‘Interpolating’ between Hilbert space operators, and real positivity for operator algebras

---

David Blecher
University of Houston

June 2014

Abstract

With Charles Read we have introduced and studied a new notion of (real) positivity in operator algebras, with an eye to extending certain C-algebraic results and theories to more general algebras. As motivation note that the 'completely' real positive maps on C^*-algebras or operator systems are precisely the completely positive maps in the usual sense; however with real positivity one may develop a useful order theory for more general spaces and algebras. This is intimately connected to new relationships between an operator algebra and the C^*-algebra it generates, and in particular to what we call noncommutative peak interpolation, and noncommutative peak sets. We report on the state of this theory (joint work with Read, and some with Matt Neal, some in progress at the time of writing) and on the parts of it that generalize further to certain classes of Banach algebras (joint work with Narutaka Ozawa).

Part I. Noncommutative topology and ‘interpolation’

• We make a noncommutative generalization of function theory, and in particular the theory of algebras of continuous functions on a topological space (function algebras/uniform algebras), where historically there was an interesting kind of ‘relative topology’ going on (peak sets).

Part I. Noncommutative topology and 'interpolation'

• We make a noncommutative generalization of function theory, and in particular the theory of algebras of continuous functions on a topological space (function algebras/uniform algebras), where historically there was an interesting kind of 'relative topology' going on (peak sets).

Noncommutative function algebra: an algebra of continuous linear operators $T : H \to H$ on a Hilbert space H. Henceforth: an operator algebra. Equivalently, a subalgebra A of a C^*-algebra.

Unital if there is an identity of norm 1. Approximately unital if there is a contractive approximate identity (cai).

- We will describe a way to merge theories:

$[C^*$-algebra$]$ + $[$function algebras$]$ $\rightsquigarrow$ nc func algebras/nc func theory

- We will describe a way to merge theories:

$[C^*$-algebra$] + [$function algebras$] \rightsquigarrow$ nc func algebras/nc func theory

Remarks. 1) Do not think of this as producing a 'poor mans C^*-algebra theory', but rather as a noncommutative function theory

2) Which theory...which hundreds of theorems... one ends up with, really depends on which nc topology is used. What are the open/closed/compact sets, topological theorems, etc?

• There have been many approaches over the last century to noncommutative topology. Commonly, 'noncommutative topology' is the study of noncommutative algebras with the same algebraic structure as $C_0(K)$, namely C^*-algebras.

• But then: what are the open/closed/compact sets? What theorems from your topology course generalize?

• We take what I think of as the most literal approach to the above good question: Akemann's noncommutative topology which we describe later. As opposed to other approaches, such as the spectrum etc., which have different advantages/difficulties. This distinction is key, as we said on the last slide.

• What is classical 'interpolation'? And then what is the 'interpolating between Hilbert space operators' of the title?

• What is classical 'interpolation'? And then what is the 'interpolating between Hilbert space operators' of the title?

• If A is a function algebra, I will define classical 'interpolation' as 'building functions' in A which 'do what you want'.

• If A is a operator algebra, I will define 'interpolation' as 'building operators' in A which 'do what you want'.

• What is classical 'interpolation'? And then what is the 'interpolating between Hilbert space operators' of the title?

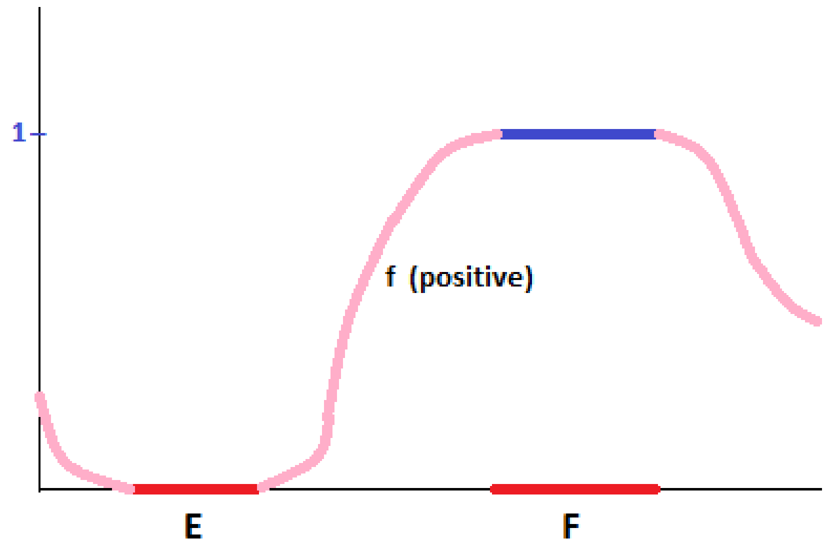
• If A is a function algebra, I will define classical 'interpolation' as 'building functions' in A which 'do what you want'.

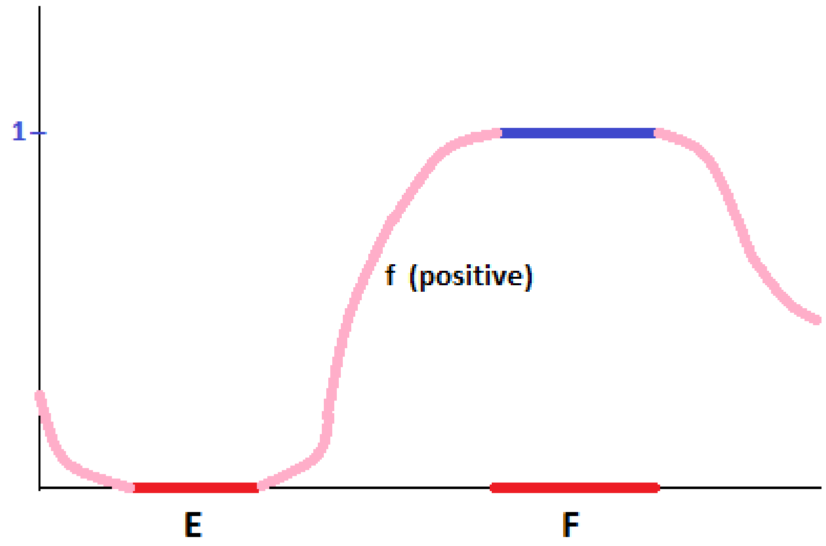
• If A is a operator algebra, I will define 'interpolation' as 'building operators' in A which 'do what you want'.

• Lets look at some examples of 'which do what you want': Urysohn lemma, Tietze extension, Bishop's peak interpolation, 'order-interpolation'/Brown-Akemann-Pedersen's C^*-algebraic interpolation/semicontinuity theory.

The 'grand-daddy' interpolation result: Urysohn's lemma

Given: disjoint closed subsets E, F of compact K ... there exists $f \in C(K)$ with $0 \leq f \leq 1$ and $f = 0$ on E and $f = 1$ on F.





• This is perhaps the most important result in topology for analysts, certainly the most important result in topology for function-algebraists, and constitutes the first steps in building more complicated functions with prescribed values or behaviors on given subsets of K.

Three 'what if's'

• What if we insist f is in a fixed given subalgebra A of $C(K)$? For example, the disk algebra of continuous functions on a disk that are analytic in the interior.

This is the starting point of a subject called peak interpolation, which we will survey quickly on the next slides

Three ‘what if’s’

• What if we insist f is in a fixed given subalgebra A of $C(K)$? For example, the disk algebra of continuous functions on a disk that are analytic in the interior.

This is the starting point of a subject called peak interpolation, which we will survey quickly on the next slides

• What if we want to replace functions by operators on a Hilbert space? Or replace $C(K)$ above by a C^*-algebra (this is the starting point of noncommutative C^*-algebraic interpolation (Akemann, Pedersen, Brown, ...).

E.g. Akemann's Urysohn lemma for C^*-algebras is a noncommutative interpolation result of a selfadjoint flavor, and this result plays a role in recent approaches to the important Cuntz semigroup.

