Completely bounded isomorphisms and similarity to complete isometries

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There exists an invertible matrix X such that XTX^{-1} is in Jordan form.

Functional model for Jordan cells

Let $J \in M_n(\mathbb{C})$ be the usual Jordan cell with eigenvalue 0,

$$J = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

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Consider the Hardy space $H^2 = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$. The unilateral shift S acts on H^2 as $(Sf)(z) = zf(z)$.

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Up to unitary equivalence, we have that $J = P_{K_\theta} S|K_\theta$.
Allowing for functions θ with more than one root, we see that any linear operator on a finite dimensional Hilbert space is similar to such a functional model.

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Let $T \in B(\mathcal{H})$ be a completely non-unitary contraction. Define

$$D_T = (I - T^*T)^{1/2}, \mathcal{D}_T = \overline{D_T \mathcal{H}}$$

$$D_{T^*} = (I - TT^*)^{1/2}, \mathcal{D}_{T^*} = \overline{D_{T^*} \mathcal{H}}.$$

The characteristic function of T is the contractive operator-valued holomorphic function

$$\Theta_T : \mathbb{D} \to B(\mathcal{D}_T, \mathcal{D}_{T^*})$$

defined as

$$\Theta_T(\lambda) = (-T + \lambda D_{T^*}(1 - \lambda T^*)^{-1} D_T)| \mathcal{D}_T .$$

We also have the pointwise defect function

$$\Delta_T : \mathbb{T} \to B(\mathcal{D}_T)$$

such that

$$\Delta_T(\zeta) = (I - \Theta_T(\zeta)^* \Theta_T(\zeta))^{1/2}.$$

One check that Δ_T is essentially bounded. Finally, put

$$K_{\Theta_T} = (H^2(\mathcal{D}_{T^*}) \oplus \overline{\Delta_T L^2(\mathcal{D}_T)}) \ominus \{\Theta_T u \oplus \Delta_T u : u \in H^2(\mathcal{D}_T)\}$$

$$S_{\Theta_T} = P_{K_{\Theta_T}}(S \oplus U)|K_{\Theta_T}.$$

Then, T is unitarily equivalent to S_{Θ_T} (this whole machinery is known as the Sz.-Nagy–Foias model theory).

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Definition

A (completely non-unitary) contraction $T \in B(\mathcal{H})$ is said to be of *class* C_0 if the associated H^∞-functional calculus has non-trivial kernel.

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Theorem (Sz.-Nagy–Foias, Bercovici,...)

Let $T \in B(\mathcal{H})$ be a C_0 contraction. Then, there exists a unique Jordan operator $J \in B(\mathcal{K})$ which is quasisimilar to T: there exist two bounded linear injective operators $W : \mathcal{H} \to \mathcal{K}, Z : \mathcal{K} \to \mathcal{H}$ with dense range and the property that $WT = JW, ZJ = TZ$.

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The relation of quasisimilarity is rather weak...Can this be improved?

Unitary equivalence

- (Arveson 1967, C. 2013) Let T_1 and T_2 be two quasisimilar C_0 contractions (satisfying some mild technical conditions). Assume that there exists a completely isometric algebra isomorphism

$$\varphi : \{T_1\}' \to \{T_2\}'$$

such that $\varphi(T_1) = T_2$. Then, T_1 and T_2 are unitarily equivalent.

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- What about similarity between T_1 and T_2? Can it be obtained under the weaker assumption that φ be only a completely bounded homomorphism with completely bounded inverse?
- Possible strategy: up to similarity, reduce to the situation addressed by the theorem

Paulsen's similarity theorem

Theorem (Paulsen 1984)

Let $\mathcal{A}$ be a unital operator algebra and $\varphi : \mathcal{A} \to B(\mathcal{H})$ be a unital completely bounded homomorphism. Then, there exists an invertible operator X with

$$\|X\|^2 = \|X^{-1}\|^2 = \|\varphi\|_{cb}$$

and such that map

$$a \mapsto X\varphi(a)X^{-1}$$

is completely contractive.

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QUESTION Let $\mathcal{A}, \mathcal{B}$ be unital operator algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a unital completely bounded homomorphism with completely bounded inverse ("completely bounded isomorphism"). Can we find two invertible operators X and Y with the property that the map

$$XaX^{-1} \mapsto Y\varphi(a)Y^{-1}$$

is completely isometric?

