_A$.
- Let $\mu \in F^\times$, $V = A \oplus A \oplus A$ and $x = (a_0, a_1, a_2) \in V$. Then the unit element, cubic form and adjoint

$$\begin{aligned} 1 &:= (1, 0, 0), \\ N(x) &:= N_A(a_0) + \mu N_A(a_1) + \mu^{-1} N_A(a_2) - T_A(a_0 a_1 a_2), \\ x^\# &:= (a_0\# - \mu a_1 a_2,\ \mu a_2^\# - a_0 a_1,\ \mu^{-1} a_1^\# - a_2 a_0) \end{aligned}$$

define a cubic norm structure on V. We denote the corresponding Jordan algebra by $J(A, \mu)$. This is the *First Tits Construction*.

First Tits Construction

Theorem

The Jordan algebra $J(A, \mu)$ is an Albert algebra. It is a division algebra if and only if $\mu \notin N_A(A^\times)$. A^+ is isomorphic to the subalgebra $(A, 0)$ of $J(A, \mu)$. Conversely if an Albert algebra $\mathcal{A}$ contains a subalgebra isomorphic to A^+ then $\mathcal{A}$ is isomorphic to $J(A, \mu)$ for a suitably chosen μ.

Second Tits Construction

- Let B/E be a central simple associative algebra of degree 3. Assume B has an involution of the second kind such that $\mathcal{C}(B,*) = F$, E/F a separable field extension.
- Let $u \in \mathcal{H}(B,*)$ and $\beta \in E^{\times}$ such that $N_B(u) = \beta\beta^*$, $V = \mathcal{H}(B,*) \oplus B$ and $x = (a,b) \in V$. Then the unit element, cubic form and adjoint

$$\begin{aligned} 1 &:= (1,0), \\ N(x) &:= N_B(a) + \beta N_B(b) + \beta^* N_B(b)^* - T_B(a, bub^*), \\ x^{\#} &:= (a_0^{\#} - bub^*,\ \beta^*(b^*)^{\#}u^{-1} - ab) \end{aligned}$$

define a cubic norm structure on V. We denote the corresponding Jordan algebra by $J(B,*,u,\beta)$. This is the *Second Tits Construction*.

Second Tits Construction

Theorem

The Jordan algebra $J(B, *, u, \beta)$ is an Albert algebra. It is a division algebra if and only if $\beta \notin N_B(B^\times)$. $\mathcal{H}(B, *)$ is isomorphic to the subalgebra $(\mathcal{H}(B, *), 0)$ of $J(B, *, u, \beta)$. Conversely if an Albert algebra $\mathcal{A}$ contains a subalgebra isomorphic to $\mathcal{H}(B, *)$ then $\mathcal{A}$ is isomorphic to $J(B, *, u, \beta)$ for suitably chosen u and β.

Albert Algebras

Theorem

Albert algebras coincide with simple exceptional Jordan algebras. The two Tits Constructions yield all Albert algebras.

Using $A^+ \cong \mathcal{H}(A \oplus A^{op}, *)$, $*$ the exchange involution, it is easy to subsume the First Tits construction into a generalized Second Tits construction.

The Automorphism Group

Definition (Automorphism)

A map $\eta \in \mathrm{GL}(\mathcal{J})$ is an *automorphism* of $\mathcal{J}$ if $\eta(1) = 1$ and $\eta \mathrm{U}_a = \mathrm{U}_{\eta(a)}\eta$.

The second condition says $\eta(\mathrm{U}_a b) = \mathrm{U}_{\eta(a)}\eta(b)$.

Definition (Derivation)

A map $D \in \mathrm{End}_F(\mathcal{J})$ is a *derivation* of $\mathcal{J}$ if $D(1) = 0$ and $[D, \mathrm{U}_a] = \mathrm{U}_{a,Da}$.

The second condition says $D\mathrm{U}_a b = \mathrm{U}_{a,Da} b + \mathrm{U}_a Db$. If D is a derivation of an associative algebra A,
$D(\mathrm{U}_a b) = D(aba) = (Da)ba + a(Db)a + ab(Da) = (\mathrm{U}_{a,Da} + \mathrm{U}_a D)b$.