Three 'what if's'

• What if we insist f is in a fixed given subalgebra A of $C(K)$? For example, the disk algebra of continuous functions on a disk that are analytic in the interior.

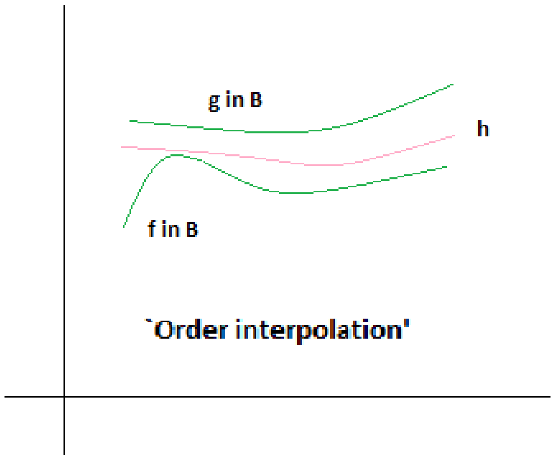
This is the starting point of a subject called peak interpolation, which we will survey quickly on the next slides

• What if we want to replace functions by operators on a Hilbert space? Or replace $C(K)$ above by a C^*-algebra (this is the starting point of noncommutative C^*-algebraic interpolation (Akemann, Pedersen, Brown, ...).

E.g. Akemann's Urysohn lemma for C^*-algebras is a noncommutative interpolation result of a selfadjoint flavor, and this result plays a role in recent approaches to the important Cuntz semigroup.

• What if we want to do both? So now want f in an operator algebra, that is, in a subalgebra A of a C^*-algebra (noncommutative peak interpolation, Blecher-Hay-Neal-Read)

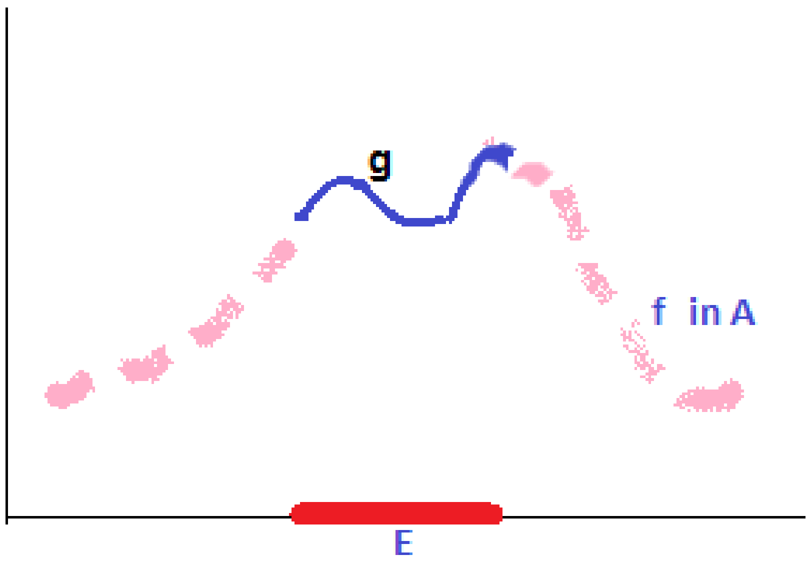
• One can do the same 'what ifs' for e.g. the Tietze extension theorem, or 'order-interpolation' (fitting a function from A (or its real part), between two given functions).



Setting for classical peak interpolation:

Given: a fixed algebra A of continuous scalar functions on a compact (for convenience in this talk) Hausdorff space K, ...

... and one tries to build functions in A which have prescribed values or behaviour on a fixed closed subset E of K (or on several disjoint subsets), without increasing the sup norm.

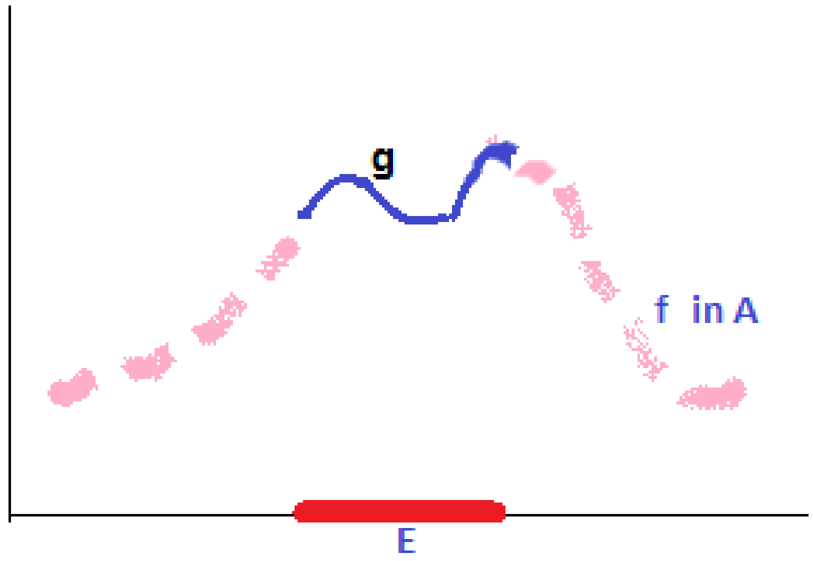


• The sets E that 'work' for this are the p-sets, namely the closed sets whose characteristic functions are in the 'second annihilator' $A^{\perp\perp}$ (or weak* closure) of A in $C(K)^{**}$

Tietze-type extension:

Given: again A is a fixed algebra of continuous scalar functions on K, and E is a p-set. Suppose g is given on E

... and one tries to build a function $f \in A$ extending g, without increasing the sup norm, and whose 'values' lie in the same convex set as the values of g.



• The sets E that 'work' for peak interpolation are the p-sets, namely the closed sets whose characteristic functions are in the 'second annihilator' $A^{\perp\perp}$ (or weak* closure) of A in $C(K)^{**}$

Glicksberg's peak set theorem characterizes these sets as the intersections of peak sets, i.e. sets $f^{-1}(\{1\})$ for a norm 1 function f in A.

• In the separable case, they are just the peak sets (one doesnt need intersections)

Peak set: $E = f^{-1}(\{1\})$ for a norm 1 function f in A. One may rechoose f such that $|f| < 1$ on E^c, in which case $f^n \to \chi_E$.

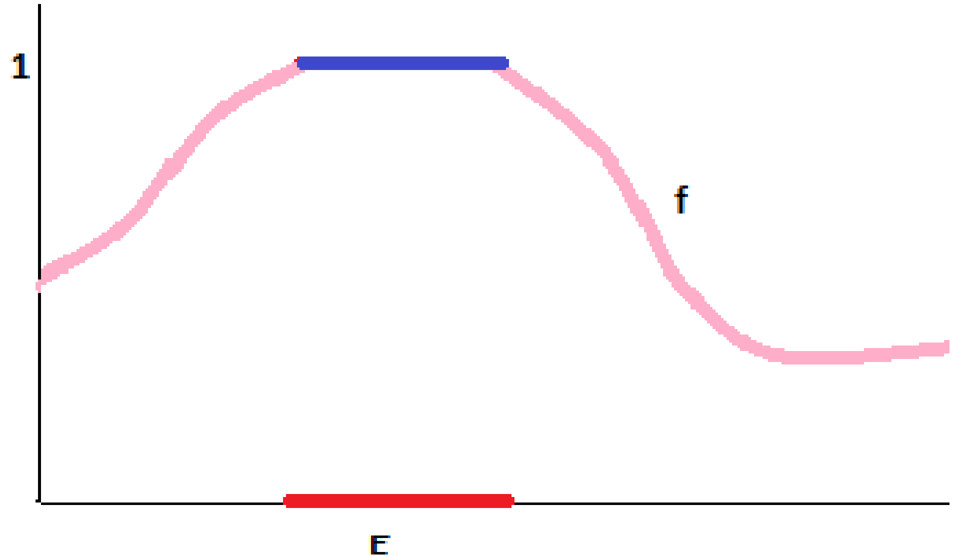


Figure 1: A peak set E

A primary example of a peak interpolation result, which originated in results of Errett Bishop, and continued by Gamelin, says:

Theorem If h is a continuous strictly positive scalar valued function on K, then the continuous functions on E which are restrictions of functions in A, and which are dominated in modulus by the 'control function' h on E, have extensions f in A with $|f(x)| \leq h(x)$ for all $x \in K$.