A general result

Theorem (C., 2014)

Let $\mathcal{A} \subset B(\mathcal{H}_1)$ and $\mathcal{B} \subset B(\mathcal{H}_2)$ be unital operator algebras. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a unital completely bounded isomorphism. Then, for any $\varepsilon > 0$ and any finite set $\mathcal{A}_0 \subset \mathcal{A}$, there exist two invertible operators $X \in B(\mathcal{H}_1)$ and $Y \in B(\mathcal{H}_2)$ such that the map

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is a complete contraction and such that

$$\|XaX^{-1}\| \leq (1+\varepsilon)\,(1+\varepsilon\,/\rho(\varepsilon))\,\|Y\varphi(a)Y^{-1}\|$$

for every $a \in \mathcal{A}_0$, where $\rho(\varepsilon)$ is a positive constant depending only on ε.

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for every $a \in \mathcal{A}_0$, where $\rho(\varepsilon)$ is a positive constant depending only on ε.
Moreover, if the subset $\mathcal{A}_0$ contains no non-trivial quasi-nilpotent element, then we have the sharper inequality

$$\|XaX^{-1}\| \leq (1+\varepsilon\,/\rho)\,\|Y\varphi(a)Y^{-1}\|$$

for every $a \in \mathcal{A}_0$, where

$$\rho = \inf_{a\in\mathcal{A}_0} r(a).$$

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Paulsen's theorem does not give lower bounds. Can we do better? Can we get a complete isometry?

Special case

Theorem (C.,2014)

Let $\mathcal{A} \subset B(\mathcal{H}_1)$ and $\mathcal{B} \subset B(\mathcal{H}_2)$ be unital operator algebras. Assume that there exists a unital completely bounded isomorphism $\theta : \mathcal{C} \to \mathcal{A}$ where $\mathcal{C}$ is either a C^*-algebra or a uniform algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a unital completely bounded isomorphism. Then, there exist two invertible operators $X \in B(\mathcal{H}_1)$ and $Y \in B(\mathcal{H}_2)$ such that the map

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What about general amenable operator algebras?

Example (Choi-Farah-Ozawa, 2013)

Let $\mathcal{C} = \ell^\infty(\mathbb{N}, M_2(\mathbb{C}))$ and $\mathcal{J} = c_0(\mathbb{N}, M_2(\mathbb{C}))$. Denote by $Q : \mathcal{C} \to \mathcal{C} \,/\, \mathcal{J}$ the quotient map. Let Γ be an abelian group and $\pi : \Gamma \to Q(\mathcal{C})$ be a uniformly bounded representation. A clever choice of Γ and π yields that the operator algebra

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We can answer the question in the affirmative for the algebra $\mathcal{A}$ (C.-Marcoux 2014)

Back to the classification problem: application to quotient algebras of H^∞

Theorem (C.,2014)

Let $\theta \in H^\infty$ be an inner function.

(i) *The algebra $H^\infty/\theta H^\infty$ contains no non-trivial quasi-nilpotent elements if and only if θ is a Blaschke product with simple roots.*

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(ii) *The algebra $H^\infty/\theta H^\infty$ is a uniform algebra if and only if θ is an automorphism of the disc. In that case, the algebra is isomorphic to $\mathbb{C}$. In particular, $H^\infty/\theta H^\infty$ is a C^*-algebra if and only if it is a uniform algebra.*

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  (a) *There exists a unital completely bounded isomorphism*

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(c) *the function θ is a Blaschke product whose roots $\{\lambda_n\}_n \subset \mathbb{D}$ satisfy the Carleson condition*

$$\inf_n \left\{ \prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{1 - \overline{\lambda_k}\lambda_n} \right| \right\} > 0.$$

The general case?

A counter-example (suggested by Ken Davidson) shows that this stronger version does not hold in general, and answers the original question in the negative.

Idea behind the counterexample

Consider the operator space $\mathscr{D} \subset M_2(\mathbb{C})$ consisting of elements of the form

$$\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

where z_1, z_2 are complex numbers, along with the operator space $\mathscr{R} \subset M_2(\mathbb{C})$ consisting of elements of the form

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The map $\psi : \mathscr{R} \to \mathscr{D}$ defined as

$$\psi \begin{pmatrix} z_1 & z_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

is easily seen to be a completely bounded linear isomorphism with completely bounded inverse. Intuitively, it is clear that this cannot be made similar to a complete isometry: $\|\cdot\|_2$ gives rise to Hilbert space structure while $\|\cdot\|_\infty$ does not.

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Embedding these operator spaces in the upper-right corner of an operator algebra together with some easy but tedious computations yields the desired counter-example.

Thank you!