The Derivation Algebra of the Split Albert Algebra

- Recall $\mathrm{V}_{a,b}c = \mathrm{U}_{a,c}b$. One checks that for any Jordan algebra $\mathcal{J}$, $\mathrm{D}_{a,b} := \mathrm{V}_{a,b} - \mathrm{V}_{b,a}$, a, $b \in \mathcal{J}$, is a derivation, a *standard derivation*.
- In A^+, $\mathrm{D}_{a,b}c = [[a,b],c]$.
- If $D \in \mathrm{Der}(\mathcal{J})$ then $[D, \mathrm{D}_{a,b}] = \mathrm{D}_{Da,b} + \mathrm{D}_{a,Db}$. Thus the standard derivations span an ideal of Der($\mathcal{J}$).
- Let $\mathcal{J} = \mathcal{H}_3(C) = \mathcal{H}_3(C, J_{\{1,1,1\}})$ the split Albert algebra (i.e., C the split octonions), Der($\mathcal{J}$) its derivation algebra and Der$(J)_0$ the derivations which send e_i to 0, $i = 1, 2, 3$. Der$(J)_0$ is a subalgebra of Der(J) which fixes the Peirce spaces $\mathrm{U}_{e_j,e_k}J = \{a_i[jk] \mid a_i \in C\}$.

The Derivation Algebra of the Split Albert Algebra

- For $D \in \mathrm{Der}(J)_0$, denote D_i the restriction of D to $\mathrm{U}_{e_j,e_k} J$.
- Each D_i is skew with respect to $\mathfrak{b}_n$ (i.e., $\in \mathrm{D}_4$) and satisfies $D_i(ab) = (D_j a)b + a(D_k b)$ (local triality). Recall D_1 determines D_2 and D_3 uniquely.
- Applying the automorphisms of D_4, ϕ_2 and ϕ_3, to $D_1(ab) = (D_2 a)b + a(D_3 b)$ yield the other two equations obtained by permuting (123) cyclically.
- The converse holds, namely, given E in the split Lie algebra of type D_4, triality provides an action of E on the spaces $\mathrm{U}_{e_j,e_k} J$ and one checks that this yields an element of $\mathrm{Der}(J)_0$.

The Derivation Algebra of the Split Albert Algebra

- For $D \in \mathrm{Der}(J)$, $De_1 = a[12] + b[31]$, $De_2 = -a[12] + c[23]$ and $De_3 = -b[31] - c[23]$ for some a, b, $c \in C$, since $D1 = 0$.
- One checks that $D + \mathrm{D}_{e_1, a[12]+b[31]} + \mathrm{D}_{e_2, c[23]} \in \mathrm{Der}(J)_0$.
- The dimension of $\mathrm{Der}(J)$ is $28 + 8 + 8 + 8 = 52$.

Theorem

If the characteristic of F is not 2, the Lie algebra of derivations of a split Albert algebra is simple. It is a split Lie algebra of type F_4. Since the span of the standard derivations is an ideal, they span the Lie algebra of derivations of a split Albert algebra. In characteristic 2, the derivation algebra contains an ideal of dimension 26.

Isotopes

- Let u be an invertible element of an associative algebra A and $A^{(u)}$ be the associative algebra having the same vector space structure as A and product $a_u b := aub$. The *u-isotope* $A^{(u)}$ is a unital associative algebra with unit u^{-1}.
- Consider the map $\mathrm{L}_u : A \to A$, $\mathrm{L}_u a := ua$. $\mathrm{L}_u u^{-1} = 1_A$, $\mathrm{L}_u(a_u b) = uaub = \mathrm{L}_u a \mathrm{L}_u b$. So $A^{(u)} \cong A$.
- Let $(A, *)$ be an associative algebra with involution. If $u \in \mathcal{H}(a, *)$ is invertible, it determines another involution of A, $a^{*_u} := ua^* u^{-1}$. One checks that $\mathcal{H}(A, *_u) = u\mathcal{H}(A, *) = \mathrm{L}_u \mathcal{H}(A, *)$.
- In $A^{(u)}$ the U operator $\mathrm{U}_a^{(u)} b = aubua$ which corresponds to $\mathrm{U}_a \mathrm{U}_u b$.