Figure 2: Errett Bishop

We will return from time to time to this result, so we shall refer to it as the Bishop-Gamelin theorem

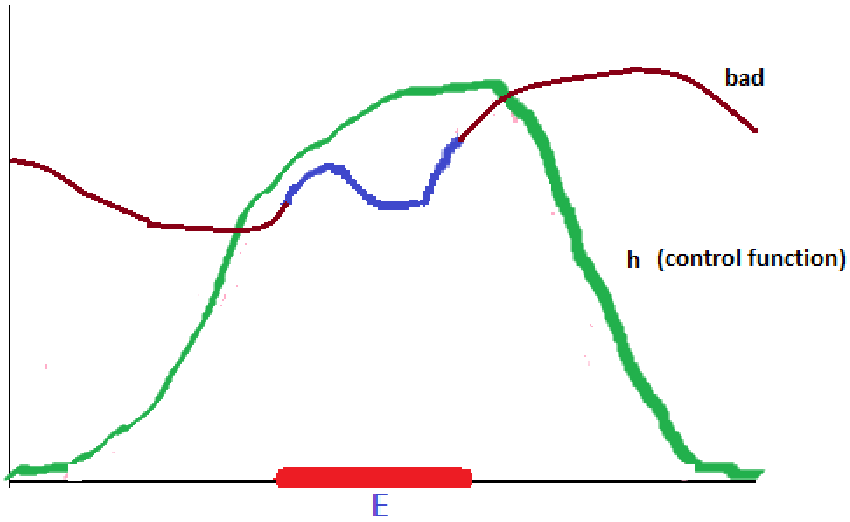


Figure 3: Extension dominated by control function

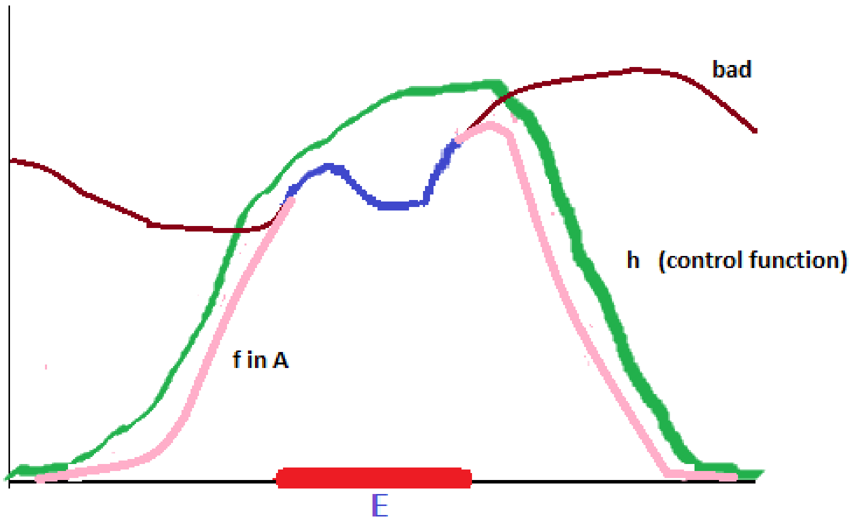
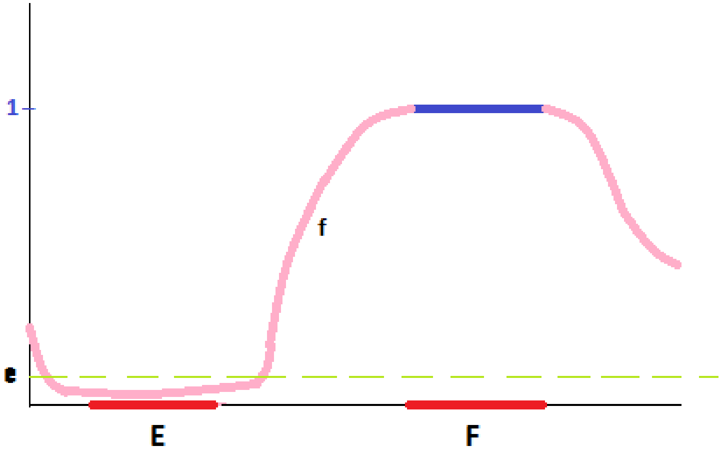


Figure 4: Extension dominated by control function

• A special case of interest is when $h = 1$; for example when this is applied to the disk algebra one obtains the well known Rudin-Carleson theorem which tells you exactly when one can extend a continuous function on a subset of the circle, to a function in the disk algebra (so continuous on the circle and analytic inside the circle), without increasing the supremum norm of the function.

• ‘Classical peak interpolation’ also yields ‘Urysohn type lemmas’ in which we find functions in A which are 1 on F and zero on a closed set E disjoint from F (or close to zero, depending on the type of closed set).



• These constitute the first steps in building more complicated functions in A with prescribed values or behaviors on given closed subsets of K.

• The above theory ‘goes noncommutative’ – in only one way right now.

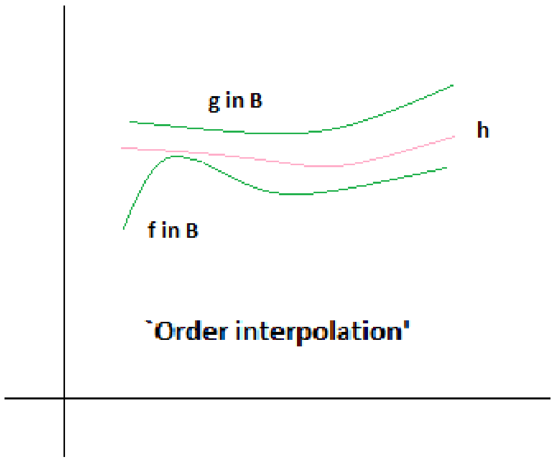
Final example of what I mean by 'interpolation': order interpolation, and Brown-Akemann-Pedersen's C^*-algebraic interpolation/semicontinuity theory.

- Here in the classical variant one is given two functions $f \leq g$ in $B = C(K)_{\rm sa}$ and one wishes to find a function $k \in A$ (or maybe its real part) 'between' f and g

- In Brown's variant (see e.g. Canad. J. Math. (1988) and many recent papers on the ArXiv), A and B are C^*-algebras, usually $A = B$, and f and g are respectively uppersemicontinuous and lowersemicontinuous elements in A^{**}

- Akemann's Urysohn lemma (next) is a special case of the latter, here f, g are also projections

- Similarly, our Urysohn lemmas (later) are such ‘order interpolation’ results, and we will see other examples later

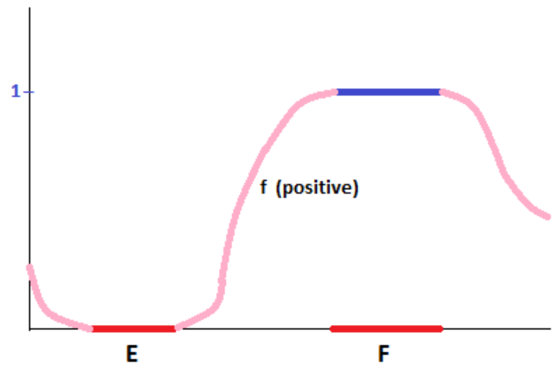


Noncommutative topology and Urysohn

The C^*-algebra case. If one rephrases Urysohns lemma algebraically, in terms of the algebra $B = C(K)$, one gets Akemann's Urysohns lemma:

- subsets E of K, are replaced by its characteristic function $p = \chi_E$, a projection (namely $p = p^2 = p^*$)

Given: p, q closed (uppersemicontinuous) projections in B^{**}, with $pq = 0$ there exists $f \in B$ with $0 \leq f \leq 1$ and $fp = 0$ and $fq = q$.



