ABSTRACTS OF TALKS.
WHITNEY WORKSHOP, TORONTO 2012

Charlie Fefferman, Arie Israel, Kevin Luli. *Mini-course*: "Whitney's extension problem in Sobolev spaces".
Abstract: Let X be the Sobolev space of functions F on $\mathbf{R}^n$ whose m-th derivatives belong to L^p. Given a real-valued function f defined on a (given) arbitrary set $E \subset \mathbf{R}^n$, we ask: How can we decide whether f can be extended to a function F belonging to X? If such an F exists, then how small can we take its norm in X? What can we say about F and its derivatives at a given point? Can we take an F to depend linearly on f while keeping the order of magnitude of the Sobolev norm of F as small as possible? If E is finite, can we compute an F whose Sobolev norm is of least possible order of magnitude? If so, how many computer operations does it take? The analogues of these questions for $X = C^m(R^n)$ are rather well understood. This mini-course explains what we know when X is a Sobolev space, sketches the proofs of some of the main results, and works out a few examples.

Matthew Hirn. A general theorem of existence of quasi absolutely minimal Lipschitz extensions.
Abstract: We present a general notion of minimal Lipschitz extensions, and then show that under certain assumptions one can prove the existence of what we call a quasi absolutely minimal Lipschitz extension. The assumptions are general enough that they include the well known scalar valued function case, as well as vector valued functions and 1-fields (joint with **Erwan Le Gruyer**).

Nahum Zobin. Quantization of Whitney problems.
Abstract: Quantitative versions of Whitney problems require construction of functions with prescribed values on a given finite subset $E \subset \mathbf{R}^n$, which minimize a preferred functional norm. After solving this minimization problem we want to compute some natural functionals of the minimizer, e.g., its values at other points. Quantization is an art of replacing a minimization problem by a problem of computing certain amplitudes (similar to expected values) for a system where the preferred functional norm is treated as an action functional. There is an interesting connection between the computation of amplitudes (which are represented as functional integrals) and computations of convolutions of functions on some important unipotent Lie groups, similar to the Heisenberg-Weyl groups.

Piotr Hajlasz. Sobolev Spaces and the Whitney Extension Theorem
Abstract: This is a survey talk in which I will discuss various results for Sobolev spaces that involve ideas around the Whitney extension theorem. I will discuss the Lusin type approximation of Sobolev functions, a version of the Sard theorem for Sobolev mappings, the Sobolev extension domains and the Lusin theorem for higher order derivatives.

Vladimir Gol'dshtein. Extension of Sobolev Functions to Capacitory Boundaries
Abstract: Using the concept of the p-capacity, associated with uniform Sobolev spaces $L^1_p(\Omega)$, we introduce a notion of a p-capacitory boundary for an arbitrary

domain $\Omega \subset \mathbb{R}^n$, $n-1 < p \leq n$ that represent "ideal boundaries" of domains $\Omega \subset R^n$ for p-capacitory metrics. The Sobolev classes $L^1_p(\Omega)$ can be extended to the p-capacitory boundaries under some natural assumptions on Ω. The p-capacitory topology is equivalent to the Euclidean one into Ω, but the p-capacitory boundary depends on p and can be very far from the Euclidean one for non-regular domains. An analog of p-capacitory boundaries can be introduced for $p > n$ also using a more delicate procedure. Joint work with **Alexander Ukhlov**.

Andreea Nicoara. Properties of sheaves in the Kohn algorithm
Abstract: In 1979 Joseph J. Kohn defined ideal sheaves of multipliers and an algorithm for producing these in order to investigate the subellipticity of the $\overline{\partial}$ Neumann problem on pseudoconvex domains in $\mathbf{C}^n$. I will be discussing the properties of these sheaves in the cases when the boundary is smooth, real-analytic, and Denjoy-Carleman. I will show that in the smooth case these ideal sheaves are quasi-flasque, and I will discuss coherence in the real-analytic case. The Denjoy-Carleman case is intermediate between the two, and I will show to what extent the nice properties of the real-analytic case transfer over.