Isotopes of Jordan Algebras

- Let u be an invertible element of a Jordan algebra $\mathcal{J}$ and $\mathcal{J}^{(u)}$ be the Jordan algebra having the same vector space structure as $\mathcal{J}$ and U operator $\mathrm{U}_a^{(u)} := \mathrm{U}_a \mathrm{U}_u$. The *u-isotope* $\mathcal{J}^{(u)}$ is a unital Jordan algebra with unit u^{-1}.
- If $\mathcal{J}_i$, $i = 1, 2$ are Jordan algebras such that $\mathcal{J}_2 \cong \mathcal{J}_1^{(u)}$ we say that they are *isotopic*.

Example

If $\mathcal{J} = \mathcal{H}(A, *)$ then $\mathcal{J}^{(u)} \cong \mathcal{H}(A, *_u)$, $x^{*_u} = u^{-1} x^* u$. They are not in general isomorphic.

Isotopes of Jordan Algebras

Example

If $u \in J(V, N, \#, 1)^{\times}$ let

$$1^{(u)} = u^{-1},$$
$$x^{\#^{(u)}} := N(u)^{-1}\mathrm{U}_{u^{-1}}x^{\#},$$
$$N^{(u)}(x) := N(u)N(x).$$

The above defines a Norm Structure and
$J(V, N^{(u)}, \#^{(u)}, 1^{(u)}) = J(V, N, \#, 1)^{(u)}$.

The Structure Group

Definition (Structure Group)

Let $\mathcal{J}/F$ be a Jordan algebra. The following are equivalent for all $\eta \in \mathrm{GL}(\mathcal{J})$:
i) η is an isomorphism of $\mathcal{J}$ onto $\mathcal{J}^{(u)}$, for some $u \in \mathcal{J}^{\times}$,
ii) There exists an $\eta^{\#} \in \mathrm{GL}(\mathcal{J})$ such that $\mathrm{U}_{\eta(x)} = \eta \mathrm{U}_x \eta^{\#}$ for all $x \in \mathcal{J}$.

The elements of $\mathrm{GL}(\mathcal{J})$ which satisfy one and hence both of these conditions form a group the *structure group* denoted $\mathrm{Str}(\mathcal{J})$. By the fundamental formula U_x, $x \in \mathcal{J}^{\times}$ belong to $\mathrm{Str}(\mathcal{J})$. They generate a subgroup, the *inner structure group* $\mathrm{Instr}(\mathcal{J})$.

The Structure Group

One can show that $\eta^{\#} = \eta^{-1}\mathrm{U}_{\eta(1)}$. The inner structure group is a normal subgroup of the structure group $\mathsf{Instr}(\mathcal{J}) \triangleleft \mathrm{Str}(\mathcal{J})$. The automorphism group $\mathsf{Aut}(\mathcal{J})$ is a subgroup of $\mathsf{Str}(\mathcal{J})$,
$\mathsf{Aut}(\mathcal{J}) = \{\eta \in \mathrm{Str}(\mathcal{J}) \mid \eta(1) = 1\}$. The *inner automorphism group*
$\mathsf{Inaut}(\mathcal{J}) = \mathsf{Instr}(\mathcal{J}) \cap \mathrm{Aut}(\mathcal{J})$
$= \{\mathrm{U}_{a_1}\mathrm{U}_{a_2}\cdots\mathrm{U}_{a_\ell} \mid a_i \in \mathcal{J}^\times, \mathrm{U}_{a_1}\mathrm{U}_{a_2}\cdots\mathrm{U}_{a_\ell}1 = 1\}$.

Theorem

If $\mathcal{J}$ is an Albert algebra, $\mathsf{Str}(\mathcal{J}) = \mathit{Instr}(\mathcal{J})$ is the norm preserving group and is of type E_6.