Akemann's noncommutative Urysohn lemma: Given p, q closed projections in B^{**}, with $pq = 0$ there exists $f \in B$ with $0 \leq f \leq 1$ and $fp = 0$ and $fq = q$.

• A projection $q \in B^{**}$ is open iff $q^{\perp}$ is closed (these are lowersemicontinuous projections in B^{**}).

• A projection $q \in B^{**}$ is compact iff there exists $b \in B$ with $qb = q$.

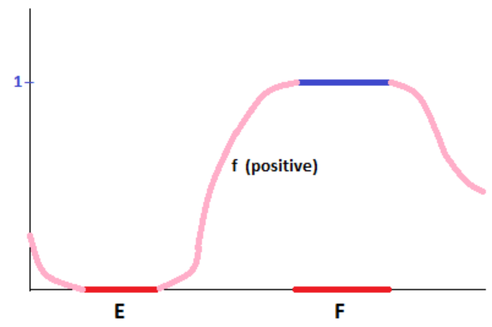
Thus topology has become the theory of a certain class of projections in the second dual B^{**}. This is Akemann's noncommutative topology.

• Of course, B^{**} is a von Neumann algebra. Can work in a smaller space than B^{**} if you want to.

• Nc Urysohn is a nice tool for C^*-algebras, and it obviously generalizes the classical Urysohn

Next: Replace the C^*-algebra B with a closed subalgebra $A \subset B$.

Question: What replaces 'positivity' in Urysohn's lemma?

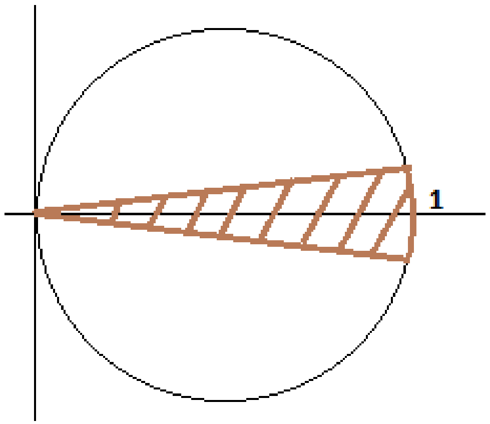


Answer: The 'positivity' or 'near-positivity' of [B-Read], which we will discuss more later in the talk.

• In the theory of C^*-algebras, positivity and the existence of positive approximate identities is crucial.

In place of positivity for an operator algebra A we use real positive elements: $a \in A : a + a^* \geq 0$ (and certain subsets of these with more powerful properties)

• n-th roots of such a have spectrum and numerical radius within a cigar which is as thin as we like, and are as close as we like to an operator (namely Re a) which is positive in the usual sense



Relative noncommutative topology

Our setting is a closed subalgebra A of a C^*-algebra B. We define open, closed, compact projections in A^{**}, and develop their theory analogously to the C^*-algebra case (that is, reprise the noncommutative variants of the facts and theorems from topology).

Relative noncommutative topology

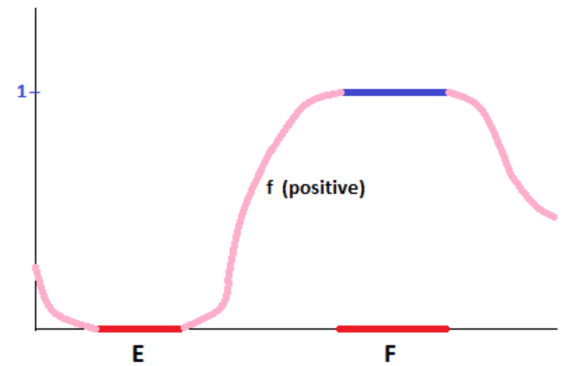
Our setting is a closed subalgebra A of a C^*-algebra B. We define open, closed, compact projections in A^{**}, and develop their theory analogously to the C^*-algebra case (that is, reprise the noncommutative variants of the facts and theorems from topology).

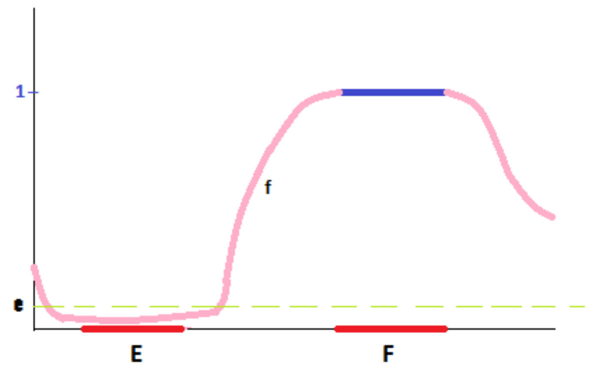
• I will not go into this theory too much today. There are intrinsic definitions, but e.g. p is open in A^{**} iff $p \in A^{**}$ and p is open in B^{**}. It turns out to be not so relative; nothing depends for example on which particular containing C^*-algebra you use.

• For example, the disk algebra $A(D)$ generates $C(S^1), C(\bar{D})$, and the Toeplitz C^*-algebra

B-Neal-Read noncommutative Urysohn lemma Let A be an operator algebra (unital for simplicity). Given p, q closed projections in A^{**}, with $pq = 0$ there exists $f \in \mathrm{Ball}(A)$ 'nearly positive' and $fp = 0$ and $fq = q$.

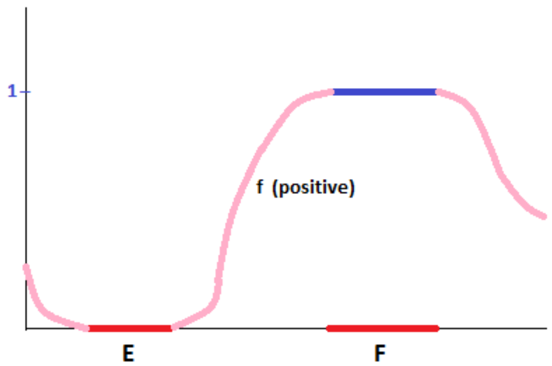
• Can also do this with q closed in B^{**}, where B is the containing C^*-algebra, but now need an $\epsilon > 0$ (i.e. f 'close to zero' on p; that is $||fp|| < \epsilon$).

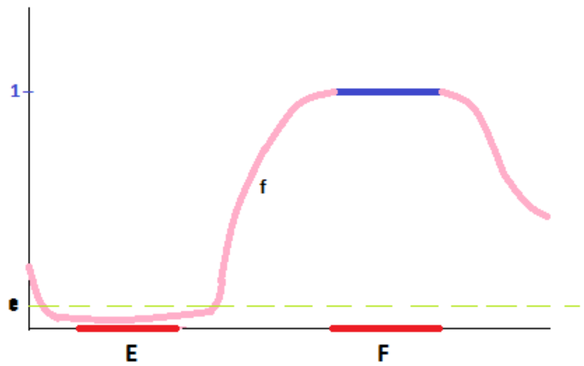




B-Neal-Read noncommutative Urysohn lemma Let A be an operator algebra (unital for simplicity). Given p, q closed projections in A^{**}, with $pq = 0$ there exists $f \in \mathrm{Ball}(A)$ 'nearly positive' and $fp = 0$ and $fq = q$.