Assaf Naor. The Lipschitz Extension Problem, 1,2,3.
Abstract: The Lipschitz extension problem asks for geometric conditions on a pair of metric spaces X and Y implying that there exists a positive constant K such that for every subset A of X, every L-Lipschitz function f from A to Y can be extended to a (KL)-Lipschitz function defined on all of X. When Y is the real line then this is always possible with $K = 1$ (the nonlinear Hahn-Banach theorem), in which case one asks for an extension of f with additional desirable properties. For general metric spaces X, Y it is usually the case that no such K exists. However, many deep investigations over the past century have revealed that in important special cases the Lipschitz extension problem does have a positive answer. Proofs of such theorems involve methods from a variety of mathematical disciplines, and when available, a positive solution to the Lipschitz extension problem often has powerful applications. The first talk will be an introduction intended for non-experts, giving an overview of the known Lipschitz extension theorems, and an example or two of the varied methods with which such theorems are proved. The following two lectures will deal with more specialized topics, including the use of probabilistic methods, some illuminating counterexamples, examples of applications, and basic problems that remain open.

Yuri Brudnyi. Covering of a dyadic sieve and nonlinear approximation of BV functions.
Abstract: A dyadic N-sieve is the unit cube in $\mathbb{R}^d$, $d \geq 2$, with removed N dyadic subcubes. The next geometric-combinatorial problem is rooted in some questions of nonlinear approximation by splines and wavelets. How many ε-linked cubes $(0 \leq \varepsilon \leq 1)$ is required to cover the N-sieve? The family of cubes is ε-linked if it can be linearly ordered so that measure of intersection for two adjacent cubes is bigger than ε multiplying by measure of their union. If at least one of removed cubes is not dyadic, the upper bound maybe arbitrarily large but here the upper bound depends on d and N (exponentially if $\varepsilon > 0$, and linearly in N if $\varepsilon = 0$). We explain in this lecture graph theoretic ideas behind the proof and application to a Sobolev-Whitney type inequality playing an essential role in nonlinear approximation.

Alf Jonsson. Spaces of Besov type defined on fractals.

Abstract: Besov, or Lipschitz, spaces have been defined on closed subsets F of $\mathbf{R}^n$, or on classes of closed subsets, in such a way that they are restrictions to F of Besov or Sobolev spaces on $\mathbf{R}^n$. In this talk we discuss some spaces closely related to these Besov spaces, which have appeared in connection with analysis on fractals or wavelets on fractals, and give an overview of some applications of them to e.g. trace theorems.

Pavel Shvartsman. Sobolev L^2_p-functions on closed subsets of $\mathbf{R^2}$, 1,2.
Abstract: For each $p > 2$ we give intrinsic characterizations of the restriction of the homogeneous Sobolev space $L^2_p(\mathbf{R^2})$ to an arbitrary finite subset E of $\mathbf{R^2}$. The trace criterion is expressed in terms of certain weighted oscillations of the second order with respect to a measure generated by the Menger curvature of triangles with vertices in E.

Lizaveta Ihnatyeva. Smoothness spaces on Ahlfors regular sets.
Abstract: The theory of local polynomial approximation provides a unified framework for describing various spaces of smooth functions on Euclidean space. We use an analogous theory to define smoothness spaces on Ahlfors d-regular subsets of $\mathbb{R}^n$, $n-1 < d < n$. This class of sets includes many interesting Cantor-type sets and self-similar sets. We give a characterization in terms of local polynomial approximations for traces of Besov spaces and Triebel–Lizorkin spaces on Ahlfors d-regular sets, $n-1 < d < n$. We also focus on Sobolev-type spaces on such subsets and the relation between these spaces and traces of classical Sobolev spaces. This work extends the results of P. Shvartsman on characterizing traces of classical spaces of smooth functions on Ahlfors n-regular subsets of $\mathbb{R}^n$. The talk is based on joint work with **A.V. Vähäkangas** and on joint work with **R. Korte**.