Structure Lie Algebras

Definition (Structure Lie algebra)

Let $\mathcal{J}$ be a Jordan algebra. The *structure Lie algebra* $\mathrm{str}(\mathcal{J}) = \{\mathrm{H} \in \mathrm{End}_F(\mathcal{J}) \mid \mathrm{U}_{a,\mathrm{H}a} = \mathrm{H}\mathrm{U}_a - \mathrm{U}_a\overline{\mathrm{H}}\}$, where $\overline{\mathrm{H}} = \mathrm{H} - \mathrm{V}_{\mathrm{H}1}$.

- The structure Lie algebra is the Lie algebra of the structure group.
- The *inner structure Lie algebra* $\mathrm{instr}(\mathcal{J}) = \{\sum \mathrm{V}_{a_i,b_i} \mid a_i, b_i \in \mathcal{J}\}$ and the *inner derivation algebra* $\mathrm{inder}(\mathcal{J}) = \{\sum \mathrm{V}_{a_i,b_i} \mid a_i, b_i \in \mathcal{J},\ \sum a_i \circ b_i = 0\}$. In particular, $\mathrm{V}_{a,b} - \mathrm{V}_{b,a}$ is an inner derivation.

The Structure Algebra of the Split Albert Algebra

- Let $\mathcal{J}$ be an Albert algebra and $\mathcal{J}_0$ the elements of trace 0.The inner structure Lie algebra instr$(\mathcal{J}) = \mathrm{V}_{\mathcal{J}} \oplus \mathrm{Der}\mathcal{J}$. Its dimension of is $27 + 52 = 79$.

Theorem

The derived algebra of the structure Lie algebra of a split Albert algebra is simple. It is a split Lie algebra of type E_6. Since the span of the standard derivations is an ideal, they span the Lie algebra of derivations of a split Albert algebra.

The Tits Kantor Koecher Lie Algebra

Definition ($TKK(\mathcal{J})$)

The *Tits Kantor Koecher Lie Algebra* of a Jordan algebra $\mathcal{J}$,
$TKK(\mathcal{J}) = \mathcal{J} \oplus \mathrm{str}(\mathcal{J}) \oplus \overline{\mathcal{J}}$, $\overline{\mathcal{J}}$ another copy of $\overline{\mathcal{J}}$, with product
$[a_1 + \mathrm{H}_1 + \overline{b_1},\ a_2 + \mathrm{H}_2 + \overline{b_2}] :=$
$(\mathrm{H}_1 a_2 - \mathrm{H}_2 a_1) + (\mathrm{V}_{a_1,b_2} - \mathrm{V}_{a_2,b_1} + [\mathrm{H}_1, \mathrm{H}_2]) + (\overline{\overline{\mathrm{H}_1} b_2 - \overline{\mathrm{H}_2} b_1})$.

If $\mathcal{J}$ is a split Albert algebra then the dimension of
$TKK(\mathcal{J}) = 27 + 27 + 52 + 27 = 133$.

Theorem

If $\mathcal{J}$ is a split Albert algebra then $TKK(\mathcal{J})$ is a simple Lie algebra of type E_7.

A Construction of Freudenthal and Tits

- Let C be a composition algebra and $\mathcal{J}$ a Jordan algebra of a cubic norm over a field F of characteristic not 2 or 3, C_0, $\mathcal{J}_0$ their elements of trace 0. For a, $b \in C$, and x, $y \in \mathcal{J}$, $a * b := ab - \frac{1}{2}t(ab)1_C$ and $x * y := x \cdot y - \frac{1}{3}T(x \cdot y)1_{\mathcal{J}}$ define products on C_0 and $\mathcal{J}_0$ respectively.
- Take $\mathfrak{L}(C, \mathcal{J}) = \mathrm{Der}\, C \oplus C_0 \otimes \mathcal{J}_0 \oplus \mathrm{Der}\mathcal{J}$. $\mathrm{Der}\, C$ and $\mathrm{Der}\mathcal{J}$ are Lie algebras.
- We wish to define a product on $\mathfrak{L}(C, \mathcal{J})$ to make it into a Lie algebra: For a, $b \in C$, and x, $y \in \mathcal{J}$, $D \in \mathrm{Der}\, C$, $D' \in \mathrm{Der}\mathcal{J}$,