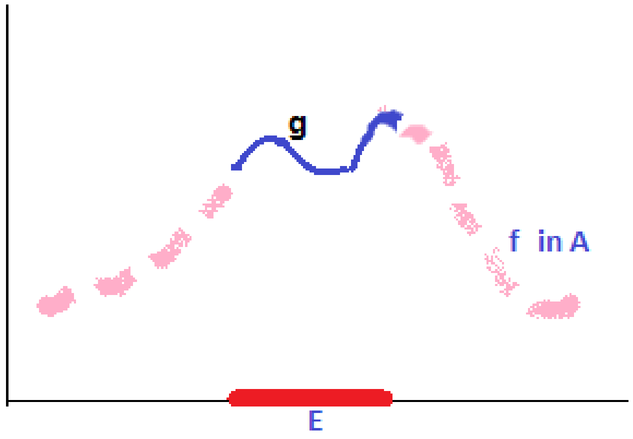
• Can also do this with q closed in B^{**}, where B is the containing C^*-algebra, but now need an $\epsilon > 0$ (i.e. f 'close to zero' on p; that is $||fp|| < \epsilon$).





B-Read Strict noncommutative Urysohn lemma This is the variant that under a natural (and necessary) countability hypothesis on the projections, one can find such f as above with also $0 < f < 1$ 'on' $q - p$.

What is noncommutative peak interpolation?:



$f = g$ on E becomes $fq = gq$

where q is the closed projection playing the role of (the characteristic function of) E

Our idea: So we want to take the classical interpolation results (like the Bishop-Gamelin theorem), and replace $A \subset C(K)$ by a subalgebra A of a C^*-algebra, replace closed sets E by closed projections q, and replace 'set statements' with 'algebra statements' like $f = g$ on E by $fq = gq$.

Exercise: What does $|f| \leq h$ on E become?

Answer: $f^* q f \leq h^* q h$, or ...

• This is the only theory at this point in time that literally and simultaneously generalizes both the classic function theoretic peak interpolation, and the C^*-algebraic interpolation I mentioned.

Noncommutative peak interpolation started in the PhD thesis of student Damon Hay

• Over the years we with coauthors (particularly Hay, Neal, and Read) have developed many noncommutative peak interpolation results, which when specialized to the case $B = C(K)$ collapse to classical peak interpolation theorems.

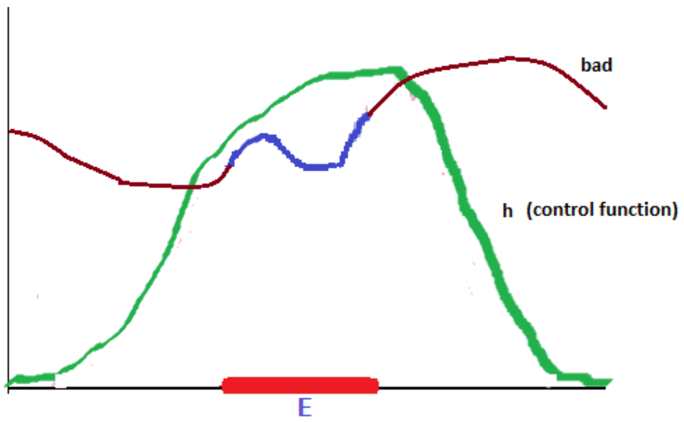
Noncommutative peak interpolation started in the PhD thesis of student Damon Hay

• Over the years we with coauthors (particularly Hay, Neal, and Read) have developed many noncommutative peak interpolation results, which when specialized to the case $B = C(K)$ collapse to classical peak interpolation theorems.

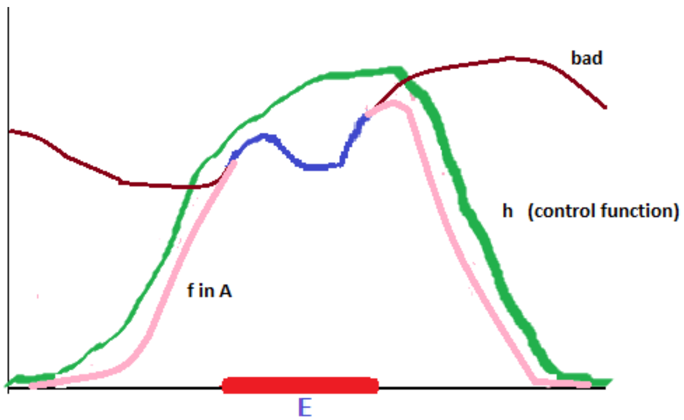
• Moreover, in the course of this investigation important applications have emerged to the theory of one-sided ideals or hereditary subalgebras of operator algebras, the theory of approximate identities, noncommutative topology, noncommutative function theory, etc.

There should be many more such applications.

Theorem (Noncommutative Bishop-Gamelin) Suppose that A is a unital (resp. not necessarily unital) operator algebra, a subalgebra of a unital C^*-algebra B. Suppose that q is a closed (resp. compact) projection in A^{**}. If $b \in A$ with $bq = qb$, and $qb^*bq \leq qh$ for an invertible positive $h \in B$ which commutes with q, then there exists an element $f \in A$ with $fq = qf = bq$, and $f^*f \leq h$.



Theorem (Noncommutative Bishop-Gamelin) Suppose that A is a unital (resp. not necessarily unital) operator algebra, a subalgebra of a unital C^*-algebra B. Suppose that q is a closed (resp. compact) projection in A^{**}. If $b \in A$ with $bq = qb$, and $qb^*bq \leq qh$ for an invertible positive $h \in B$ which commutes with q, then there exists an element $f \in A$ with $fq = qf = bq$, and $f^*f \leq h$.



Very recently: Such peak interpolation theorems but with the interpolating element 'positive' in the new senses, or with 'prescribed' numerical range (Tietze theorem).

Very recently: Such peak interpolation theorems but with the interpolating element ‘positive’ in the new senses, or with ‘prescribed’ numerical range (Tietze theorem).

- Such interpolation results are ‘new relationships between an operator algebra and the C^*-algebra it generates’

Sample application: As a corollary, one obtains the theorem of Read on contractive approximate identities (cai's), which is one of the more important results in the theory (and actually is essentially equivalent to many of the results in the talk):

Read's theorem If A is an operator algebra with a cai, then A has a cai (e_t) with positive real parts, indeed satisfying $\|1 - 2e_t\| \leq 1$ (indeed nearly positive, i.e in the thinnest of cigars) for all t.

Sample application: As a corollary, one obtains the theorem of Read on contractive approximate identities (cai's), which is one of the more important results in the theory (and actually is essentially equivalent to many of the results in the talk):

Read's theorem If A is an operator algebra with a cai, then A has a cai (e_t) with positive real parts, indeed satisfying $\|1 - 2e_t\| \leq 1$ (indeed nearly positive, i.e in the thinnest of cigars) for all t.

In turn, from Read's theorem it is not hard to show the noncommutative version of Glicksberg's peak set theorem which we mentioned earlier.

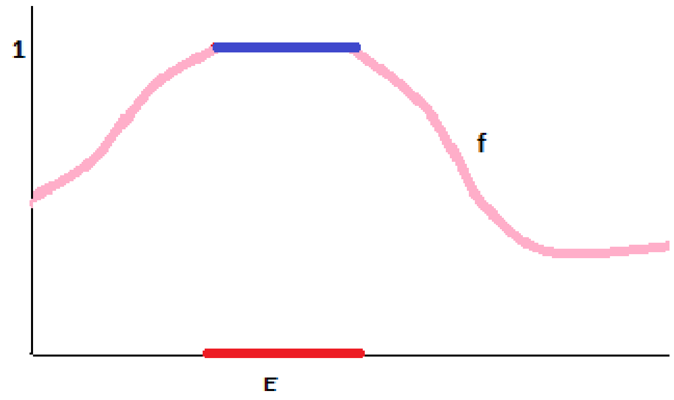
Reminder: The sets E that 'work' for classical peak interpolation are the p-sets, namely the closed sets whose characteristic functions are in the 'second annihilator' (or weak* closure) of A in $C(K)^{**}$

Glicksberg's peak set theorem characterizes these sets as the intersections of peak sets, i.e. sets $E = f^{-1}(\{1\})$ for a norm 1 function f in A.