Yosef Yomdin. Reconstruction of Piecewise C^k - Functions From Point-wise and Fourier Samples.
Abstract: A C^k-function on an n-dimensional torus is approximated by its Fourier polynomial of degree N with the accuracy of order $\frac{C}{N^k}$. However, for piecewise C^k-functions the approximation accuracy drops to the order of $\frac{C}{\sqrt{N}}$. The following conjecture by Eckhoff (1995) has been intensively studied:

There is a non-linear algebraic procedure reconstructing any piecewise C^k-function (of one or several variables) from its first N Fourier coefficients, with the overall accuracy of order $\frac{C}{N^k}$. This includes the discontinuities' positions, as well as the smooth pieces over the continuity domains.

Recently we have shown that for functions in one variable at least half the conjectured accuracy (i.e. an order of $C/N^{\frac{k}{2}}$) can be achieved. However, the multidimensional question remains open, as well as the question of maximal possible reconstruction accuracy. In this talk we present some recent results towards this reconstruction problem. We also discuss the case of "mixed measurements" where a part of the samples is taken in the geometric domain, and a part in the Fourier one. In this last setting the Whitney C^k - extension problem naturally enters the considerations. This is a joint work with **D. Batenkov**.

Chen Li. Gradual Variations and Partial Differential Equations
Abstract: Numerically solving partial differential equations (PDE) has made a significant impact on science and engineering since the last century. The finite difference method and finite elements method are two main methods for numerical

PDEs. This talk presents a new method that uses gradually varied functions to solve partial differential equations, specifically in groundwater flow equations. We first introduce basic partial differential equations including elliptic, parabolic, and hyperbolic differential equations and the brief descriptions of their numerical solving methods. Second, we establish a mathematical model based on gradually varied functions for parabolic differential equations, then we use this method for groundwater data reconstruction. This model can also be used to solve elliptic differential equations. Lastly, we present a case study for solving hyperbolic differential equations using our new method.

Alex Brudnyi. On the approximation properties of spaces predual to H^∞-spaces.
Abstract: The famous open problem asks about the (Grothendieck) approximation property for the Banach space H^∞ of bounded analytic functions on the unit disk equipped with supremum norm. In the talk we discuss some recent results related to this problem and establish approximation property for a large class of space predual to H^∞-spaces.

Michal Wojciechowski. On the dependence of isomorphic type of spaces of smooth functions on the domain.
Abstract: We construct a three dimensional domain D with the property that at least one of the following statements holds true 1) $C^1(D)$ is not isomorphic to $C^1(I^k)$ for some $k > 1$, 2) the space $\mathrm{Lip}\,(I^k)$, $k > 1$, does not have bounded approximation property. Here I^k stands for the k-dimensional cube, $C^1(D)$ is the space of smooth functions with gradient continuously extending to the closure of D, and $\mathrm{Lip}\,(I^k)$ stands for the space of Lipschitz functions on I^k.

Wieslaw Plesniak. Subanalytic geometry methods in polynomial approximation
Abstract: In multivariate polynomial approximation one has to manage with a complicated geometry of sets. In particular, serious obstacles are caused by cuspidal sets. It appears that a lot of such problems may be surmounted by using tools provided by the Hironaka–Lojasiewicz–Gabrielov subanalytic geometry, and more general by its far-going generalizations, that are polynomially bounded o-minimal structures. The goal of my talk will be to demonstrate the effectiveness of such an approach in the case of polynomial approximation of analytic and in- finitely differentiable functions on compact subsets of $\mathbf{C}^n$ or $\mathbf{R}^n$.

Krzysztof Kurdyka. Approximation of o-minimal maps satisfying a Lipschitz condition (after A. Fischer).
Abstract: In a recent paper Andreas Fischer shows that definable (in o-minimal expansion of the real field) Lipschitz continuous maps can be definably fine approximated by definable continuously differentiable Lipschitz maps whose Lipschitz constant is close to that of the original map. His construction uses lambda-regular decompositions.

Athipat Thamrongthanyalak. Definable Version of Whitney's Extension Theorem.
Abstract: In 1934, Whitney asked how can we determine whether a jet of order m on a closed subset E of $\mathbf{R}^n$ is a jet of order m of a C^m function. It is well-known that the answer is such jets must be Whitney fields. Later, Pawłucki and Kurdyka proved that subanalytic Whitney fields of order m are restrictions of subanalytic C^m functions. Here, we discuss Whitney's Extension Theorem in o-minimal structures (joint work with **Matthias Aschenbrenner**).