C*-algebras of Matricially Ordered *-Semigroups

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Preface

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Universal C*-algebras involving an automorphism realized via an implementing unitary, or an endomorphism via an isometry, have played a fundamental role in operator algebras. Such maps preserve algebraic structure.

A map of a C*-algebra defined via an implementing partial isometry does not preserve algebra structure. It is, however, a completely positive *-linear map.

We consider *-semigroups S, matricial partial order orders on S, along with a universal C*-algebra associated with S and a matricial ordering on S.

For a particular example of a matricially ordered *-semigroup S along with complete order map on S, we obtain a C*-correspondence over the associated C*-algebra of S. The complete order map is implemented by a partial isometry in the Cuntz-Pimsner C*-algebra associated with the correspondence.

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It is known that $\mathcal{P}$ is nonunital, nonexact, residually finite dimensional, and Morita equivalent to the universal C*-algebra generated by a contraction.

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For B a C*-algebra, the contractions (or strict contractions) in B viewed as a semigroup under multiplication, with * the usual involution. In particular, for $\mathcal{H}$ a Hilbert space and $B = \mathcal{B}(\mathcal{H})$.

Matricial order

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For $k \in \mathbb{N}$, let $[n_i]$ denote an element $[n_1, ..., n_k] \in \mathrm{M}_{1,k}(S)$, the $1 \times n$ matrices with entries in S.

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For $k \in \mathbb{N}$, let $[n_i]$ denote an element $[n_1, ..., n_k] \in \mathrm{M}_{1,k}(S)$, the $1 \times n$ matrices with entries in S.

Then $[n_i]^* \in \mathrm{M}_{k,1}(S)$, a $k \times 1$ matrix over S,
and the element $[n_i]^*[n_j] = [n_i^* n_j] \in \mathrm{M}_k(S)^{sa}$.

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For example, if

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

is positive in $\mathrm{M}_2(B)^{sa}$ then

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,2} \\ a_{2,1} & a_{2,2} & a_{2,2} \\ a_{2,1} & a_{2,2} & a_{2,2} \end{pmatrix}$$

is also positive in $\mathrm{M}_3(B)^{sa}$.

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Notation:
For $d, k \in \mathbb{N}$ and $d \leq k$, set

$$\mathcal{P}(d, k) = \left\{ (t_1, ..., t_d) \in (\mathbb{N}_0)^d \mid \sum_{r=1}^{d} t_r = k \right\}.$$

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Each $\tau = (t_1, \ldots, t_d) \in \mathcal{P}(d,k)$ yields a *-map $\iota_\tau : \mathrm{M}_d(B) \to \mathrm{M}_k(B)$. For $[a_{i,j}] \in \mathrm{M}_d(B)$ the element $\iota_\tau([a_{i,j}]) := [a_{i,j}]_\tau \in \mathrm{M}_k(B)$ is the matrix obtained using matrix blocks; the i, j block of $[a_{i,j}]_\tau$ is the $t_i \times t_j$ matrix with the constant entry $a_{i,j}$.

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Lemma

For $\tau = (t_1, ..., t_d) \in \mathcal{P}(d,k)$ *and* $[b_{i,j}] \in \mathrm{M}_{r,d}(B)$. *There is* $[c_{i,j}] \in \mathrm{M}_{r,k}(B)$, *whose entries appear in* $[b_{i,j}]$, *such that*

$$\iota_\tau([b_{i,j}]^* [b_{i,j}]) = [c_{i,j}]^* [c_{i,j}] .$$

Proof.

For $1 \leq i \leq r$ let the $r \times k$ matrix $[c_{i,j}]$ have i-th row

$$[b_{i1}, ..., b_{i1}, b_{i2}, ..., b_{i2}, ..., b_{id}, ..., b_{id}]$$

where each element b_{ij} appears repeated t_j consecutive times. □

Note that the maps ι_τ are defined even if the matrix entries are from a set, so in particular for matrices with entries from a *-semigroup S, and although there is no natural 'positivity' for matrices with entries in S one can still use partial orderings.