- In the separable case, they are just the peak sets (one doesnt need intersections)

Theorem (Noncommutative Glicksberg peak set theorem of B-Read) The closed (resp. compact) projections in $A^{\perp\perp}$ are precisely the decreasing limits (or infima) of peak projections. If A is separable, they are just the peak projections.

• There are many equivalent ways to define peak projections (see Hay's thesis, etc). In fact they are the weak* limits of f^n for $f \in \mathrm{Ball}(A)$ in the cases that such limit exists.



• We now understand these noncommutative peak sets and how they work.

Section II. Real positivity (B-Read 2011-2013, B-Ozawa 2014)

• In a possibly nonunital operator algebra A, say that $x \in A$ is real positive (or accretive) if $x + x^* \geq 0$. More generally, an element x in a Banach algebra A is real positive if Re $\varphi(x) \geq 0$ for every state φ on A^1. (Below 1 is the identity of the unitization.)

• Write $\mathfrak{r}_A$ for the set of real positive elements.

• This contains $\mathfrak{F}_A = \{a \in A : ||1 - a|| \leq 1\}$, and the cone $\mathbb{R}^+\mathfrak{F}_A$. Let us write $\mathfrak{C}_A$ for either of these cones.

Section II. Real positivity (B-Read 2011-2013, B-Ozawa 2014)

• In a possibly nonunital operator algebra A, say that $x \in A$ is real positive (or accretive) if $x + x^* \geq 0$. More generally, an element x in a Banach algebra A is real positive if Re $\varphi(x) \geq 0$ for every state φ on A^1. (Below 1 is the identity of the unitization.)

• Write $\mathfrak{r}_A$ for the set of real positive elements.

• This contains $\mathfrak{F}_A = \{a \in A : ||1 - a|| \leq 1\}$, and the cone $\mathbb{R}^+\mathfrak{F}_A$. Let us write $\mathfrak{C}_A$ for either of these cones.

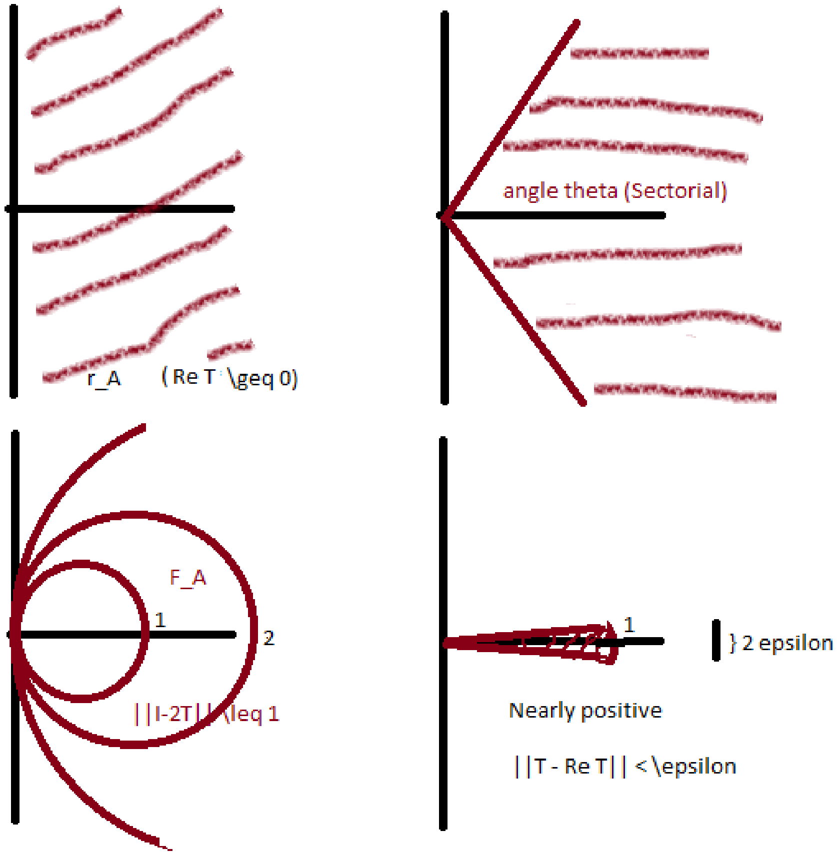
Proposition If A has a cai then $\overline{\mathbb{R}^+\mathfrak{F}_A} = \mathfrak{r}_A$

• These will play the role for us of positive elements in a C^*-algebra; the main goal is to generalize certain nice C^*-algebraic results, or nice function space results, which use positivity or positive cai's. For example, we saw this in our new Urysohn lemmas above.

• As we said earlier, if $T+T^* \geq 0$ then $T^{\frac{1}{n}}$ has numerical range in 'cigar' centered on $[0,1]$ that thins to $[0,1]$ as $n \to \infty$.

• Fact: If the numerical range $W(T) \subset [0,1] \times [-\epsilon, \epsilon]$ then $\mathrm{Re}(T) \geq 0$ and $\|T - \mathrm{Re}(T)\| \leq \epsilon$

'Nearly positive' : if in a given theorem you can choose the element as close as one wishes to a positive in the usual sense.



Four variants of 'positivity' (these are pictures of the region containing the numerical range of T)

Recall that $T : A \to B$ between C^*-algebras (or operator systems) is completely positive if $T(A_+) \subset B_+$, and similarly at the matrix levels

Definition (Bearden-B-Sharma) A linear map $T : A \to B$ between operator algebras or unital operator spaces is *real completely positive*, or RCP, if $T(\mathfrak{r}_A) \subset \mathfrak{r}_B$ and similarly at the matrix levels. (Later variant of a notion of B-Read.)

Recall that $T : A \to B$ between C^*-algebras (or operator systems) is completely positive if $T(A_+) \subset B_+$, and similarly at the matrix levels

Definition (Bearden-B-Sharma) A linear map $T : A \to B$ between operator algebras or unital operator spaces is *real completely positive*, or RCP, if $T(\mathfrak{r}_A) \subset \mathfrak{r}_B$ and similarly at the matrix levels. (Later variant of a notion of B-Read.)

Theorem A (not necessarily unital) linear map $T : A \to B$ between C^*-algebras or operator systems is completely positive in the usual sense iff it is RCP

(Extension and Stinespring-type) Theorem A linear map $T : A \to B(H)$ on an approximately unital operator algebra or unital operator space is RCP iff T has a completely positive (usual sense) extension $\tilde{T} : C^*(A) \to B(H)$

This is equivalent to being able to write T as the restriction to A of $V^*\pi(\cdot)V$ for a $*$-representation $\pi : C^*(A) \to B(K)$, and an operator $V : H \to K$.

The induced ordering is obviously $b \preccurlyeq a$ iff $\mathrm{Re}(a-b) \geq 0$ (or equivalently, or for Banach algebras, iff $a-b$ accretive (i.e. numerical range in right half plane))

The induced ordering is obviously $b \preccurlyeq a$ iff $\mathrm{Re}(a-b) \geq 0$ (or equivalently, or for Banach algebras, iff $a - b$ accretive (i.e. numerical range in right half plane))

- Of course many C^*-algebra results involving order dont generalize, or are harder largely because our functional calculus is not as good

The induced ordering is obviously $b \preccurlyeq a$ iff $\mathrm{Re}(a-b) \geq 0$ (or equivalently, or for Banach algebras, iff $a-b$ accretive (i.e. numerical range in right half plane))

- Of course many C^*-algebra results involving order dont generalize, or are harder largely because our functional calculus is not as good

An 'order interpolation result': If an approximately unital operator algebra A generates a C^*-algebra B, then A is *order cofinal* in B: given $b \in B_+$ there exists $a \in A$ with $b \preccurlyeq a$. Indeed can do this with $b \preccurlyeq a \preccurlyeq \|b\| + \epsilon$

Indeed can do this with $b \preccurlyeq Ce_t \preccurlyeq \|b\| + \epsilon$, for a nearly positive cai (e_t) for A

(This and the results on next page are trivial if A unital)

Theorem Let A be an operator algebra which generates a C^*-algebra B, and let $\mathcal{U}_A = \{a \in A : \|a\| < 1\}$. The following are equivalent:

(1) A is approximately unital.