$$\begin{aligned}
[D, a \otimes x] &:= Da \otimes x, \\
[D', a \otimes x] &:= a \otimes D'x, \\
[a \otimes x, b \otimes y] &:= \frac{1}{12} T(x \cdot y)\mathrm{D}_{a,b} + (a * b) \otimes (x * y) + \frac{1}{2} t(ab)\mathrm{D}_{x,y}, \\
[D, D'] &:= 0.
\end{aligned}$$

Freudenthal Tits Magic Square

This product defines a Lie algebra structure on $\mathfrak{L}(C, \mathcal{J})$.

	F	$F \times F \times F$	$\mathcal{H}_3(F, \mathrm{J}_\gamma)$	$\mathcal{H}_3(E, \mathrm{J}_\gamma)$	$\mathcal{H}_3(\mathcal{Q}, \mathrm{J}_\gamma)$	$\mathcal{H}_3(\mathcal{O}, \mathrm{J}_\gamma)$
F	0	0	A_1	A_2	C_3	F_4
E	0	$\mathfrak{A}$	A_2	$\mathrm{A}_2 \oplus \mathrm{A}_2$	A_5	E_6
$\mathcal{Q}$	A_1	$\mathrm{A}_1 \oplus \mathrm{A}_1 \oplus \mathrm{A}_1$	C_3	A_5	D_6	E_7
$\mathcal{O}$	G_2	D_4	F_4	E_6	E_7	E_8

$\mathfrak{A}$ is an abelian Lie algebra of dimension 2. For a discussion of the real forms of exceptional Lie algebras, see [J1].

Cohomological Invariants

- Recall:
- Two composition algebras are isomorphic if and only if their norm forms are isometric.
- If F is of characteristic not 2, the norm form of an octonion algebra C is a Pfister form $<< \lambda_1, \lambda_2, \lambda_3 >>$.

Theorem

Two reduced Albert algebras $\mathcal{J} = \mathcal{H}_3(C, J_\gamma)$ and $\mathcal{J}' = \mathcal{H}_3(C', J_{\gamma'})$ are isomorphic if and only if their coefficient algebras $C \cong C'$ and their quadratic traces are isometric.

Associated Quadratic Forms

- In other words, two reduced Albert algebras $\mathcal{J}$ and $\mathcal{J}'$ are isomorphic if and only if two associated quadratic forms n_C, $S_{\mathcal{J}}$ and $n_{C'}$, $S_{\mathcal{J}'}$ are isometric.
- For $\sum_{(123)}(\alpha_i e_{ii} + a_i[jk]) = \begin{bmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3 \overline{a_2} \\ \gamma_1 \overline{a_3} & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2 \overline{a_1} & \alpha_3 \end{bmatrix}$, the quadratic trace $S(x) = T(x^{\#}) = \sum_{(123)}((\alpha_j \alpha_k - \gamma_j \gamma_k n(a_i))$.
- This is the form $[-1] \oplus \mathbf{h} \oplus \langle -1 \rangle . \langle \gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2 \rangle \otimes n_C$, where $[-1]$ is the one dimensional form $-\alpha^2$ and $\mathbf{h}$ the hyperbolic plane. Writing $Q_{\mathcal{J}}$ for $\langle \gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2 \rangle \otimes n_C$, we have $Q_{\mathcal{J}}$ determines $S_{\mathcal{J}}$ and vice versa.