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Definition

A *-semigroup S is matricially ordered, write $(S, \preceq, \mathcal{M})$, if there is a sequence of partially ordered sets $(\mathcal{M}_k(S), \preceq)$, $\mathcal{M}_k(S) \subseteq \mathrm{M}_k(S)^{sa}$ $(k \in \mathbb{N})$, with $\mathcal{M}_1(S) = S^{sa}$, satisfying (for $[n_i] \in \mathrm{M}_{1,k}(S)$)

a. $[n_i]^*[n_j] = [n_i^* n_j] \in \mathcal{M}_k(S)$

b. if $[a_{i,j}] \preceq [b_{i,j}]$ in $\mathcal{M}_k(S)$ then $[n_i^* a_{i,j} n_j] \preceq [n_i^* b_{i,j} n_j]$ in $\mathcal{M}_k(S)$

c. the maps $\iota_\tau : \mathcal{M}_d(S) \to \mathcal{M}_k(S)$ are order maps for all $\tau \in \mathcal{P}(d, k)$.

The lemma above showed that a C*-algebra B has a matricial order where $\mathcal{M}_k(B)$ is the usual partially ordered set $\mathrm{M}_k(B)^{sa}$.

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We may define *-maps $\beta : S \to T$ of matricially ordered *-semigroups S and T that are *complete order maps* - so $\beta_k : \mathcal{M}_k(S) \to \mathcal{M}_k(T)$ is defined, and an order map of partially ordered sets. A completely positive map of C*-algebras is then a complete order map.

A *complete order representation* of a matricially ordered *-semigroup S into a C*-algebra is a *-homomorphism which is a complete order map.

C*-algebras of S

If F is a specified collection of *-representations of S in C*-algebras, for example *-representations, contractive *-representations, or complete order *-representations, then the universal C*-algebra of S is a C*-algebra $C^*_F(S)$ along with a *-semigroup homomorphism $\iota : S \to C^*_F(S)$ in F satisfying the universal property

$$\begin{array}{lcl} S & & \\ \downarrow \iota & \searrow \gamma & \in F \\ C^*(S) & \pi_\gamma \dashrightarrow & C \end{array}$$

Given $\gamma : S \to C$, $\gamma \in F$, there is a unique *-homomorphism $\pi_\gamma = \pi : C^*_F(S) \to C$ with $\pi_\gamma \circ \iota = \gamma$.

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If F is a specified collection of *-representations of S in C*-algebras, for example *-representations, contractive *-representations, or complete order *-representations, then the universal C*-algebra of S is a C*-algebra $C_F^*(S)$ along with a *-semigroup homomorphism $\iota : S \to C_F^*(S)$ in F satisfying the universal property

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Given $\gamma : S \to C$, $\gamma \in F$, there is a unique *-homomorphism $\pi_\gamma = \pi : C_F^*(S) \to C$ with $\pi_\gamma \circ \iota = \gamma$.

For an arbitrary *-semigroup one can also form the universal C*-algebra where F is the collection of contractive *-representations.

Hilbert modules

Definition

Let $\beta : S \to T$ be a *-map of a *-semigroup S to a matricially ordered *-semigroup $(T, \preceq, \mathcal{M})$. The map β_k has the Schwarz property for k, if

$$\beta_k([n_i])^* \beta_k([n_j]) \preceq \beta_k([n_i]^*[n_j])$$

in $\mathcal{M}_k(T)$ for $[n_i] \in \mathrm{M}_{1,k}(S)$. Here $\beta_k([n_i])^* \beta_k([n_j])$ is the selfadjoint element $[\beta(n_i)^* \beta(n_j)]$ in $\mathcal{M}_k(T)$.

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A *-homomorphism $\sigma : S \to T$ of *-semigroups has the Schwarz property (since $\sigma_k([n_i])^* \sigma_k([n_j]) = \sigma_k([n_i]^*[n_j])$ for $[n_i] \in \mathrm{M}_{1,k}(S)$).

Note that if $\beta : R \to S$ and $\sigma : S \to T$ are complete order maps, β with the Schwarz property and σ a *-semigroup homomorphism, then $\sigma\beta$ is a complete order map with the Schwarz property.