(2) For any positive $b \in \mathcal{U}_B$ there exists $a \in \mathfrak{c}_A$ with $b \preccurlyeq a$.

(2') Same as (2), but also $a \in \frac{1}{2}\mathfrak{F}_A$ and nearly positive.

(3) For any pair $x, y \in \mathcal{U}_A$ there exist nearly positive $a \in \frac{1}{2}\mathfrak{F}_A$ with $x \preccurlyeq a$ and $y \preccurlyeq a$.

(4) For any $b \in \mathcal{U}_A$ there exist nearly positive $a \in \frac{1}{2}\mathfrak{F}_A$ with $-a \preccurlyeq b \preccurlyeq a$.

(5) For any $b \in \mathcal{U}_A$ there exist $x, y \in \frac{1}{2}\mathfrak{F}_A$ with $b = x - y$.

(6) $\mathfrak{C}_A$ is a generating cone (that is, $A = \mathfrak{C}_A - \mathfrak{C}_A$).

• An operator algebra or function algebra may have no positive elements in the usual sense, but we saw in Read's theorem that if it has a cai then it has a cai in $\frac{1}{2}\mathfrak{F}_A$, and even 'nearly positive'

We also just saw: An operator algebra A has a cai iff $A = \mathfrak{c}_A - \mathfrak{c}_A$

Theorem If operator algebra A has no cai then $D = \mathfrak{c}_A - \mathfrak{c}_A$ is the biggest subalgebra with a cai. It is a HSA (that is, $DAD \subset D$).

• An operator algebra or function algebra may have no positive elements in the usual sense, but we saw in Read's theorem that if it has a cai then it has a cai in $\frac{1}{2}\mathfrak{F}_A$, and even 'nearly positive'

We also just saw: An operator algebra A has a cai iff $A = \mathfrak{C}_A - \mathfrak{C}_A$

Theorem If operator algebra A has no cai then $D = \mathfrak{C}_A - \mathfrak{C}_A$ is the biggest subalgebra with a cai. It is a HSA (that is, $DAD \subset D$).

• So for operator algebras, cai's are manifestations of our cone $\mathfrak{C}_A$, just as we will see that a nice class of one-sided ideals are manifestations of $\mathfrak{C}_A$

• $x \mapsto x(1+x)^{-1}$ maps $\mathfrak{r}_A$ into $\mathfrak{F}_A$, with inverse map $x \mapsto x(1-x)^{-1}$ (This is related to the Cayley transform). For operator algebras the range of this map is $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$

• We recall that the positive part of the open unit ball $\mathcal{U}_B$ of a C^*-algebra B is a directed set, and indeed is a net which is a positive cai for B. The following generalizes this to operator algebras:

Corollary If A is an approximately unital operator algebra, then $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is a directed set in the $\preccurlyeq$ ordering, and with this ordering $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is an increasing cai for A.

• The first part is also true for a large class of Banach algebras

- How about for Banach algebras?

In fact variants of much of this goes through, for 'nice' Banach algebras. First,

Theorem A Banach algebra A with a sequential cai and a smoothness property, has a sequential cai in $\mathfrak{F}_A$. Under a stronger property than smoothness, e.g. if A is an M-ideal in its unitization A^1, then A has a cai in $\frac{1}{2}\mathfrak{F}_A$.

- How about for Banach algebras?

In fact variants of much of this goes through, for 'nice' Banach algebras. First,

Theorem A Banach algebra A with a sequential cai and a smoothness property, has a sequential cai in $\mathfrak{F}_A$. Under a stronger property than smoothness, e.g. if A is an M-ideal in its unitization A^1, then A has a cai in $\frac{1}{2}\mathfrak{F}_A$.

Proposition If A has a cai then $\mathfrak{r}_A$ and $\mathfrak{F}_A$ are closed under roots

Theorem For 'nice' Banach algebras the cone $\mathfrak{C}_A$ is generating again (and has others of the nice order properties in the earlier 6 part theorem)

- 'Nice' includes all unital Banach algebras, or those with a countable cai and/or weak* closed quasistate space, or those which are Hahn-Banach smooth in A^1, etc

Application: understanding ideal structure of an algebra

An r-ideal is a closed right ideal with a left cai

An ℓ-ideal is a closed left ideal with a right cai

- For Banach algebras we ask that these cai are in $\mathfrak{r}_A$. For operator algebras it is automatic that one can choose them in $\mathfrak{r}_A$.

In earlier work with Read, we completely classified these ideals, and with Ozawa we extended much of this to Banach algebras:

Application: understanding ideal structure of an algebra

We now discuss how r-ideals are built for Banach algebras A from our 'positive' elements

Application: understanding ideal structure of an algebra

We now discuss how r-ideals are built for Banach algebras A from our 'positive' elements

- For operator algebras, and separable Banach algebras, there is a non-trivial correspondence between the r-ideals and ℓ-ideals

Application: understanding ideal structure of an algebra

We now discuss how r-ideals are built for Banach algebras A from our 'positive' elements

- For operator algebras, and separable Banach algebras, there is a non-trivial correspondence between the r-ideals and ℓ-ideals

Actually, these one-sided ideals turn out to be 'manifestations' of our cone $\mathfrak{S}_A$, or if preferred, 'nearly positive' elements:

Theorem The r-ideals in an operator algebra A are exactly $\overline{EA}$ for some subset $E \subset \mathfrak{C}_A$. Then the matching ℓ-ideal is $\overline{AE}$

- If A (or the r-ideal) is separable can take E singleton

Theorem Let A be any Banach algebra. The r-ideals in A, are precisely the closures of increasing unions of ideals of the form $\overline{xA}$, for $x \in \mathfrak{C}_A$.

Theorem Let A be a commutative approximately unital Banach algebra. The closed ideals in A with a bai in $\mathfrak{r}_A$ are precisely the ideals of the form $\overline{EA}$ for some subset $E \subset \mathfrak{F}_A$. They are also the closures of increasing unions of ideals of the form $\overline{xA}$ for $x \in \mathfrak{F}_A$. Under a countability condition, only one such x is needed (and can drop the 'commutative' hypothesis).

- For all operator algebras can drop the 'commutative' and the 'approximately unital' assumption ([BR] 2011)

Corollary If A is a Banach algebra, then A has a sequential real positive cai iff there exists an $x \in \mathfrak{C}_A$ with $A = \overline{xA} = \overline{Ax} = \overline{xAx}$.

- A separable Banach algebra with a real positive cai of course is of this form

- Of course a cai is $(x^{\frac{1}{n}})$

- There are nice connections to the classical theory of ordered spaces (Krein, Ando, Alfsen, etc)

Theorem If A is an approximately unital operator algebra, or 'nice' Banach algebra, then the real positive $f : A \to \mathbb{C}$ are just the nonnegative multiples of states of A

- In fact the latter is essentially equivalent to the quasistate space being weak* closed

Corollary (Kaplansky density type result) The ball of $\mathfrak{r}_A$ is weak* dense in the ball of $\mathfrak{r}_{A^{**}}$

Remark. We use states a lot .... However for a general approximately unital Banach algebra A with cai (e_t), the definition of 'state' is problematic. There are several natural notions, and which is best depends on the situation. We haven't noticed this discussed in the literature though....?

- Under a smoothness hypothesis though they all coincide.