The mod 2 Invariants

- Multiplying J_γ by γ_1^{-1}, we may assume $\gamma_1 = 1$ and $\langle \gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2 \rangle = \langle \gamma_2\gamma_3, \gamma_3, \gamma_2 \rangle$. In that case $n_C \oplus Q_{\mathcal{J}} = << -\gamma_2, -\gamma_3 >> \otimes n_C$.
- To include characteristic 2, (following [EKM]) we would need to consider Pfister forms $<< \alpha_1, \ldots, \alpha_n >> \otimes n_E$, E a quadratic étale algebra
- The forms $<< \alpha_1, \ldots, \alpha_n >>$ have cup product $(\alpha_1) \cup \cdots \cup (\alpha_n) \in H^n(F, \mathbb{Z}/2\mathbb{Z})$, the n^{th} cohomology group.
- We therefore have two so-called invariants mod 2:

  $f_3(\mathcal{J}) = (\lambda_1) \cup (\lambda_2) \cup (\lambda_3) \in H^3(F, \mathbb{Z}/2\mathbb{Z})$ and
  $f_5(\mathcal{J}) = f_3(\mathcal{J}) \cup (-\gamma_2) \cup (-\gamma_3) \in H^5(F, \mathbb{Z}/2\mathbb{Z})$.

Reduced Albert Algebras

Theorem

The invariants $f_3(\mathcal{J})$ and $f_5(\mathcal{J})$ classify reduced Albert algebras.

- If $\mathcal{J}$ is an Albert division algebra then $\mathcal{J}_E = \mathcal{J} \otimes_F E$ is a reduced Albert algebra for a suitable odd-degree reducing extension E/F. Since the two Pfister forms over E afforded by $\mathcal{J}_E$ are obtained by tensoring Pfister forms over F, the invariants $f_3(\mathcal{J})$ and $f_5(\mathcal{J})$ are also defined for division Albert algebras.
- Note that $f_3(\mathcal{J}) = 0$ implies $f_5(\mathcal{J}) = 0$.

A mod 3 Invariant?

- If A/F is a central simple associative algebra, denote by $[A]$ the class of A in the Brauer group; so $[A] \in Br(F) = H^2(F, F_s)$, F_s the separable closure of F.
- If A/F is of degree 3 then $[A] \in {}_3Br(F) \cong H^2(F, \mu_3)$, μ_3 the cube roots of unity. For $\alpha \in F^\times$, denote by (α) the image of α in $F^\times/F^{\times 3} \cong H^1(F, \mu_3)$. Since $\mu_3 \otimes \mu_3$ is canonically isomorphic to $\mathbb{Z}/3\mathbb{Z}$, the cup product $[A] \cup (\alpha) \in H^2(F, \mu_3) \cup H^1(F, \mu_3) \cong H^3(F, \mathbb{Z}/3\mathbb{Z})$.

The mod 3 Invariant

Theorem

There is an invariant of the isomorphism class of an Albert algebra $\mathcal{J}/F$, $g_3(\mathcal{J}) \in H^3(F, \mathbb{Z}/3\mathbb{Z})$ which
1) is compatible with base change, i.e., $g_3(\mathcal{J} \otimes_F E) = res_{E/F}(g_3(\mathcal{J}))$, for any field extension E/F, for the restriction map $res_{E/F} : H^i(F, \mathbb{Z}/3\mathbb{Z}) \to H^i(E, \mathbb{Z}/3\mathbb{Z})$,
2) characterizes division algebras, i.e., $\mathcal{J}$ is reduced if and only if $g_3(\mathcal{J}) = 0$,
3) satisfies $g_3(J(A, \alpha)) = [A] \cup (\alpha)$.

First Tits Algebras Containing a Copy of A^+

Theorem

The first Tits construction algebras $J(A, \alpha_1)$, $J(A, \alpha_2)$ are isomorphic if and only if $\alpha_1 = \alpha_2 N_A(u)$ for some $u \in A^\times$.

- If $\alpha_1 = \alpha_2 N_A(u)$ for $u \in A^\times$, it is not hard to show that $J(A, \alpha_1) \cong J(A, \alpha_2)$. If $J(A, \alpha_1) \cong J(A, \alpha_2)$ then $g_3(J(A, \alpha_1)) = g_3(J(A, \alpha_2))$ and $[A] \cup (\alpha_1) = [A] \cup (\alpha_2)$. Therefore $[A] \cup (\alpha_1) - [A] \cup (\alpha_2) = [A] \cup (\alpha_1 \alpha_2^{-1}) = 0$. So $J(A, \alpha_1 \alpha_2^{-1})$ is reduced and by the criterion for a first Tits construction to be a division algebra $\alpha_1 \alpha_2^{-1} \in N_A(A^\times)$.