A (complete) Schwarz map to a C*-algebra C is necessarily completely positive:

Definition

A *-map $\beta : S \to C$ from a *-semigroup S into a C*-algebra C is completely positive if the matrix $[\beta(n_i^* n_j)]$ is positive in $\mathrm{M}_k(C)$ for any finite set $n_1, ..., n_k$ in S.

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Completely positive maps yield Hilbert modules; so for $\beta : S \to C$ completely positive from a *-semigroup S into a C*-algebra C then $X = \mathbb{C}[S] \otimes_{alg} C$ has a C valued (pre) inner product (for $x = s \otimes c$, $y = t \otimes d$, with $s, t \in S, c, d$ in C

set $\langle x, y \rangle = \langle c, \beta(s^* t) d \rangle = c^* \beta(s^* t) d)$,

After moding out by 0 vectors and completing obtain a right Hilbert module $\mathcal{E}_C$.

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Assume there is a *-map $\alpha : S \to S$ which is a complete order map satisfying the Schwarz inequality for all $k \in \mathbb{N}$.

Then since $\iota : S \to C^*((S, \preceq, \mathcal{M}))$ is a complete order representation, the composition $\beta = \iota \circ \alpha : D_1 \to C^*((D_1, \preceq, \mathcal{M}))$ is a complete order map satisfying the (complete) Schwarz inequality.

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The map β is therefore completely positive and we can form the Hilbert module $\mathcal{E}_{C^*(S, \preceq, \mathcal{M}))}$.

Furthermore, if the left action of S extends to an action by adjointable maps on the Hilbert module $\mathcal{E}_C$, and if

$$I : S \to \mathcal{L}(\mathcal{E}_{C^*((S,\preceq,\mathcal{M}))})$$

is additionally a complete order representation of the matricially ordered *-semigroup S, the universal property yields a *-representation

$$\phi : C^*((S,\preceq,\mathcal{M})) \to \mathcal{L}(\mathcal{E}_{C^*((S,\preceq,\mathcal{M}))})$$

defining a correspondence $\mathcal{E}$ over the C*-algebra $C^*((S,\preceq,\mathcal{M}))$.

There is a *-semigroup D_1 for which one can describe an ordering, and matricial ordering, where the steps in this process hold. It is nonunital, and not left cancellative, so existing procedures for forming C*-algebras from semigroups, which seem largely motivated by versions of a 'left regular representation', do not apply.

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A relative Cuntz-Pimsner C*-algebra associated with the above C*-correspondence over the C*-algebra $C^*((D_1, \preceq, \mathcal{M}))$ is isomorphic to the universal C*-algebra $\mathcal{P}$ generated by a partial isometry.

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Let S be the two element unital (unit u) two element *-semigroup $\{u, s\}$ with s a selfadjoint idempotent and α the map sending both elements to u. The above Cuntz-Pimsner algebra over the C*-algebra of this semigroup is the universal C*-algebra generated by an isometry.

The free *-semigroup generated by a single element is $A_c \cong \mathbb{N}^+ * \mathbb{N}^-$, it consists of reduced words of nonzero integers $(n_0, n_1, ..., n_k)$ alternating in sign, multiplication is concatenation, and $(n_0, n_1, ..., n_k)^* = (-n_k, -n_{k-1}, ..., -n_0)$.

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The *-semigroup A is a quotient of A_c. Form the equivalence relation generated by the relation

$$(n_0, n_1, ..., n_k) \sim (n_0, n_1, ..., n_{i-1} \pm 1 + n_{i+1}, ...n_k)$$

whenever $n_i = \pm 1$ for $1 \leq i \leq k-1$.

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The map $\alpha : A \to A$ is defined by $\alpha(n) = (-1)n(1)$.
The elements $(-1, 1)$ and $(1, -1)$ of A^0 are idempotents, and $\alpha(1. - 1)) = (-1, 1)$.

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The *-semigroup D_1 is the smallest α-closed (*-)subsemigroup of A containing the element $(1, -1)$.