There is a nonselfadjoint 'Tietze' extension theorem, a noncommutative version of:

Theorem Suppose that A is a function algebra on a compact Hausdorff space K, and E is a peak (or p-) set for A. If $f \in A$ with $f(E) \subset F$, where F is closed convex set F in the plane, then there exists a function $g \in A$ which agrees with f on E, which has norm $\|g\|_K = \|f_{|E}\|_E$, and which has range $g(K) \subset F$ (or $g(K) \subset \hat{L}$ if $\mathrm{conv}(f(E))$ is a line segment L, where $\hat{L}$ is a thin triangle given in advance, whose one side is L).

- Essentially a result of Smith et al, and this one generalizes to the case A is a Banach algebra satisfying a reasonable condition.

Corollary Can lift 'positives' in quotients A/J to 'positives' in A (if J is nice)

When xA is closed/pseudoinvertibility

The C^*-algebra result:

We recall that 'well supported' operators are those operators x that have a 'spectral gap' for $|x|$ at 0, that is 0 is absent from, or is isolated in, the spectrum of $|x|$.

Theorem (Harte-Mbekhta) An element x of a C^*-algebra A is well supported iff xA is closed, and iff there exists $y \in A$ with $xyx = x$.

When xA is closed/pseudoinvertibility

The C^*-algebra result:

We recall that 'well supported' operators are those operators x that have a 'spectral gap' for $|x|$ at 0, that is 0 is absent from, or is isolated in, the spectrum of $|x|$.

Theorem (Harte-Mbekhta) An element x of a C^*-algebra A is well supported iff xA is closed, and iff there exists $y \in A$ with $xyx = x$.

Such a y is called a *generalized inverse* or *pseudoinverse*.

We get a similar result, about pseudoinvertibility in nonselfadjoint operator algebras, and with 'spectral gap' for x not $|x|$, for our cone.

Theorem For any Banach algebra A with cai, if $x \in \mathfrak{C}_A$, then the following are equivalent:

(i) xA is closed.

(ii) Ax is closed.

(iii) There exists $y \in A$ with $xyx = x$.

(iv) $\overline{alg}(x)$ is unital.

Also, the latter conditions imply

(v) 0 is isolated in, or absent from, $\mathrm{Sp}_A(x)$.

Finally, if further $\overline{alg}(x)$ is semisimple, then conditions (i)–(v) are all equivalent.

Theorem These xA are equal to eA for a projection in A.

In fact more is true. A closed right ideal J in A is *algebraically finitely generated* if there exist $x_1, \cdots, x_n \in A$ with $J = x_1 A + \cdots + x_n A$.

Theorem The algebraically finitely generated closed right ideals in A that have a left cai, actually are singly generated, indeed equal eA for an idempotent e in $\mathfrak{F}_A$ (in $\frac{1}{2}\mathfrak{F}_A$ if A is an operator algebra).

Section IV. Where can this lead?

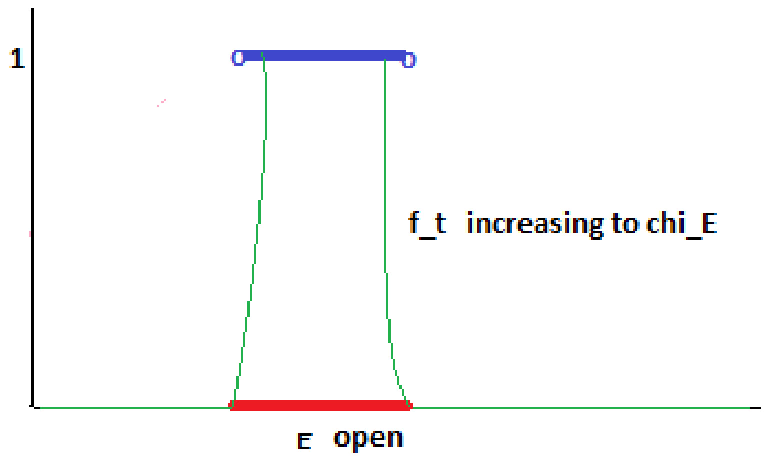
• Main answer at this point: Generalizing results and theories that are only known for C^*-algebras, to more general algebras of operators on a Hilbert space, thus expanding the theory of such algebras in useful ways. I call this C^*-theory for operator algebras, and we already have several examples of this.

• Also, generalizing more of the classical applications from function algebra theory/function theory of peak sets, to operator algebras.

Akemann's noncommutative topology

Question: What kind of projections correspond exactly to closed sets in K?

Answer: E is an open set in K iff χ_E is a increasing (weak) limit of positive elements in $B = C(K)$ (a good word here is 'semicontinuous')



We can view χ_E as a projection in the second dual B^{**}, which is known to be a C^*-algebra. (This is because by the Riesz representation theorem $C(K)^*$ is a space of measures μ on K, and any such measure μ may be paired with E.)

Thus it is natural to declare a projection $q \in B^{**}$ to be *open* if it is a increasing (weak*) limit of positive elements in B

- Define a projection q to be closed if $1 - q$ is open.

Exercise: A projection $q \in C(K)^{**}$ is open (resp. closed) iff it is the image as above of the characteristic function of an open (resp. closed) set in E.

Thus it is natural to declare a projection $q \in B^{**}$ to be *open* if it is a increasing (weak*) limit of positive elements in B

- Define a projection q to be closed if $1 - q$ is open.

Exercise: A projection $q \in C(K)^{**}$ is open (resp. closed) iff it is the image as above of the characteristic function of an open (resp. closed) set in E.

Thus topology has become the theory of a certain class of projections in the second dual B^{**}. This is Akemann's noncommutative topology.

- As we said, B^{**} is known to be a C^*-algebra, indeed it is a von Neumann algebra. Can work in a smaller space than B^{**} if you want to.

We will not discuss Akemann's noncommutative topology much today, but with the definitions above one can now try to prove noncommutative versions of the basic results in topology. E.g. 0 and 1 are both open and closed projections, Unions of sets are replaced by suprema $\vee_i p_i$ of projections, Intersections of sets are replaced by infima $\wedge_i p_i$ of projections.

We will not discuss Akemann's noncommutative topology much today, but with the definitions above one can now try to prove noncommutative versions of the basic results in topology. E.g. 0 and 1 are both open and closed projections, Unions of sets are replaced by suprema $\vee_i p_i$ of projections, Intersections of sets are replaced by infima $\wedge_i p_i$ of projections.

(If we are thinking of A^{**} as linear operators on H, then a projection $p \in A$ corresponds to a closed subspace $p(H)$ of H. Then $\vee_i p_i$ is the projection onto the closure of the span of the $p_i(H)$. And $\wedge_i p_i$ is the projection onto $\cap_i p_i(H)$.)

• Open projections arise naturally in functional analysis. For example, they come naturally out of the ' spectral theorem/functional calculus': the spectral projections of a selfadjoint operator T corresponding to open (resp. closed) sets in the spectrum of T, are open (resp. closed) projections.

• Open projections arise naturally in functional analysis. For example, they come naturally out of the ‘ spectral theorem/functional calculus’: the spectral projections of a selfadjoint operator T corresponding to open (resp. closed) sets in the spectrum of T, are open (resp. closed) projections.

The range projection of an operator a, that is the smallest projection p with $pa = a$, is an open projection.

Conversely, every open projection is a supremum (increasing limit) of range projections.

• Open projections arise naturally in functional analysis. For example, they come naturally out of the ' spectral theorem/functional calculus': the spectral projections of a selfadjoint operator T corresponding to open (resp. closed) sets in the spectrum of T, are open (resp. closed) projections.

The range projection of an operator a, that is the smallest projection p with $pa = a$, is an open projection.

Conversely, every open projection is a supremum (increasing limit) of range projections.

• A final very cool thing about open projections: they are in one-to-one correspondence with the closed right ideals in B, via the support projection. (Or the left ideals.) So you can interpret the results above in terms of the one-sided ideal structure in B