Second Tits Algebras Containing a Copy of $\mathcal{H}(B, *)$

Theorem

The second Tits construction algebras $J(B, *, u_1, \beta_1)$, $J(B, *, u_2, \beta_2)$ are isomorphic if and only if $u_2 = vu_1v^*$ and $\beta_2 = \beta_1 N_B(v)$ for some $v \in B^\times$.

Let E/F be a separable field extension whose degree is not divisible by 3. By considering the restriction and corestriction maps, one sees that a non zero mod 3 invariant remains non trivial under that base change.

Theorem

If $\mathcal{J}/F$ is a division Albert algebra and E/F is a separable field extension whose degree is not divisible by 3, then $\mathcal{J} \otimes_F E$ is a division algebra.

Reduced Models

Realizing an Albert algebra $\mathcal{J}$ as a generalized Second Tits construction and considering the corresponding quadratic forms $S_{\mathcal{J}}$ and $Q_{\mathcal{J}}$, one obtains the following

Theorem

If $\mathcal{J}/F$ is a division Albert algebra then there exists a reduced Albert algebra $\mathcal{H}_3(C, J_\gamma)$ over F such that for any extension E/F that reduces $\mathcal{J}$, $\mathcal{J} \otimes_F E \cong \mathcal{H}_3(C, J_\gamma) \otimes_F E$. $\mathcal{H}_3(C, J_\gamma)$ is unique up to isomorphism and is called the *reduced model* of $\mathcal{J}$.

$f_3(\mathcal{J}) = 0$

A careful look at cubic subfields of first Tits algebras allows one to obtain

Theorem

If $\mathcal{J}/F$ is an Albert algebra, TFAE
1) $\mathcal{J}$ is a first Tits construction algebra,
2) The reduced model of $\mathcal{J}$ is split,
3) $f_3(\mathcal{J}) = 0$.

Theorem

If $\mathcal{J}/F$ is a first Tits construction algebra and $\mathcal{J}'$ is isotopic to $\mathcal{J}$ then $\mathcal{J}'$ is isomorphic to $\mathcal{J}$.

Isotopy Invariants

Let $\mathcal{J}$ be an Albert algebra. The definition of $f_5(\mathcal{J})$ shows that passing to an isotope may change f_5. If $\mathcal{J}$ is reduced then isotopes have isomorphic coefficient algebras. So f_3 will be the same for isotopes. If $\mathcal{J}$ is a division algebra and E/F a cubic subfield of $\mathcal{J}$ then $\mathcal{J} \otimes_F E$ is reduced and again f_3 is an isotopy invariant. By the previous Theorem, if $\mathcal{J}$ is a first Tits algebra then all isotopes are isomorphic so g_3 is an isotopy invariant. If $\mathcal{J}$ is not a first Tits algebra then tensoring with an appropriate quadratic extension yields a first Tits algebra.

Theorem

The invariants $f_3(\mathcal{J})$ and $g_3(\mathcal{J})$ are isotopy invariants.

Do the Invariants Determine an Albert Algebra?

- Do $f_3(\mathcal{J})$ and $g_3(\mathcal{J})$ determine $\mathcal{J}$ up to isotopy?
- Do the invariants mod 2 and mod 3 classify Albert algebras up to isomorphism?

Theorem

Let $\mathcal{J}/F$ and $\mathcal{J}'/F$ be Albert algebras having the same mod 2 and mod 3 invariants. If F is of characteristic not 2 or 3 then there exists a finite extension E/F whose degree is not divisible by 3 and a finite extension K/F whose degree divides 3 such that $\mathcal{J} \otimes E \cong \mathcal{J}' \otimes E$ and $\mathcal{J} \otimes K \cong \mathcal{J}' \otimes K$.

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